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SHAKEDOWN ANALYSIS OF 3D FRAMES WITH AN EFFECTIVE TREATMENT OF THE LOAD COMBINATIONS

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Abstract. Using the static theorem and an algorithm based on dual decomposition, an efficient formulation for the shakedown analysis of 3D frame is proposed. An efficient treatment of the load combinations and an accurate and simple definition of the yield function of cross-section are proposed to increase effectiveness and to make shakedown analysis an affordable design tools.

1 INTRODUCTION

Shakedown analysis furnishes, in a direct and elegant manner, a reliable safety factor against plastic collapse, loss in functionality due to excessive deformation (ratcheting) or collapse due to low cycle fatigue (plastic shakedown) [1]. Due to its importance as a tool for designers [2, 3, 4], in the last years a notable efforts has been done to propose efficient numerical algorithms of analysis. The interest in direct methods for application to limit and shakedown analysis has been encouraged by the availability of new and efficient optimization algorithms [5, 6, 7] such as the Interior Point method which is employed for solving very large non-linear problems, [8, 9], as those obtained in the Finite Element discretization of real-scale engineering structures. An alternative approach is represented by the specialized direct method proposed in[10, 11, 12] which can be used to evaluate the shakedown safety factors of structures. This approach is based on a strain-driven strategy of analysis hinged on closest point projection return mapping schemes and Riks arc-length solution techniques. It can be seen as the application of the proximal point algorithm to the static shakedown or limit analysis theorem and the solution of the resulting problem is performed by means of the dual decomposition strategy [12]. For its relation with standard elastoplastic analysis, to which it reduce in the limit analysis case, it will be named pseudo elastoplastic analysis and denoted as SD-CPP (Strain Driven - Closest Point Projection).

Shakedown analysis in structural design is still confined to the research community despite its important implications. This is, in part, still due to a series of problems regarding the efficiency and the robustness of the algorithms of analysis used. In the case of 3D frame structures, when considering standard rules as the Eurocodes ones, shakedown analysis requires a preliminary fine tuning o two important aspects to be used by designers: i) a suitable treatment of the usually large number of load combinations; ii) an accurate and simple definition of the yield function which defines the nonlinear behavior of the generic cross-section.

With respect to point i) it is well known how the number of basic actions to be considered heavily affects the computational costs of the analysis especially when an approach based on the static theorem is used. In the more simple case of load domain defined as a combination of basic actions varying between a minimum and a maximum value the *elastic envelope*, that is the set of elastic stresses due to all possible load combinations, becomes a convex polytope with 2^p vertexes being p the number of basic loads. For typical values of p to impose the plastic admissibility for all these vertexes strongly affect the efficiency of the analysis. This problem becomes also more important if the load domain definition is not so trivial, as for the case of the actual design rules adopted in Eurocodes. Referring to the *local level* of the analysis (i.e. the finite element or the generic Gauss point of the element) where the plastic admissibility has to be tested, the larger part of the elastic stresses associated to the load domain vertexes are in the interior of the elastic envelope. This means that, in order to improve efficiency without affecting the accuracy, we can use only the convex hull vertexes, to check plastic admissibility and ignore those that, at the local level, are in the interior. In the paper an efficient and effective strategy that directly evaluate the significant vertexes of the convex hull is presented and validated.

With respect to point ii) the yield function of 3D frames is usually evaluated only considering flexural failures. In spite of this simplifying assumption, computing accurate yield surfaces with combined axial force and bending moments is not an easy task and has received increasing attention in the literature [4, 3, 2]. A piecewise linearization often requires a large number of polyhedral facets to obtain a sufficiently accurate approximation, which can have an important effect on the quality of the estimated bounds [4] but also on the efficiency of the algorithm. Since the yield criterion has to be verified for a large number of points throughout the whole structure, a compromise between accuracy and computational efficiency is required in the case of large-scale problems [10, 11].

Recently a strategy for approximating the true nonlinear yield surfaces by using a Minkowsky sum of ellipsoids, has been proposed for limit analysis problems in [3] and the resulting Second Order Cone Programming problem has been solved with the commercial code MOSEK. A similar approach is adopted in the present work where, however, the SD-CPP approach is proposed for the solution. The Minkowsky sums of ellipsoids allows us to accurately describe the section elastic domain using only few analytical functions while the decomposition strategy we propose allows us to perform the return mapping by closest point projection (CPP) for each ellipsoid in a separate fashion so increasing the computational performance and the robustness of the analysis. The approach we propose is general and could also be effectively employed in standard path–following elasto-plastic analysis of 3D frame.

A series of numerical test confirm the effectiveness and accuracy of our proposal.

2 THE 3D BEAM MODEL

In the following the beam model and its discrete finite element version is presented. We refers to [13, 14] for more details.

2.1 The beam model

Let us consider a cylinder occupying a reference configuration $\mathcal{B}$ of length ℓ confined by the lateral boundary denoted by $\partial\mathcal{B}$ and two terminal bases Ω_0 and Ω_ℓ. The cylinder is referred to a Cartesian frame $(\mathcal{O}, s, x_2, x_3)$ with unit vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\boldsymbol{e}_1$ aligned with the cylinder axis. In this system, see Figure 1, we denote with $\boldsymbol{X} = \boldsymbol{X}_0 + \boldsymbol{x}$ the position of a point P, $\boldsymbol{X}_0 = s\boldsymbol{e}_1$ is the position of P with respect to the beam axis, s being an abscissa which identifies the generic cross-section $\Omega[s]$ of the beam, while $\boldsymbol{x} = x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$ is the position of P inside $\Omega[s]$.

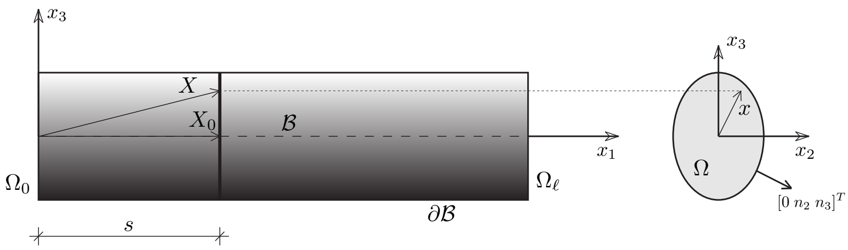


Figure 1: The cylindrical solid.

The displacement field $\boldsymbol{u}[\boldsymbol{X}]$ of the model is expressed, as usual, as a section rigid motion

$$\boldsymbol{u}[\boldsymbol{X}] = \boldsymbol{u}_0[s] + \boldsymbol{\varphi}[s] \wedge \boldsymbol{x} \tag{1}$$

where $\boldsymbol{u}_0[s]$ and $\boldsymbol{\varphi}[s]$ are the mean translation and rotation of the section, $\wedge$ denote the cross product. The kinematics assumed in Eq.(1) allows to evaluate, using a standard linear 3D Cauchy continuum, the stress strain work $\mathcal{W}$ in terms of the generalized strains and stresses on the section (see [13]) as

$$\mathcal{W} \equiv \int_\ell (\boldsymbol{N}[s] \cdot \boldsymbol{\varepsilon}_L[s] + \boldsymbol{M}[s] \cdot \boldsymbol{\chi}_L[s]) ds = \int_\ell \mathbf{t}[s]^T \boldsymbol{\rho}[s] ds \tag{2}$$

where $\boldsymbol{\rho}[s] = \{\boldsymbol{\varepsilon}[s], \boldsymbol{\chi}[s]\}$ collects the generalized strain parameters $\boldsymbol{\varepsilon}$ and $\boldsymbol{\chi}$ defined as

$$\boldsymbol{\varepsilon}[s] = \boldsymbol{u}_{0,s}[s] + \boldsymbol{e}_1 \wedge \boldsymbol{\varphi}[s] \quad , \quad \boldsymbol{\chi}_L = \boldsymbol{\varphi}[s]_{,s} \tag{3}$$

and $\mathbf{t}[s] = \{\boldsymbol{N}[s], \boldsymbol{M}[s]\}$ are the resultant force $\boldsymbol{N}[s]$ and moment $\boldsymbol{M}[s]$ defined as

$$\boldsymbol{N}[s] = \int_{\Omega} \boldsymbol{s} d\Omega \quad , \quad \boldsymbol{M}[s] = \int_{\Omega} \boldsymbol{x} \wedge \boldsymbol{s} d\Omega. \tag{4}$$

where $\boldsymbol{s} = \boldsymbol{\sigma} \boldsymbol{e}_1$ is the traction applied to the generic cross section. In Eq. (3) a comma stands for derivative. Finally the elastic constitutive laws is expressed as

$$\boldsymbol{\rho}[s] = \boldsymbol{H}\mathbf{t}[s] \quad , \quad \boldsymbol{H}[s] = \begin{bmatrix} \boldsymbol{H}_{NN} & \boldsymbol{H}_{NM} \\ \boldsymbol{H}_{NM}^T & \boldsymbol{H}_{MM} \end{bmatrix} \tag{5}$$

where the coefficients of the cross-section compliance matrix $\boldsymbol{H}$ are explicitly reported in [15].

2.2 The finite element for the beam

The beam equilibrium equation for zero body forces

$$\boldsymbol{N}_{,s} = \mathbf{0}, \qquad \boldsymbol{M}_{,s} + \boldsymbol{e}_1 \wedge \boldsymbol{N} = \mathbf{0} \tag{6}$$

states that $\boldsymbol{N}$ and the torsional moment M_1 are constant while the two flexural components $M_2[s]$ and $M_3[s]$ of $\boldsymbol{M}[s]$ are linear with s. The internal work become then

$$\mathcal{W} = \boldsymbol{N} \cdot (\boldsymbol{u}_0[\ell] - \boldsymbol{u}_0[0]) + \boldsymbol{M}[\ell] \cdot \boldsymbol{\varphi}[\ell] - \boldsymbol{M}[0] \cdot \boldsymbol{\varphi}[0] = \boldsymbol{d}_e^T \boldsymbol{Q}_e^T \boldsymbol{\beta}_e \tag{7}$$

so allowing to directly obtain the discrete form of $\mathcal{W}$ without need to use any FEM interpolation for the kinematic variables. In Eq. (7) the vectors collecting the finite element generalized parameters and the compatibility operator $\boldsymbol{Q}_e$ are

$$\boldsymbol{\beta}_e = \begin{bmatrix} N \\ M_2[0] \\ M_3[0] \\ M_2[\ell] \\ M_3[\ell] \\ M_1 \end{bmatrix} , \quad \boldsymbol{d}_e = \begin{bmatrix} \boldsymbol{u}_0[0] \\ \boldsymbol{\varphi}[0] \\ \boldsymbol{u}_0[\ell] \\ \boldsymbol{\varphi}[\ell] \end{bmatrix} , \quad \boldsymbol{Q}_e = \begin{bmatrix} -\boldsymbol{e}_1 & \mathbf{0} & \boldsymbol{e}_1 & \mathbf{0} \\ \frac{\boldsymbol{e}_3}{\ell} & -\boldsymbol{e}_2 & -\frac{\boldsymbol{e}_3}{\ell} & \mathbf{0} \\ -\frac{\boldsymbol{e}_3}{\ell} & \mathbf{0} & \frac{\boldsymbol{e}_3}{\ell} & \boldsymbol{e}_2 \\ -\frac{\boldsymbol{e}_2}{\ell} & -\boldsymbol{e}_3 & \frac{\boldsymbol{e}_2}{\ell} & \mathbf{0} \\ \frac{\boldsymbol{e}_2}{\ell} & \mathbf{0} & -\frac{\boldsymbol{e}_2}{\ell} & \boldsymbol{e}_3 \\ \mathbf{0} & -\boldsymbol{e}_1 & \mathbf{0} & \boldsymbol{e}_1 \end{bmatrix} \tag{8}$$

Eq. (8) allows us to write the discrete form of the equilibrium equations as

$$\boldsymbol{Q}^T \boldsymbol{\beta} - \lambda \boldsymbol{p} = \mathbf{0} \quad \text{with} \quad \boldsymbol{Q}^T \boldsymbol{\beta} = \mathcal{A}_e \{\boldsymbol{Q}_e^T \boldsymbol{\beta}_e\} \tag{9}$$

where $\boldsymbol{\beta}$ denotes the global vector collecting all the stress parameters $\boldsymbol{\beta}_e$ and $\boldsymbol{p}$ is the load vector. The global equilibrium matrix $\boldsymbol{Q}^T$ is obtained as usual by means of the contribution of each finite element and $\mathcal{A}_e$ is the standard assembling operator which takes into account the inter–element continuity conditions on $\boldsymbol{u}$ and $\boldsymbol{\varphi}$. From now on a subscript e denote the finite element counterpart of a global vector or matrix.

The element elastic compliance matrix, is obtained from the equivalence

$$\int_{\ell} \mathbf{t}[s] \cdot \boldsymbol{H}\mathbf{t}[s] ds = \boldsymbol{\beta}_e^T \boldsymbol{H}_e \boldsymbol{\beta}_e \quad \boldsymbol{H}_e = \int_{\ell} \boldsymbol{D}_t[s]^T \boldsymbol{H} \boldsymbol{D}_t[s] \, ds \tag{10}$$

being $\boldsymbol{D}_t[s]$ the matrix collecting the functions used to interpolate $\mathbf{t}[s]$.

3 THE ELASTIC DOMAIN OF THE BEAM SECTION

In this section we shortly present the construction of the elastic domain, extending the procedure used in Malena and Casciaro [4] with the approach recently proposed by Bleyer and De Buhan [3], see also [2].

3.1 Evaluation of the true support functions of the beam elastic domain

We decompose the beam section domain Ω as the union of n_d subdomains Ω_i in which the material is homogeneous. For each Ω_i, the plastic admissibility condition is expressed in terms of the normal stress only as $-\sigma_{ci} \leq \sigma_{11} \leq \sigma_{ti}$ where σ_{ti} is the ultimate normal stress in tension (positive) and σ_{ci} in compression (negative). This corresponds to assume, as usual for technical application, frame members infinitely resistant with respect to shear effects as well as torsion. Hence, the cross-section yield surfaces will be described in the 3D space involving axial force N_1 and bending moments M_2 and M_3 collected in vector $\mathbf{t}^\sigma[s] = \{N_1, M_2, M_3\}$.

Due to the section rigid motion hypothesis, a generic *section collapse mechanism* will be defined by the position of the neutral axis, i.e. by the condition

$$\varepsilon_1 + x_3\,\chi_2 - x_2\,\chi_3 = 0 \tag{11}$$

Denoting with $\dot{\boldsymbol{\varepsilon}}_k = \{\epsilon_1, \chi_2, \chi_3\}$ the collapse mechanism direction we can evaluate the associated ultimate section strength $\mathbf{t}^\sigma{}_k$ as

$$\mathbf{t}^\sigma{}_k = \begin{bmatrix} N_k \\ M_{k2} \\ M_{k3} \end{bmatrix} \quad \text{with} \quad \begin{cases} N_k = \sum_i \left(\int_{\Omega_i^+} \sigma_{ti} d\Omega_i - \int_{\Omega_i^-} \sigma_{ci} d\Omega_i \right) \\ M_{k2} = \sum_i \left(\int_{\Omega_i^+} x_3 \sigma_{ti} d\Omega_i - \int_{\Omega_i^-} x_3 \sigma_{ci} d\Omega_i \right) \\ M_{k3} = \sum_i \left(\int_{\Omega_i^+} x_2 \sigma_{ti} d\Omega_i - \int_{\Omega_i^-} x_2 \sigma_{ci} d\Omega_i \right) \end{cases} \tag{12}$$

where we assumed positive the action if in the positive direction of the axis while Ω_i^+ and Ω_i^- are the portion of dS_i in traction or compression for the given mechanism $\dot{\boldsymbol{\varepsilon}}_k$.

This definition states that, for each position of the neutral axis, i.e for each $\dot{\boldsymbol{\varepsilon}}_k$, the corresponding generalized stress $\mathbf{t}^\sigma{}_k$ on the yield function is obtained by considering uniaxial stress fields reaching their maximum strength capacity in each region, either in tension or in compression. We refer to [4] for a suitable choice for the collapse mechanics $\dot{\boldsymbol{\varepsilon}}_k$ for RC beams.

The collapse resultants defined in (12) belong to the boundary of the section yield function for construction and they are then characterized by the following condition

$$\pi_{\mathbb{E}_s}[\dot{\boldsymbol{\varepsilon}}_k] = \sup\{\dot{\boldsymbol{\varepsilon}}_k^T \mathbf{t}^\sigma : \mathbf{t}^\sigma \in \mathbb{E}_s\}, \tag{13}$$

Eq.(13) is the definition of the support function of the section elastic domain $\mathbb{E}_s$ in the direction $\dot{\boldsymbol{\varepsilon}}_k$ (see [3]).

3.2 The approximation of $\mathbb{E}_s$ using a Minkowsy sum of ellipsoids

Having obtained the support function $\pi_{\mathbb{E}_s}[\dot{\boldsymbol{\varepsilon}}_k]$ for a series of direction $\dot{\boldsymbol{\varepsilon}}_k$, we use them to approximate $\mathbb{E}_s$ by means of a Minkowsky sum of ellipsoids with the Ith of them defined as

$$\mathcal{E}[\boldsymbol{C}_I, \boldsymbol{c}_I] = \left\{ \mathbf{t}^\sigma :\ \|\boldsymbol{J}_I^{-1}(\mathbf{t}^\sigma - \boldsymbol{c}_I)\| - 1 \leq 0 \right\} \tag{14}$$

where $\|\cdot\|$ stand for the Euclidean norm, the symmetric and definite positive matrix $\boldsymbol{C}_I = \boldsymbol{C}_I^T = \boldsymbol{J}_I^T \boldsymbol{J}_I$ gives the orientation and $\boldsymbol{c}_I$ the origin of the ellipsoid. The support function of the Minkowsky sum of $N_\mathcal{E}$ ellipsoid is the sums of their support functions $\pi_{\mathcal{E}_I}$ [3], so

$$\pi_\mathcal{E}[\dot{\boldsymbol{\varepsilon}}_k] = \sum_{I=1}^{N_\mathcal{E}} \pi_{\mathcal{E}_I}[\dot{\boldsymbol{\varepsilon}}_k] = \sum_{I=1}^{N_\mathcal{E}} \|\boldsymbol{J}_I \dot{\boldsymbol{\varepsilon}}_k\| + \dot{\boldsymbol{\varepsilon}}_k^T \boldsymbol{c} \quad \text{with} \quad \pi_{\mathcal{E}_I}[\dot{\boldsymbol{\varepsilon}}_k] = \|\boldsymbol{J}_I \dot{\boldsymbol{\varepsilon}}_k\| + \dot{\boldsymbol{\varepsilon}}_k^T \boldsymbol{c}_I \tag{15}$$

and $\boldsymbol{c} = \sum_I \boldsymbol{c}_I$. The values of the unknowns that define $\boldsymbol{J}_I$ and $\boldsymbol{c}$ are obtained by minimizing the difference error between $\pi_{\mathbb{E}_s}[\dot{\boldsymbol{\varepsilon}}_k]$ and $\pi_\mathcal{E}[\dot{\boldsymbol{\varepsilon}}_k]$ for a series of N_p direction $\dot{\boldsymbol{\varepsilon}}_k$

$$\text{minimize} \sum_{k=1}^{N_p} \left(\pi_{\mathbb{E}_s}[\dot{\boldsymbol{\varepsilon}}_k] - \sum_{I=1}^{N_\mathcal{E}} \|\boldsymbol{J}_I \dot{\boldsymbol{\varepsilon}}_k\| - \dot{\boldsymbol{\varepsilon}}_k^T \boldsymbol{c} \right)^2 \tag{16}$$

Eqs.(15) allow us to approximate the elastic domain as the Minkowsy sum of n ellipsoids centered in the origin and of a singleton $\boldsymbol{c}$, as

$$\mathbb{E}_s[\mathbf{t}^\sigma] \equiv \left(\bigoplus_{I=1}^{N_\mathcal{E}} \mathcal{E}_I \right) \oplus \boldsymbol{c} \quad , \quad \mathcal{E}_I \equiv \mathcal{E}[\boldsymbol{C}_I, \mathbf{0}] \tag{17}$$

The admissibility condition for $\mathbf{t}^\sigma[s]$ can be now expressed as a sum of $N_\mathcal{E} + 1$ terms

$$\mathbf{t}^\sigma = \left(\boldsymbol{c} + \sum_{I=1}^{N_\mathcal{E}} \mathbf{t}_I^\sigma \right) \in \mathbb{E}_s \quad \Longleftrightarrow \quad \mathbf{t}_I^\sigma \in \mathcal{E}_I \tag{18}$$

or in terms of the section s yield functions $\phi_I[s, \mathbf{t}^\sigma[s]]$ as

$$\boldsymbol{\Phi}_\mathcal{E}[s, \mathbf{t}^\sigma[s]] \leq \mathbf{0} \quad \Longleftrightarrow \quad \phi_I[s, \mathbf{t}_I^\sigma[s]] \equiv \|\boldsymbol{J}_I[s]^{-1} \mathbf{t}_I^\sigma[s]\| - 1 \leq 0 \quad \forall I = 1 \cdots N_\mathcal{E} \tag{19}$$

4 SHAKEDOWN ANALYSIS BASED ON PROXIMAL POINT METHOD AND DUAL DECOMPOSITION

In this Section the approach proposed in [12] is particularized to the shakedown analysis of 3D frame in case of yield domain obtained as Minkowsky sums of ellipsoids.

4.1 The elastic envelope of the stresses

We assume that the external actions $\boldsymbol{p}[t]$, variable with the time t, are expressed as a combination of p basic loads $\boldsymbol{p}_i$ belonging to the admissible closed and convex *load domain* $\mathbb{P}$. This is defined according to the Eurocode rules for *ultimate limit states*, that prescribe N_β combination of actions, each of them obtained by considering a leading variable action p_β, assumed with its characteristic value while the other variable actions $\boldsymbol{p}_j$, which may act simultaneously with the leading variable one, are taken into account as accompanying variable actions and are represented by their combination value, i.e. their characteristic value reduced by the relevant factor ψ_j. All the action are multiplied by the coefficient α_j to consider the variability in time so obtaining the load domain as

$$\mathbb{P} := \bigcup_{\beta=1}^{N_\beta} \mathbb{P}^{(\beta)} \quad , \quad \mathbb{P}^{(\beta)} := \left\{ \boldsymbol{p}[t] \equiv \alpha_\beta \boldsymbol{p}_\beta + \sum_{j=1, j\neq\beta}^{p} \psi_j \alpha_j \boldsymbol{p}_j : \ \alpha_i^{min} \leq \alpha_i \leq \alpha_i^{max} \right\}. \tag{20}$$

The *elastic envelope* $\mathbb{S}$ defines the set of the elastic stresses $\hat{\boldsymbol{\beta}}[t]$ produced by each load path contained in $\mathbb{P}$. Due to the local definition of the stress variables it can be decoupled for each finite element as

$$\mathbb{S}_e = \bigcup_{\beta=1}^{N_c} \mathbb{S}_e^{(\beta)} \quad , \quad \mathbb{S}_e^{(\beta)} := \left\{ \hat{\boldsymbol{\beta}}_e[t] \equiv \alpha_\beta \hat{\boldsymbol{\beta}}_{e\beta} + \sum_{j=1, j\neq\beta}^{p} \psi_j \alpha_j \hat{\boldsymbol{\beta}}_{ej} : \ \alpha_i^{min} \leq \alpha_i \leq \alpha_i^{max} \right\} \tag{21}$$

where $\hat{\boldsymbol{\beta}}_i$ are the elastic stress solution for $\boldsymbol{p}_i$. We have that

$$\hat{\boldsymbol{\beta}}[t] \in \mathbb{S} \ \Leftrightarrow \ \hat{\boldsymbol{\beta}}_e[t] \in \mathbb{S}_e \ \forall e \tag{22}$$

Without affecting the results we can substitute the true $\mathbb{S}_e$ with its convex hull, i.e. with a convex polytope. Its significant vertexes are, in general, a subset of the $(N_\beta \cdot 2^p)$ vertexes corresponding to the load domain in Eq.(20). Each $\hat{\boldsymbol{\beta}}_e \in \mathbb{S}_e$ can then be expressed as a convex combination of the N_v elastic envelope vertexes $\hat{\boldsymbol{\beta}}_e^\alpha$ that can be usefully referred to the reference stress $\hat{\boldsymbol{\beta}}_e^0$ so obtaining

$$\hat{\boldsymbol{\beta}}_e[t] = \hat{\boldsymbol{\beta}}_e^0 + \sum_{\alpha=1}^{N_v} s^\alpha \hat{\boldsymbol{\beta}}_e^\alpha \qquad s^\alpha \geq 0 \qquad \sum_{\alpha=1}^{N_v} s^\alpha = 1 \tag{23}$$

If the external loads increase by a real number λ, called *load domain multiplier*, the elastic envelope becomes $\lambda\mathbb{S} := \left\{ \lambda\hat{\boldsymbol{\beta}} : \ \hat{\boldsymbol{\beta}} \in \mathbb{S} \right\}$.

4.2 The shakedown elastic domain

The plastic admissibility condition, being locally defined, can be decoupled at the element *local* level, i.e. $\boldsymbol{\beta}$ will be plastically admissible if

$$\boldsymbol{\Phi}[\boldsymbol{\beta}] \leq \mathbf{0} \iff \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e] \leq \mathbf{0}, \ \forall e. \tag{24}$$

At the local level Eq. (24) can be expressed in terms of the admissibility of the end nodes generalized stresses $\mathbf{t}^\sigma[s]$ with $s = 0,\ \ell$ extracted from $\boldsymbol{\beta}_e$ as $\mathbf{t}^\sigma[0] = \mathbf{S}^\sigma[0]\boldsymbol{\beta}_e$ and $\mathbf{t}^\sigma[\ell] = \mathbf{S}^\sigma[\ell]\boldsymbol{\beta}_e$. End nodes stresses and, according to Eq.(18), are considered as as sum of ellipsoid contributions. The element plastic admissibility condition, according to Eq.(19), rewrites as

$$\boldsymbol{\Phi}_e[\boldsymbol{\beta}_e] \equiv \begin{bmatrix} \boldsymbol{\Phi}_{\mathcal{E}}[0, \mathbf{t}^\sigma[0]] \\ \boldsymbol{\Phi}_{\mathcal{E}}[\ell, \mathbf{t}^\sigma[\ell]] \end{bmatrix} \leq \mathbf{0} \tag{25}$$

Shakedown analysis requires the plastically admissible condition for all the stresses contained in the amplified elastic envelope $\lambda\mathbb{S}_e$ translated by $\bar{\boldsymbol{\beta}}_e$. Due to the convexity of $\boldsymbol{\Phi}_e$ this can be easily expressed in terms of the plastic admissibility of all the α stress vertexes $\boldsymbol{\beta}_e^\alpha = \lambda(\hat{\boldsymbol{\beta}}_e^\alpha + \hat{\boldsymbol{\beta}}_e^0) + \bar{\boldsymbol{\beta}}_e$ of the convex envelope of $\mathbb{S}_e$, regardless from the convexity of $\mathbb{S}_e$

$$\boldsymbol{\Phi}_e[\lambda\hat{\boldsymbol{\beta}}_e + \bar{\boldsymbol{\beta}}_e] \leq \mathbf{0},\ \forall\hat{\boldsymbol{\beta}}_e \in \mathbb{S}_e \iff \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e^\alpha] \leq \mathbf{0},\ \forall\alpha \tag{26}$$

where, from now on, a Greek superscript denotes the convex hull vertexes quantities.

4.3 Simplified evaluation of the elastic envelope

In the following we denote with $\mathbb{S}^\sigma[s]$ the set obtained from $\mathbb{S}_e$ considering only normal stresses $\hat{\mathbf{t}}^\sigma[s] = \mathbf{S}^\sigma[s]\hat{\boldsymbol{\beta}}_e$. The elastic envelopes for the beam node s is evaluated by referring to the significant vertexes of $\mathbb{S}^\sigma[s]$ with respect to N_h *supporting hyperplanes* of $\mathbb{E}_s[s]$ of assigned normal $\boldsymbol{n}_k$ and tangent to $\mathbb{E}_s[s]$ in $\mathbf{t}^\sigma{}_k[s]$

$$f_k[\boldsymbol{\beta}] \equiv \left\{\boldsymbol{n}_k^T\mathbf{t}^\sigma - c_k\right\} \leq 0 \quad , \quad c_k = (\mathbf{t}^\sigma{}_k)^T\boldsymbol{n}_k \qquad k = 1, \cdots, m. \tag{27}$$

The vertex associated with the kth hyperplane is obtained as the values of $\hat{\mathbf{t}}^\sigma[s] \in \mathbb{S}^\sigma[s]$ at minimum distance from f_k, that is maximizing its projection b_k through the simple and fast expression:

$$b_k := \max_{\hat{\mathbf{t}}^\sigma \in\, \mathbb{S}^\sigma[s]} \{\boldsymbol{n}_k^T\hat{\mathbf{t}}^\sigma\}. \tag{28}$$

Recalling the definition of $\mathbb{S}_e$ in Eq.(21) we have, for each β

$$b_k = \sum_{i=1}^p a_{ki}, \quad \hat{\mathbf{t}}_\alpha^\sigma = \sum_{i=1}^p a_{ki}\hat{\mathbf{t}}_i^\sigma \qquad \text{with} \qquad a_{ki} := \left\langle \begin{array}{ll} \alpha_i^{min}\boldsymbol{n}_k^T\hat{\mathbf{t}}_i^\sigma & \text{if } \boldsymbol{n}_k^T\hat{\mathbf{t}}_i^\sigma < 0 \\ \alpha_i^{max}\boldsymbol{n}_k^T\hat{\mathbf{t}}_i^\sigma & \text{if } \boldsymbol{n}_k^T\hat{\mathbf{t}}_i^\sigma \geq 0 \end{array}\right.$$

and maintaining that producing the maximum b_k for each β.

We obtain, at the end of this filtering process, $N_v \leq N_h$ vertexes of the convex hull of $\mathbb{S}_e$. As this operation is performed once and for all at the beginning of the analysis we can also use a fine mesh of controlling hyperplane to improve accuracy. Notwithstanding its simplicity the algorithm presents many advantages with respect to standard evaluation of the convex hull of $\mathbb{S}_e$

- It is available also for complex load combination, not directly expressible as Minkowsky sum of segments.

- It directly furnishes the vertexes of the elastic envelope without the need of a preliminarily evaluation of all those corresponding to the load vertexes (2^p for each β).

4.4 The pseudo-elastoplastic step for shakedown analysis

Shakedown analysis is performed using the algorithm proposed in [10, 11, 12] (see also [16]), which is based on the application of the proximal point method to the Melan static theorem. A sequences of subproblems or *steps* are obtained by adding a quadratic positive term, using the compliance matrix $\boldsymbol{H}$, to the objective function

$$\begin{aligned} \text{maximize} \quad & \Delta\xi^{(n)}\lambda^{(n)} - \frac{1}{2}\Delta\boldsymbol{\beta}^T\boldsymbol{H}\Delta\boldsymbol{\beta} \\ \text{subject to} \quad & \boldsymbol{Q}^T\boldsymbol{\beta}^{(n)} - \lambda^{(n)}\boldsymbol{p}_0 = \mathbf{0} \\ & \boldsymbol{\Phi}^{\alpha}[\boldsymbol{\beta},\lambda] \equiv \boldsymbol{\Phi}[\bar{\boldsymbol{\beta}} + \lambda_s\hat{\boldsymbol{\beta}}^{\alpha}] \leq \mathbf{0}, \quad \alpha = 1\cdots N_v \end{aligned} \tag{29}$$

where the superscript $(\cdot)^{(n)}$ will denote quantities evaluated in the nth step, the symbol $\Delta(\cdot) = (\cdot)^{(n)} - (\cdot)^{(n-1)}$ is the increment of a quantity from the previous step and $\Delta\xi^{(n)} > 0$ is an assigned real positive number, $\boldsymbol{p}_0 \equiv \boldsymbol{Q}^T\hat{\boldsymbol{\beta}}^0$ and $\boldsymbol{\beta} \equiv \bar{\boldsymbol{\beta}} + \lambda_s\hat{\boldsymbol{\beta}}^0$.

For limit analysis the first order condition of (29) exactly corresponds to a step of the arc-length algorithm used to solve the incremental elastoplastic problem [10, 11, 12]. As for elastic perfectly plastic structures the limit load can be evaluated by recovering the complete equilibrium path by means of a path-following algorithms in the same fashion the skakedown multiplier can be obtained by evaluating a sequence of states, by solving a series of problems (29). i.e defining a pseudo-elastoplastic equilibrium curve [10].

The solution is performed in two step as for standard strain driven elasto-plasticity analysis: 1) in the first step, performed at the local level, a return mapping by closest point projection process is used to evaluate the stresses in terms of the displacements; ii) an arc-length Riks method is used, at the global level, to define the pseudo-elastoplastic equilibrium path. It is possible to shown as this algorithm corresponds to use a dual decomposition to solve the proximal point step. We refer to [12, 16] for further details.

5 NUMERICAL RESULTS

In this section we present some tests regarding the accuracy and the efficiency of the new proposal in the shakedown analysis of 3D reinforced concrete frames.

5.1 Yeld function and elastic envelope vertexes for a L-shaped section

In Fig. 2 the yield function approximations for a L shaped reinforced concrete (see fig.4 of [4]) are reported on the basis of 1, 3 and 5 ellipsoids. The points correspond to the true values of $\mathbf{t}^{\sigma}{}_k$. The increase in accuracy is evident in the case of more ellipsoids.

In Fig. 3 we report, for the same section, the convex hull vertexes of $\mathbb{S}^{\sigma}$ for a sequence

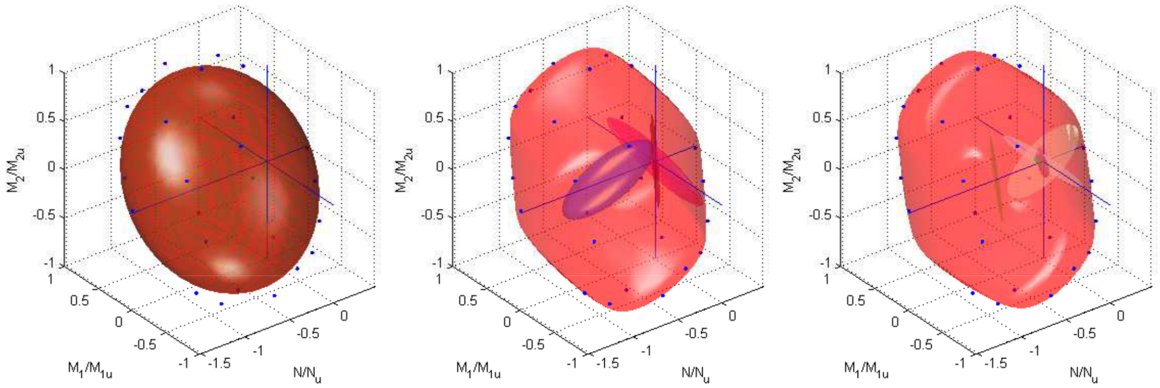


Figure 2: Section geometry and yield function approximation with 1, 3 and 5 ellipsoids

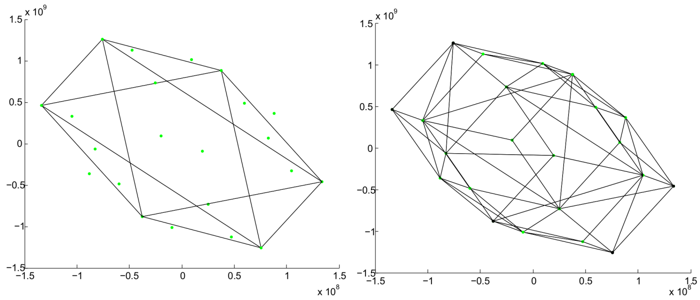
a 

Figure 3: Convex hull evaluation.

of 5 assigned $\hat{\mathbf{t}}_j^{\sigma}$ with $0 \leq \alpha_j \leq 1$. The Figure corresponds to the exact and approximate evaluation of $\mathbb{S}_e$ using the 26 hyperplanes proposed in [4].

5.2 Shakedown analysis of a 3D frame

The influence on the shakedown multiplier of an approximated evaluation of the elastic envelope is tested analyzing the structure reported in fig.4. The frame has been analyzed adopting an xx yy hyperplanes evaluation of $\mathbb{S}_e$. The results have been also compared with the results obtained maintaining all the vertexes and are reported in table 1

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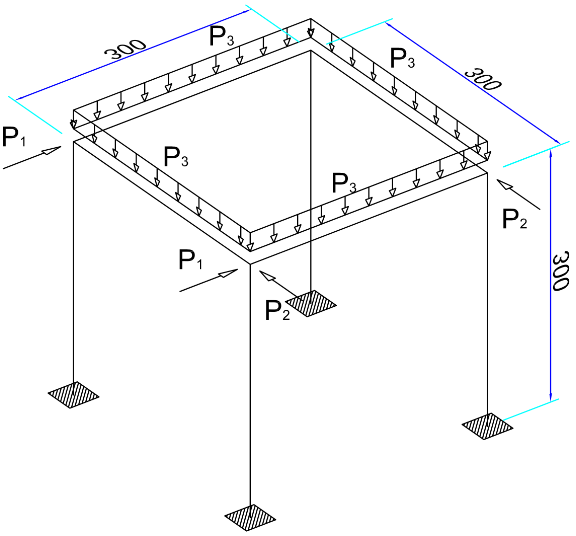

Figure 4: Loads and geometry.

	shakedown multiplies	
	$N_h = 26$	*exact*
λ_s	78.63	78.63
λ_p	234.5	213.1

Table 1: Shakedown multipliers.

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