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ABOUT THE MOVEMENT OF A SOLID BODY ON A PLANE SURFACE IN ACCORDANCE WITH ELLIPTIC CONTACT AREA AND ANISOTROPIC FRICTION

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Abstract. We examine inertial movement of a thin body on a plane surface. The contact area has an elliptical shape. We assume that friction forces are anisotropic. Equations of motion in such case are developed for situations with regular and linear normal pressure distribution. Several examples were numerically evaluated.

1 PROBLEM STATEMENT

Contact process between bodies depends on material properties and current state of each body surface layer. Large variety of effects (including wear, plasticity, mechanical operations) can lead to the anisotropy of friction forces at the contact area. Anisotropic friction depends on the direction of sliding. Short review of reasons of frictional anisotropy with experimental examples is presented in [1].

Summarising results of [2, 3, 4], we assume that for our problem the friction law can be written in the form:

$$\mathbf{T} = -N\mathcal{F}(M)\frac{\mathbf{v}}{|\mathbf{v}|}, \tag{1}$$

where $\mathbf{T}$ – friction force vector, N – normal pressure, $\mathcal{F}(M)$ – friction tensor, $\mathbf{v}$ – velocity vector.

It is shown in [4] that one can change axes in such way, that friction tensor becomes:

$$\mathcal{F}(M) = \begin{pmatrix} f_x & f \\ -f & f_y \end{pmatrix}, \tag{2}$$

where f_x, f_y, f – frictional coefficients. Let's assume that tensor components f_x, f_y, f remain constant and do not depend on interrelations between contacting surfaces. Such situation is possible when hardness of one interaction body much more than hardness of another one.

The objective of this study is to examine a movement of a thin body on a rough surface in accordance with anisotropy of dry friction forces.

Figure 1 shows coordinate system for our calculations: $C\xi\eta$ – associated with the body coordinate system, OXY – fixed coordinate system (it is chosen such that friction tensor has the form (2)). The body mass is m, I – inertia related to axes $C\zeta$ perpendicular to the plate, τ_x, τ_y – components of specific friction force, $p(\xi, \eta)$ – pressure force at the point of the contact, x, y – coordinates of plate mass center in OXY coordinate system, φ – angle between OX and $C\xi$.

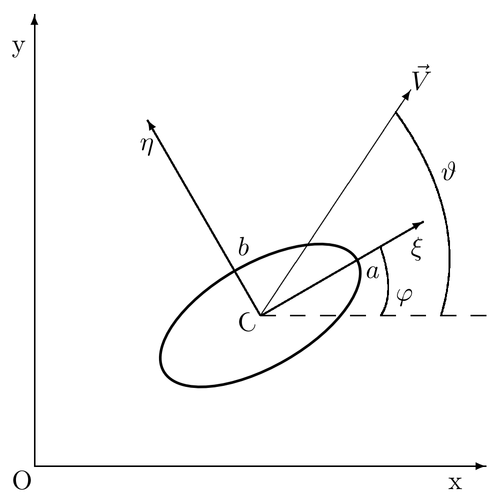

Figure 1: Coordinate system

Equation of motion are the following:

$$m\ddot{x} = \iint\limits_{\Omega} \tau_x d\xi d\eta, \tag{3}$$

$$m\ddot{y} = \iint\limits_{\Omega} \tau_y d\xi d\eta,$$

$$I\ddot{\varphi} = \iint\limits_{\Omega} mom_{C\zeta}(\vec{\tau}) d\xi d\eta,$$

where

$$\tau_x = -p(\xi,\eta)\frac{f_x(\dot{x}-\dot{\varphi}y') + f(\dot{y}+\dot{\varphi}x'))}{|v|},$$
$$\tau_y = -p(\xi,\eta)\frac{-f(\dot{x}-\dot{\varphi}y') + f_y(\dot{y}+\dot{\varphi}x'))}{|v|},$$
$$mom_{C\zeta}(\vec{\tau}) = (\tau_y x' - \tau_x y'),$$
$$x' = \xi\cos\varphi - \eta\sin\varphi, \quad y' = \xi\sin\varphi + \eta\cos\varphi,$$
$$|\vec{v}| = \sqrt{(\dot{x}-\dot{\varphi}y')^2 + (\dot{y}+\dot{\varphi}x')^2}\,.$$

2 INERTIAL MOVEMENT OF A BODY WITH ELLIPTIC CONTACT AREA.

Let's assume that contact area between the body and the surface has an elliptical shape, where ellipse semi-axes are a and b and eccentricity is e (see figure 1). We will examine several pressure distribution cases: uniform and linear, taking into account anisotropic friction forces.

The system (3) can be rewritten with variables v and ϑ, where v – velocity of the body's center (with $\dot{x} = v\cos\vartheta, \quad \dot{y} = v\sin\vartheta$), ϑ – angle between Ox and tangent to the trajectory. Besides, let's introduce variables $\beta = \dfrac{v}{\omega}$ (length to instantaneous velocity center), and $\mu = f_y - f_x$.

Therefore, we express equation system (3) as follows:

$$\dot{x} = v\cos\vartheta, \quad \dot{y} = v\sin\vartheta, \quad \dot{\varphi} = \omega, \tag{4}$$
$$\dot{v} = -\frac{1}{m}\int\limits_{-a}^{a}\int\limits_{-h(\xi)}^{h(\xi)} \frac{p(\xi,\eta)\,(A_0 + A_1\eta)}{\sqrt{\eta^2 + D_1\eta + D_0}}d\xi d\eta = -\frac{F_\tau}{m},$$
$$\dot{\vartheta} = -\frac{1}{mv}\int\limits_{-a}^{a}\int\limits_{-h(\xi)}^{h(\xi)} \frac{p(\xi,\eta)\,(B_0 + B_1\eta)}{\sqrt{\eta^2 + D_1\eta + D_0}}d\xi d\eta = -\frac{F_n}{mv},$$
$$\dot{\omega} = -\frac{1}{I}\int\limits_{-a}^{a}\int\limits_{-h(\xi)}^{h(\xi)} \frac{p(\xi,\eta)\,(C_0 + C_1\eta + C_2\eta^2)}{\sqrt{\eta^2 + D_1\eta + D_0}}d\xi d\eta = -\frac{M_{C\zeta}}{I}.$$

Here limits are

$$h(\xi) = \frac{b}{a}\sqrt{a^2 - \xi^2}, \tag{5}$$

and coefficients are given by relations:

$$\begin{aligned} A_0 = \beta(f_x + \mu\sin^2\vartheta) + \xi\left(f\cos(\vartheta-\varphi) + \mu\sin\vartheta\cos\varphi\right) + \\ +\xi\left(f_x\sin(\vartheta-\varphi)\right), \\ A_1 = f\sin(\vartheta-\varphi) - \mu\sin\vartheta\sin\varphi - \\ -f_x\cos(\vartheta-\varphi), \\ B_0 = \beta(\mu\sin\vartheta\cos\vartheta - f) + \xi(-f\sin(\vartheta-\varphi) + \mu\cos\vartheta\cos\varphi + \\ +f_x\cos(\vartheta-\varphi)), \\ B_1 = f\cos(\vartheta-\varphi) - \mu\cos\vartheta\sin\varphi + \\ +f_x\sin(\vartheta-\varphi), \\ C_0 = \xi\beta(-f\cos(\vartheta-\varphi) + \mu\sin\vartheta\cos\varphi + f_x\sin(\vartheta-\varphi)) + \\ +\xi^2(f_x + \mu\cos^2\varphi)), \\ C_1 = \beta(-f\sin(\vartheta-\varphi) - \mu\sin\vartheta\sin\varphi - f_x\cos(\vartheta-\varphi)) - \\ -2\xi\mu\cos\varphi\sin\varphi, \\ C_2 = f_x + \mu\sin^2\varphi, \\ D_0 = \beta^2 + \xi^2 + 2\beta\xi\sin(\vartheta-\varphi), \\ D_1 = -2\beta\cos(\vartheta-\varphi). \end{aligned} \tag{6}$$

2.1 Uniform pressure distribution case

Let us consider that pressure in system (4) is of the form:

$$p(\xi,\eta) = \frac{P}{S} = \frac{mg}{\pi ab}$$

and remains constant during the sliding process.

Suppose, that

$$\begin{aligned} \kappa = \sqrt{1-e^2}, \quad I = \frac{\rho\pi\kappa a^4(1+\kappa^2)}{4}, \\ \xi = a\xi_1, \quad \xi_1 \in [-1;1], \quad \eta = a\kappa\eta_1, \quad \eta_1 \in [-1;1], \\ v = v_1\sqrt{\frac{ag}{\pi}}, \quad \omega = \omega_1\sqrt{\frac{g}{a\pi}}, \\ t = t_1\sqrt{\frac{g}{\pi}}, \quad \dot\vartheta = \frac{d\vartheta_1}{dt_1}\sqrt{\frac{g}{a\pi}}. \end{aligned} \tag{7}$$

Thus, with supplementary operations, we achieve system with dimensionless variables (indices 1 are omitted), which is integrated by the variable η (see [5]):

$$\dot{x} = v\cos\vartheta, \quad \dot{y} = v\sin\vartheta, \quad \dot{\varphi} = \omega, \tag{8}$$

$$\dot{v} = -\int_{-1}^{1}\left[\left(A_0 - A_1\frac{D_1}{2}\right)I_1 + A_1\frac{1}{\kappa}I_2\right]d\xi$$

$$\dot{\vartheta} = -\frac{1}{v}\int_{-1}^{1}\left[\left(B_0 - B_1\frac{D_1}{2}\right)I_1 + B_1\frac{1}{\kappa}I_2\right]d\xi$$

$$\dot{\omega} = -\frac{4}{1+\kappa^2}\int_{-1}^{1}\left[\left(C_0 - C_1\frac{D_1}{2} + C_2\left(\frac{3D_1^2}{8} - \frac{D_0}{2}\right)\right)I_1\right]d\xi$$

$$-\frac{4}{1+\kappa^2}\int_{-1}^{1}\left[\left(C_1 - C_2\frac{3D_1}{4}\right)\frac{1}{\kappa}I_2 + \frac{C_2 h(\xi)}{2}I_3\right]d\xi,$$

where

$$I_1 = \frac{1}{\kappa}\ln\left(\frac{2\sqrt{D_0 + D_1\kappa h(\xi) + \kappa^2 h(\xi)^2} + 2\kappa^2 h(\xi) + D_1\kappa}{2\sqrt{D_0 - D_1\kappa h(\xi) + \kappa^2 h(\xi)^2} - 2\kappa^2 h(\xi) + D_1\kappa}\right), \tag{9}$$

$$I_2 = \sqrt{D_0 + D_1\kappa h(\xi) + \kappa^2 h(\xi)^2} - \sqrt{D_0 - D_1\kappa h(\xi) + \kappa^2 h(\xi)^2},$$

$$I_3 = \sqrt{D_0 + D_1\kappa h(\xi) + \kappa^2 h(\xi)^2} + \sqrt{D_0 - D_1\kappa h(\xi) + \kappa^2 h(\xi)^2},$$

and $h(\xi) = \sqrt{1-\xi^2}$, coefficients A_i, B_i, C_i, D_i remain the same as in (6).

2.2 Linear pressure distribution case

Let's suppose that pressure is linearly distributed. According to [6], one can write the following law:

$$p(\xi,\eta) = p_0 + p_1\xi + p_2\eta, \tag{10}$$

where $p_0 = \dfrac{P}{S}$ (same as in previous case) and $p_1 = \dfrac{P\xi_c}{I_{\eta\eta}}$, $\quad p_2 = \dfrac{P\eta_c}{I_{\xi\xi}}$, where inertia moments are $I_{\xi\xi} = \dfrac{ma^2(1-e^2)}{4}$ and $I_{\eta\eta} = \frac{ma^2}{4}$ and ξ_c, η_c – the body's gravity center coordinates.

Thus, equation system (4) can be rewritten to the form:

$$\dot{x} = v\cos\vartheta, \quad \dot{y} = v\sin\vartheta, \quad \dot{\varphi} = \omega, \tag{11}$$
$$\dot{v} = -\frac{1}{m}\int\limits_{-a}^{a}\int\limits_{-h(\xi)}^{h(\xi)} \frac{A_{01} + A_{11}\eta + A_{21}\eta^2}{\sqrt{\eta^2 + D_1\eta + D_0}} d\xi d\eta,$$
$$\dot{\vartheta} = -\frac{1}{mv}\int\limits_{-a}^{a}\int\limits_{-h(\xi)}^{h(\xi)} \frac{B_{01} + B_{11}\eta + B_{21}\eta^2}{\sqrt{\eta^2 + D_1\eta + D_0}} d\xi d\eta,$$
$$\dot{\omega} = -\frac{1}{I}\int\limits_{-a}^{a}\int\limits_{-h(\xi)}^{h(\xi)} \frac{C_{01} + C_{11}\eta + C_{21}\eta^2 + C_{31}\eta^3}{\sqrt{\eta^2 + D_1\eta + D_0}} d\xi d\eta,$$

where coefficients defined as following:

$$\begin{aligned} A_{01} &= p_0A_0 + \xi p_1A_0, \\ A_{11} &= p_0A_1 + \xi p_1A_1 + p_2A_0, \\ A_{21} &= p_2A_1, \\ B_{01} &= p_0B_0 + \xi p_1B_0, \\ B_{11} &= p_0B_1 + \xi p_1B_1 + p_2B_0, \\ B_{21} &= p_2B_1, \\ C_{01} &= p_0C_0 + \xi p_1C_0, \\ C_{11} &= p_0C_1 + \xi p_1C_1 + p_2C_0, \\ C_{21} &= p_0C_2 + \xi p_1C_2 + p_2C_1, \\ C_{31} &= p_2C_2, \end{aligned} \tag{12}$$

and $h(\xi)$ has the form (5), A_i, B_i, C_i, D_i are determined as in uniform pressure distribution case (6).

After integration by η with relations (7), we achieve system of equations (see [5]):

$$\dot{x} = v\cos\vartheta, \quad \dot{y} = v\sin\vartheta, \quad \dot{\varphi} = \omega, (13)$$

$$\dot{v} = -\int\limits_{-1}^{1}\left[\left(A^*_{01} - A^*_{11}\frac{D_1}{2} + A^*_{21}\left(\frac{3D_1^2}{8} - \frac{D_0}{2}\right)\right) I_1\right] d\xi$$

$$-\int\limits_{-1}^{1}\left[(A^*_{11} - A^*_{21}\frac{3D_1}{4})\frac{1}{\kappa}I_2 + \frac{A^*_{21}h(\xi)}{2}I_3\right] d\xi,$$

$$\dot{\vartheta} = -\frac{1}{v}\int\limits_{-1} 1^1\left[\left(B^*_{01} - B^*_{11}\frac{D_1}{2} + B^*_{21}\left(\frac{3D_1^2}{8} - \frac{D_0}{2}\right)\right) I_1\right] d\xi$$

$$-\frac{1}{v}\int\limits_{-1}^{1}\left[(B^*_{11} - B^*_{21}\frac{3D_1}{4})\frac{1}{\kappa}I_2 + \frac{B^*_{21}h(\xi)}{2}I_3\right] d\xi,$$

$$\dot{\omega} = -\frac{4}{1+\kappa^2}\int\limits_{-1}^{1}\left[\left(C^*_{01} - C^*_{11}\frac{D_1}{2} + C^*_{21}\left(\frac{3D_1^2}{8} - \frac{D_0}{2}\right) - C^*_{31}D_1\left(\frac{3D_0}{4} + \frac{5D_1^2}{16}\right)\right) I_1\right] d\xi$$

$$-\frac{4}{1+\kappa^2}\int\limits_{-1}^{1}\left[\left(C^*_{11} - C^*_{21}\frac{3D_1}{4} + C^*_{31}\left(\frac{5D_1^2}{8} - \frac{2D_0}{3}\right)\right)\frac{1}{\kappa}I_2 + \frac{h(\xi)}{2}\left(C^*_{21} - C^*_{31}\frac{5D_1}{6}\right) I_3\right] d\xi,$$

where index $*$ means that coefficients (12) are dimensionless and I_i have the form (9).

3 Results

3.1 Uniform pressure law

Situations with regular normal pressure distribution are widely investigated in the literature but for more simple contact areas: circle [4, 6, 7, 8, 9] and ring [4], line [10] and rectangular [11]. This study deals with elliptic contact area. We examined our problem, assuming $t_0 = 0$, $v_0 = 1$, $\omega_0 = 1$,$\vartheta_0 = \frac{\pi}{4}$, as initial conditions.

The following equation arise from the system (8):

$$\frac{dv}{d\omega} = \Phi(\beta, \vartheta_*), \tag{14}$$

where

$$\Phi(\beta, \vartheta_*) = \frac{I_* F_\tau}{M_{c\zeta}}, \tag{15}$$

with I_* – dimensionless inertia moment.

ϑ_* is a parameter in equation (15), which is zero for orthotropic friction (see [4]).

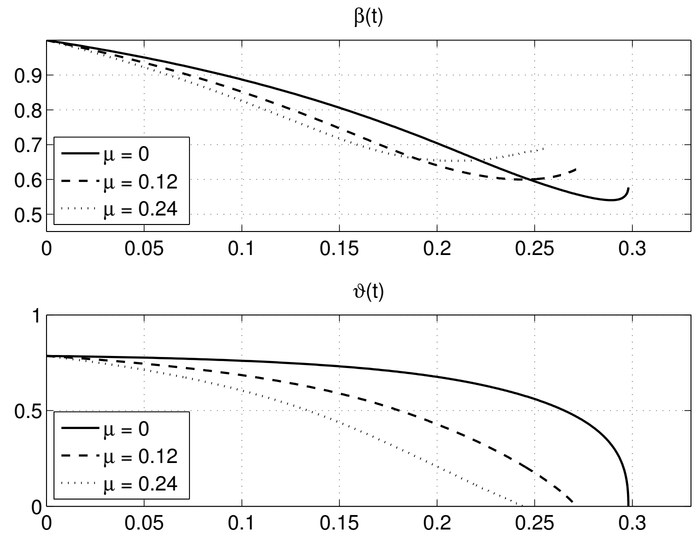

Figure 2: $\beta(t)$, $\vartheta(t)$ with $e = 0.866$, regular pressure distribution

Summarising results of [4], we obtain equation for β:

$$\beta - \Phi_i(\beta, \vartheta_*) = 0, \tag{16}$$

which was solved numerically for several φ. The results are presented in the table 1.

Table 1: $\beta(\mu)$ with $e = 0.866$. Regular pressure distribution.

μ	0.00	0.03	0.06	0.09	0.12	0.15	0.18	0.21	0.24
$\varphi = \frac{\pi}{4}$	0.404	0.446	0.486	0.524	0.561	0.599	0.642	0.704	0.787
$\varphi = \frac{\pi}{3}$	0.494	0.521	0.547	0.572	0.597	0.620	0.643	0.667	0.689
$\varphi = \frac{\pi}{2}$	0.577	0.593	0.608	0.623	0.637	0.650	0.664	0.677	0.690

It can be seen from the table, that resultant β strictly depends from μ, growing faster for lower values of φ. Figure 2 shows evolution of β and ϑ during motion. Sliding and spinning stop at the same moment with $\vartheta = 0$.

3.2 Linear pressure law

Assuming linear pressure law, we solved system (11) (which was obtained for ellipse center) for $\xi_0 = 0.1$, $\eta_0 = 0.1$ (see equation (10)) and same initial conditions as in the previous case. (Similar case for disk was studied in [12]).

Results of numerical analysis of linear pressure law are shown on figure 3: β here is a distance between ellipse center and instantaneous velocity center and it is limited at

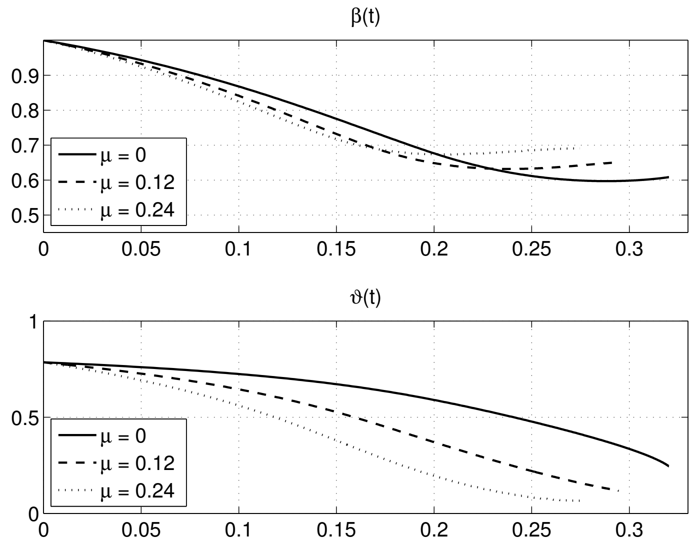

Figure 3: $\beta(t)$, $\vartheta(t)$ with $e = 0.866$, linear pressure distribution

the stopping moment, thus sliding and spinning end simultaneously. On figure 4 normal force evolution is presented. It can be seen, that for all 3 examples F_n seeks zero, thus, we achieved end of the movement.

4 APPENDIX

We shall give some remarks on the movement of a thin body resting on a circle contact area. This case was observed in [7, 8]. Assuming, that solid body rests on the circle contact area, let's look at 3 classical pressure distribution cases: uniform law, Boussinesq law, Herz law:

$$p = \frac{mg}{\pi a^2}, \quad p = \frac{p_0}{\sqrt{1-\rho^2/a^2}},$$
$$p_0 = \frac{mg}{2\pi a^2},$$
$$p = p_0\sqrt{1-\frac{\rho^2}{a^2}}, \quad p_0 = \frac{3mg}{2\pi a}.$$

Suppose that axes Ox and Oy of orthogonal coordinate system are chosen so, that the orthotropic friction law is:

$$\mathbf{T} = -p\begin{pmatrix} f_x & 0 \\ 0 & f_y \end{pmatrix}\begin{pmatrix} v_x/v \\ v_y/v \end{pmatrix}, \tag{17}$$

where $\mathbf{T}$ – friction force vector; p – pressure at the contact point; v_x, v_y, v – velocity projections on Ox and Oy axes and it's magnitude .

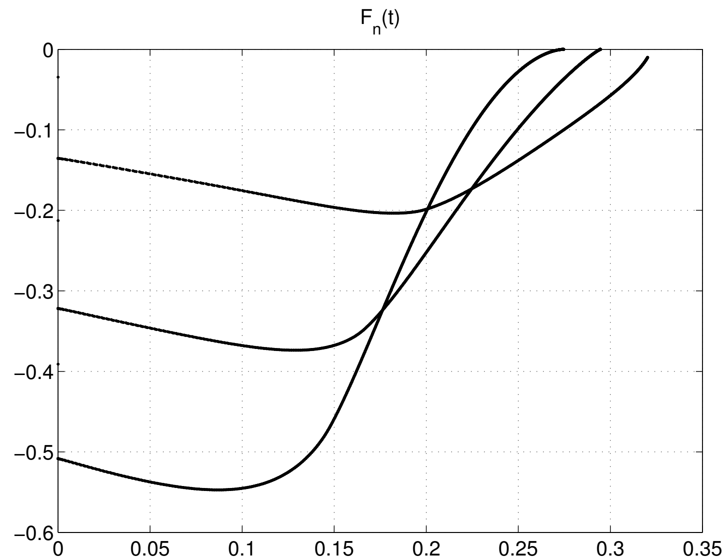

Figure 4: Normal force evolution, with $e = 0.866$, linear pressure distribution

Velocity vector of the circle area center is:

$$\mathbf{v_o} = v_o(\cos\vartheta\mathbf{i} + \sin\vartheta\mathbf{j}),$$

where v_o – magnitude of velocity vector; ϑ – directional velocity angle, starting from Ox.

Thus, several statements can be formulated (for Herz pressure distribution) [13]:

Statement 1. Let a body rests on horizontal plane surface and contact area is a circle, and pressure is distributed following Herz law. Let friction forces have orthotropic properties: f_x, f_y – friction coefficients along orthogonal axes Ox and Oy, fixed with plane and $f_y \geq f_x, \quad \mu = f_y - f_x \geq 0$. Let's assume that gravity force projection on the plane matches circle center and dimensionless inertia moment of the body about axes, passing through the center, is $I_* = I/(ma^2)$ (* is omitted in the following text). Than:

1. with $I \in (0, (f_x+\mu)/(5f_x))$, angular velocity of the body ω goes to zero faster, than linear velocity of circles center v_o, but becomes zero simultaneously with $\omega/v_o \to 0$, tangent to the phase trajectory at the point $v_o = 0, \omega = 0$ has a zero inclination angle;

2. with $I \in ((f_x+\mu)/(5f_x), (6f_x+5\mu)/(24f_x))$, ω and v becomes zero simultaneously with the rule $v_0 = k\omega, k > 1$;

3. with $I \in ((6f_x+5\mu)/(24f_x), (2f_x+\mu)/(4f_x))$, ω and v_0 turn to zero at the same time with the rule $v_0 = k\omega, k < 1$

4. with $I \in ((2f_x+\mu)/(4f_x), +\infty)$ ω and v_0 turn to zero simultaneously with $\omega/v_0 \to 0$, but v seeking zero faster, than ω and tangents to the phase trajectories at the point $v_0 = 0, \omega = 0$ have an inclination angle of $\pi/2$.

Same statements can be formulated for two other laws (see [14, 15]). Results comparison is in the table 2.

Table 2: Moment inertia intervals in wich statement 1 is true

	uniform law	Boussinesq law	Herz law
1	$(0, (f_x + \mu))/(4f_x)$	$(0, (f_x + \mu)/(3f_x))$	$(0, (f_x + \mu)/(5f_x))$
2	$((f_x + \mu)/(4f_x), (5f_x + 4\mu)/(15f_x))$	$((f_x + \mu)/(3f_x), (4f_x + 3\mu)/(8f_x))$	$((f_x + \mu)/(5f_x), (6f_x + 5\mu)/(24f_x))$
3	$((5f_x + 4\mu)/(15f_x), (2f_x + \mu)/(3f_x))$	$((4f_x + 3\mu)/(8f_x), (2f_x + \mu)/(2f_x))$	$((6f_x + 5\mu)/(24f_x), (2f_x + \mu)/(4f_x))$
4	$((2f_x + \mu)/(3f_x), +\infty)$	$((2f_x + \mu)/(2f_x), +\infty)$	$((2f_x + \mu)/(4f_x), +\infty)$

5 CONCLUSIONS

- Differential equations of the body's movement, resting on elliptical area are achieved in accordance with anisotropic friction.

- Differential equations were analysed. It is shown that direction of center of mass velocity vector matches the axes along which friction coefficient has a minimum value.

- Dependency of distance from elliptical plate center to the instantaneous velocity center at the stopping moment from components of friction tensor, ellipse eccentricity and it's orientation on the plane was identified. It is shown that sliding and spinning end at the same moment for regular and linear normal pressure distribution.

- For circle contact area statement, which connects inertia moment of the body, friction tensor and pressure distribution law with the distance to the instantaneous velocity center at the stopping moment was approved.

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