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TIME-AVERAGED SHALLOW WATER EQUATIONS BY ASYMPTOTIC ANALYSIS

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Abstract. The objective of this paper is to derive, from Navier-Stokes equations, a new bidimensional shallow water model able to reduce the high frequency oscillations that are produced when the Reynolds number is increased in turbulent flows. With this aim, Navier-Stokes equations are time-averaged and then asymptotic analysis techniques (where the small parameter considered is the quotient between the characteristic depth and the diameter of the domain) are used. Numerical experiments confirm that this new model is able to obtain, in most of the cases, a given accuracy using time steps larger than the time step needed by classical shallow water models.

1 INTRODUCTION

As it is well known, the equations governing the behavior of a fluid are the Navier-Stokes equations. Due to their strong nonlinearity, high frequency oscillations are produced when the Reynolds number is increased, and the flow becomes unstable and turbulent. The most common approach at the moment in hydraulic engineering practise is to solve the Reynolds Averaged Navier-Stokes equations, in which the effect of turbulence is modelled rather than solved. When we approximate Navier-Stokes equations using a shallow water model, if the flow is turbulent, a very small time step must be chosen.

This paper is focused on the derivation, from Navier-Stokes equations, of a new bidimensional shallow water model able to reduce these oscillations and then able to achieve good results with larger time steps. Filtering has given good results when working with turbulent Navier-Stokes equations (see [1]) and asymptotic analysis has been applied successfully to obtain and justify shallow water models (see [2, 3]). In the literature, we can found that the separation between large and small scales is traditionally assumed to be obtained by applying a spatial filter to the Navier-Stokes equations (see [4, 5]); but time filtering is also suggested by several authors (see [6, 7]). In this work, we have decided to use a time filter.

All the details of the derivation of the new model can be found in [8]. In that paper, we introduce and render non-dimensional Navier-Stokes system that serves as our starting point (see section 2). We then derive our shallow water model (section 3) and show (as in section 5) that this new model is able to obtain, in most of the cases, a given accuracy using time steps larger than the time step needed by classical shallow water models.

2 THE THREE-DIMENSIONAL MODEL EQUATIONS

The three-dimensional incompressible Navier-Stokes model [9] serves as the starting point for our subsequent development. We consider a basin with varying bottom topography and a free top surface and supplement them with appropriate boundary conditions.

2.1 Non-dimensionalization

The shallow water approximation is characterized by the smallness of a non-dimensional parameter that we can identify assuming that the typical depth of the basin (H_C) is much smaller than the typical horizontal length (L_C), i.e., that

$$\frac{H_C}{L_C} = \varepsilon \qquad \text{where} \quad \varepsilon << 1 \tag{1}$$

This small parameter is an aspect ratio.

We introduce the non-dimensional independent variables t, x, y and z by

$$t = \frac{t^\varepsilon}{T_C}, \quad x = \frac{x^\varepsilon}{L_C}, \quad y = \frac{y^\varepsilon}{L_C}, \quad z = \frac{z^\varepsilon}{\varepsilon L_C} \tag{2}$$

where T_C is a typical time. Recalling (1), the non-dimensional water height and bottom surface are defined by

$$h(t,x,y) = \frac{H^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)}{\varepsilon L_C}, \quad b(x,y) = \frac{B^\varepsilon(x^\varepsilon, y^\varepsilon)}{\varepsilon L_C} \tag{3}$$

so the non-dimensional top surface

$$s(t,x,y) = \frac{S^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon)}{\varepsilon L_C} \tag{4}$$

We shall now introduce non-dimensional dependent variables (velocity, pressure, ...),

$$u_i^\varepsilon(t,x,y,z) = \frac{T_C}{L_C} U_i^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon), \quad i = 1,2,3 \tag{5}$$

$$G = \frac{T_C^2}{L_C} g, \quad \phi = T_C \Phi \tag{6}$$

$$p^\varepsilon(t,x,y,z) = \frac{T_C^2}{\rho_0 L_C^2} P^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon), \quad p_s(t,x,y) = \frac{T_C^2}{\rho_0 L_C^2} P_s^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon) \tag{7}$$

$$\sigma_{ij}^\varepsilon(t,x,y,z) = \frac{T_C^2}{\rho_0 L_C^2} T_{ij}^\varepsilon(t^\varepsilon, x^\varepsilon, y^\varepsilon, z^\varepsilon) \quad i,j = 1,2,3 \tag{8}$$

$$\vec{f}_W^\varepsilon = \frac{T_C^2}{\rho_0 L_C^2} \vec{\mathbf{F}}_W^\varepsilon, \quad \vec{f}_R^\varepsilon = \frac{T_C^2}{\rho_0 L_C^2} \vec{\mathbf{F}}_R^\varepsilon \tag{9}$$

where g is the gravitational acceleration, $\vec{\Phi} = \Phi \left(\sin\varphi^\varepsilon \vec{k} + \cos\varphi^\varepsilon \vec{\jmath} \right)$ is the angular velocity of rotation of the Earth with $\Phi = 7.29 \times 10^{-5}$ rad/s, P_s^ε is the ambient atmospheric pressure, $\mathbf{T}^\varepsilon$ is the stress tensor, $\vec{\mathbf{F}}_R^\varepsilon$ is the friction force and $\vec{\mathbf{F}}_W^\varepsilon$ is the force of the wind.

2.2 The non-dimensional equations

We now express the three-dimensional incompressible Navier-Stokes system in terms of the above non-dimensional variables:

$$\begin{aligned} \frac{\partial u_1^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_1^\varepsilon}{\partial x} + u_2^\varepsilon \frac{\partial u_1^\varepsilon}{\partial y} + \frac{1}{\varepsilon} u_3^\varepsilon \frac{\partial u_1^\varepsilon}{\partial z} &= -\frac{\partial p^\varepsilon}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u_1^\varepsilon}{\partial x^2} + \frac{\partial^2 u_1^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2} \frac{\partial^2 u_1^\varepsilon}{\partial z^2} \right) \\ &\quad + 2\phi \left((\sin\varphi)\, u_2^\varepsilon - (\cos\varphi)\, u_3^\varepsilon \right) \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{\partial u_2^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_2^\varepsilon}{\partial x} + u_2^\varepsilon \frac{\partial u_2^\varepsilon}{\partial y^\varepsilon} + \frac{1}{\varepsilon} u_3^\varepsilon \frac{\partial u_2^\varepsilon}{\partial z} &= -\frac{\partial p^\varepsilon}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 u_2^\varepsilon}{\partial x^2} + \frac{\partial^2 u_2^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2} \frac{\partial^2 u_2^\varepsilon}{\partial z^2} \right) \\ &\quad - 2\phi (\sin\varphi)\, u_1^\varepsilon \end{aligned} \tag{11}$$

$$\begin{aligned} \frac{\partial u_3^\varepsilon}{\partial t} + u_1^\varepsilon \frac{\partial u_3^\varepsilon}{\partial x} + u_2^\varepsilon \frac{\partial u_3^\varepsilon}{\partial y} + \frac{1}{\varepsilon} u_3^\varepsilon \frac{\partial u_3^\varepsilon}{\partial z} &= -\frac{1}{\varepsilon}\frac{\partial p^\varepsilon}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 u_3^\varepsilon}{\partial x^2} + \frac{\partial^2 u_3^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2} \frac{\partial^2 u_3^\varepsilon}{\partial z^2} \right) \\ &\quad - G + 2\phi (\cos\varphi)\, u_1^\varepsilon \end{aligned} \tag{12}$$

$$\frac{\partial u_1^\varepsilon}{\partial x} + \frac{\partial u_2^\varepsilon}{\partial y} + \frac{1}{\varepsilon}\frac{\partial u_3^\varepsilon}{\partial z} = 0 \tag{13}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_b^s u_1^\varepsilon dz + \frac{\partial}{\partial y}\int_b^s u_2^\varepsilon dz = 0 \tag{14}$$

where $\nu \dfrac{T_C}{L_C^2} = \dfrac{1}{Re}$.

3 DERIVATION OF THE SHALLOW WATER MODEL

The formal derivation of our shallow water model has two steps. We first obtain the time-averaged Navier-Stokes equations and then make the shallow water approximation by formally expanding solutions of the model in powers of ε.

3.1 Time averaging process

We define:

$$\bar{u}_i^\varepsilon(t,x,y,z;\eta) = \frac{1}{2\eta}\int_{t-\eta}^{t+\eta} u_i^\varepsilon(r,x,y,z)dr \quad i=1,2,3 \tag{15}$$

If we assume that $0 < \eta << 1$ and we approximate u_i^ε by its Taylor's expansion for $r \in (t-\eta, t+\eta)$:

$$\begin{aligned} u_i^\varepsilon(r,x,y,z) &= u_i^\varepsilon(t,x,y,z) + \frac{\partial u_i^\varepsilon}{\partial t}(t,x,y,z)(r-t) + \frac{1}{2}\frac{\partial^2 u_i^\varepsilon}{\partial t2}(t,x,y,z)(r-t)^2 \\ &+ \frac{1}{3!}\frac{\partial^3 u_i^\varepsilon}{\partial t^3}t,x,y,z)(r-t)^3 + \cdots \end{aligned} \tag{16}$$

the we have:

$$\bar{u}_i^\varepsilon(t,x,y,z;\eta) = u_i^\varepsilon(t,x,y,z) + \frac{\partial^2 u_i^\varepsilon}{\partial t^2}(t,x,y,z)\frac{\eta^2}{3!} + \frac{\partial^4 u_i^\varepsilon}{\partial t^4}(t,x,y,z)\frac{\eta^4}{5!} + \cdots \tag{17}$$

It is easy to demonstrate that, for regular enough funtions f^ε and g^ε, when η is small enough:

$$\overline{\frac{\partial f^\varepsilon}{\partial t}}(t,x,y,z;\eta) = \frac{\partial \bar{f}^\varepsilon}{\partial t}(t,x,y,z;\eta) \tag{18}$$

$$f^\varepsilon(t,x,y,z) = \bar{f}^\varepsilon(t,x,y,z;\eta) - \frac{\partial^2 \bar{f}^\varepsilon}{\partial t^2}(t,x,y,z;\eta)\frac{\eta^2}{6} + O(\eta^4) \tag{19}$$

$$\begin{aligned} \overline{f^\varepsilon g^\varepsilon}(t,x,y,z;\eta) &= \overline{f^\varepsilon}(t,x,y,z;\eta)\overline{g^\varepsilon}(t,x,y,z;\eta) + \frac{\partial f^\varepsilon}{\partial t}\frac{\partial g^\varepsilon}{\partial t}(t,x,y,z)\frac{\eta^2}{3} \\ &+ O(\eta^4) \end{aligned} \tag{20}$$

$$\overline{f^\varepsilon g^\varepsilon} = \bar{f}^\varepsilon g^\varepsilon + \frac{\eta^2}{6}\left(2\frac{\partial \bar{f}^\varepsilon}{\partial t}\frac{\partial g^\varepsilon}{\partial t} + \bar{f}^\varepsilon\frac{\partial^2 g^\varepsilon}{\partial t^2}\right) + O(\eta^4) \tag{21}$$

$$\overline{f^\varepsilon(t,x,y,s(t,x,y))} = \left.\bar{f}^\varepsilon\right|_{z=s} \text{ en } z=s \tag{22}$$

The averaging process, applying (18)-(22), provides, for example, from equations (10) and (13):

$$\begin{aligned} &\frac{\partial \bar{u}_1^\varepsilon}{\partial t} + \bar{u}_1^\varepsilon\frac{\partial \bar{u}_1^\varepsilon}{\partial x} + \left(\frac{\partial \bar{u}_1^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial t\partial x}\right)\frac{\eta^2}{3} + \bar{u}_2^\varepsilon\frac{\partial \bar{u}_1^\varepsilon}{\partial y} + \left(\frac{\partial \bar{u}_2^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial t\partial y}\right)\frac{\eta^2}{3} \\ &\quad + \frac{1}{\varepsilon}\left[\bar{u}_3^\varepsilon\frac{\partial \bar{u}_1^\varepsilon}{\partial z} + \left(\frac{\partial \bar{u}_3^\varepsilon}{\partial t}\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial t\partial z}\right)\frac{\eta^2}{3}\right] + O(\eta^4) = -\frac{\partial \bar{p}^\varepsilon}{\partial x} \\ &\quad + \frac{1}{Re}\left(\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial x^2} + \frac{\partial^2 \bar{u}_1^\varepsilon}{\partial y^2} + \frac{1}{\varepsilon^2}\frac{\partial^2 \bar{u}_1^\varepsilon}{\partial z^2}\right) + 2\phi\left((\sin\varphi)\,\bar{u}_2^\varepsilon - (\cos\varphi)\,\bar{u}_3^\varepsilon\right) \end{aligned} \tag{23}$$

$$\frac{\partial \bar{u}_1^\varepsilon}{\partial x} + \frac{\partial \bar{u}_2^\varepsilon}{\partial y} + \frac{1}{\varepsilon}\frac{\partial \bar{u}_3^\varepsilon}{\partial z} = 0 \tag{24}$$

Similar expressions are obtained from the other equations and conditions.

3.2 Asymptotic analysis

Our shallow water model derive from the assumption that the aspect ratio ε is small. We make the shallow water approximation by formally expanding solutions of the model in powers of ε. We seek a formal solution in the form of an asymptotic series:

$$\begin{cases} \bar{u}_i^\varepsilon = \bar{u}_i^0 + \varepsilon \bar{u}_i^1 + \varepsilon^2 \bar{u}_i^2 + \cdots, \quad i = 1,2,3 \\ \bar{p}^\varepsilon = \bar{p}^0 + \varepsilon \bar{p}^1 + \varepsilon^2 \bar{p}^2 + \cdots \\ \bar{\sigma}_{ij}^\varepsilon = \bar{\sigma}_{ij}^0 + \varepsilon \bar{\sigma}_{ij}^1 + \varepsilon^2 \bar{\sigma}_{ij}^2 + \cdots \quad (i,j = 1,2,3) \\ \bar{f}_{R_i}^\varepsilon = \varepsilon \bar{f}_{R_i}^1 + \varepsilon^2 \bar{f}_{R_i}^2 + \cdots \quad (i = 1,2) \\ \bar{f}_{W_i}^\varepsilon = \varepsilon \bar{f}_{W_i}^1 + \varepsilon^2 \bar{f}_{W_i}^2 + \cdots \quad (i = 1,2) \end{cases} \tag{25}$$

The non-dimensional time-averaged equations are written using (25). As an example, the value of $\bar{u}_3^0$ can be found from the incompressibility condition (24) written at the leading order $O(\varepsilon^{-1})$:

$$\frac{\partial \bar{u}_3^0}{\partial z} = 0 \Rightarrow \bar{u}_3^0 = \bar{u}_3^0(t,x,y) \tag{26}$$

we now consider the non-penetration boundary condition that, to the leading order, becomes:

$$\bar{u}_3^0 = 0 \text{ for } z = b \tag{27}$$

so

$$\bar{u}_3^0 = 0 \tag{28}$$

3.3 First order approximation

We can now consider a first order approximation:

$$\tilde{u}_i(\varepsilon) = \bar{u}_i^0 + \varepsilon \bar{u}_i^1 \quad (i = 1,2) \quad \tilde{u}_3(\varepsilon) = \bar{u}_3^0 + \varepsilon \bar{u}_3^1 + \varepsilon^2 \bar{u}_3^2, \quad \tilde{p}(\varepsilon) = \bar{p}^0 + \varepsilon \bar{p}^1 \tag{29}$$

Once the terms $\bar{u}_3^k$ $(k = 0,1,2)$ and $\bar{p}^k$ $(k = 0,1)$ are known:

$$
\begin{aligned}
\tilde{u}_3^\varepsilon &= \varepsilon \bar{u}_3^1 + \varepsilon^2 \bar{u}_3^2 = \varepsilon \left[(b-z) \left(\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) + \bar{u}_1^0 \frac{\partial b}{\partial x} + \bar{u}_2^0 \frac{\partial b}{\partial y} \right] \\
&+ \varepsilon^2 \left[(b-z) \left(\frac{\partial \bar{u}_1^1}{\partial x} + \frac{\partial \bar{u}_2^1}{\partial y} \right) + \bar{u}_1^1 \frac{\partial b}{\partial x} + \bar{u}_2^1 \frac{\partial b}{\partial y} \right] \\
&= \varepsilon \left[(b-z) \left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial y} \right) + \tilde{u}_1^\varepsilon \frac{\partial b}{\partial x} + \tilde{u}_2^\varepsilon \frac{\partial b}{\partial y} \right] \qquad (30) \\
\tilde{p}^\varepsilon &= \bar{p}_s - \frac{2}{Re} \left(\frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) + \varepsilon \left[(s-z)(g - 2\phi(\cos\varphi)\bar{u}_1^0) - \frac{2}{Re} \left(\frac{\partial \bar{u}_1^1}{\partial x} + \frac{\partial \bar{u}_2^1}{\partial y} \right) \right] \\
&= \bar{p}_s + \varepsilon(s-z)(g - 2\phi(\cos\varphi)\bar{u}_1^0) - \frac{2}{Re} \left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial y} \right) \qquad (31) \\
&= \bar{p}_s + \varepsilon(s-z)(g - 2\phi(\cos\varphi)\tilde{u}_1^\varepsilon) - \frac{2}{Re} \left(\frac{\partial \tilde{u}_1^\varepsilon}{\partial x} + \frac{\partial \tilde{u}_2^\varepsilon}{\partial y} \right) + O(\varepsilon^2) \qquad (32)
\end{aligned}
$$

For $\bar{u}_1^0$ and $\bar{u}_2^0$ we obtain a coupled system of equations. The first one of the system is:

$$
\begin{aligned}
&\frac{\partial \bar{u}_1^0}{\partial t} + \bar{u}_1^0 \frac{\partial \bar{u}_1^0}{\partial x} + \bar{u}_2^0 \frac{\partial \bar{u}_1^0}{\partial y} + \frac{\eta^2}{3} \left(\frac{\partial \bar{u}_1^0}{\partial t} \frac{\partial^2 \bar{u}_1^0}{\partial t \partial x} + \frac{\partial \bar{u}_2^0}{\partial t} \frac{\partial^2 \bar{u}_1^0}{\partial t \partial y} \right) \\
&\quad = -\frac{\partial \bar{p}_s}{\partial x} + \frac{1}{Re} \left(4 \frac{\partial^2 \bar{u}_1^0}{\partial x^2} + \frac{\partial^2 \bar{u}_1^0}{\partial y^2} + 3 \frac{\partial^2 \bar{u}_2^0}{\partial x \partial y} \right) + 2\phi \left(\sin\varphi \right) \bar{u}_2^0 \\
&\quad + \frac{1}{hRe} \left\{ 2 \left(2 \frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) \frac{\partial h}{\partial x} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x} \right) \frac{\partial h}{\partial y} \right. \\
&\quad + \frac{\eta^2}{6} \left[4 \frac{\partial}{\partial t} \left(2 \frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) \frac{\partial^2 h}{\partial t \partial x} + 2 \left(2 \frac{\partial \bar{u}_1^0}{\partial x} + \frac{\partial \bar{u}_2^0}{\partial y} \right) \frac{\partial^3 h}{\partial t^2 \partial x} \right. \\
&\quad \left. \left. + 2 \frac{\partial}{\partial t} \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x} \right) \frac{\partial^2 h}{\partial t \partial y} + \left(\frac{\partial \bar{u}_1^0}{\partial y} + \frac{\partial \bar{u}_2^0}{\partial x} \right) \frac{\partial^3 h}{\partial t^2 \partial y} \right] \right\} \\
&\quad + \frac{1}{h} \left(\bar{f}_{W_1}^1 + \bar{f}_{R_1}^1 \right) + O(\eta^4) \qquad (33)
\end{aligned}
$$

We repeat the process from the equations for the terms $\bar{u}_i^1$ $(i = 1, 2)$. The first one is:

$$
\begin{aligned}
&\frac{\partial \bar{u}_1^1}{\partial t} + \bar{u}_1^0 \frac{\partial \bar{u}_1^1}{\partial x} + \bar{u}_1^1 \frac{\partial \bar{u}_1^0}{\partial x} + \bar{u}_2^0 \frac{\partial \bar{u}_1^1}{\partial y} + \bar{u}_2^1 \frac{\partial \bar{u}_1^0}{\partial y} \\
&\quad + \frac{\eta^2}{3} \left(\frac{\partial \bar{u}_1^0}{\partial t} \frac{\partial^2 \bar{u}_1^1}{\partial t \partial x} + \frac{\partial \bar{u}_1^1}{\partial t} \frac{\partial^2 \bar{u}_1^0}{\partial t \partial x} + \frac{\partial \bar{u}_2^0}{\partial t} \frac{\partial^2 \bar{u}_1^1}{\partial t \partial y} + \frac{\partial \bar{u}_2^1}{\partial t} \frac{\partial^2 \bar{u}_1^0}{\partial t \partial y} \right) \\
&= -\frac{\partial s}{\partial x} g + 2\phi \left(\sin\varphi \right) \bar{u}_2^1 \\
&\quad + \frac{1}{Re} \left(4 \frac{\partial^2 \bar{u}_1^1}{\partial x^2} + \frac{\partial^2 \bar{u}_1^1}{\partial y^2} + 3 \frac{\partial^2 \bar{u}_2^1}{\partial x \partial y} \right) + \frac{2}{hRe} \left(2 \frac{\partial \bar{u}_1^1}{\partial x} + \frac{\partial \bar{u}_2^1}{\partial y} \right) \frac{\partial h}{\partial x}
\end{aligned}
$$

$$+\frac{1}{hRe}\left(\frac{\partial\bar{u}_1^1}{\partial y}+\frac{\partial\bar{u}_2^1}{\partial x}\right)\frac{\partial h}{\partial y}+\frac{\eta^2}{6h}\left[2\frac{\partial h}{\partial t}(2\phi(\cos\varphi)\bar{u}_1^0-g)\frac{\partial^2 h}{\partial t\partial x}\right.$$
$$+\frac{4}{Re}\frac{\partial}{\partial t}\left(2\frac{\partial\bar{u}_1^1}{\partial x}+\frac{\partial\bar{u}_2^1}{\partial y}\right)\frac{\partial^2 h}{\partial t\partial x}+\frac{2}{Re}\left(2\frac{\partial\bar{u}_1^1}{\partial x}+\frac{\partial\bar{u}_2^1}{\partial y}\right)\frac{\partial^3 h}{\partial t^2\partial x}$$
$$\left.+\frac{2}{Re}\frac{\partial}{\partial t}\left(\frac{\partial\bar{u}_1^1}{\partial y}+\frac{\partial\bar{u}_2^1}{\partial x}\right)\frac{\partial^2 h}{\partial t\partial y}+\frac{1}{Re}\left(\frac{\partial\bar{u}_1^1}{\partial y}+\frac{\partial\bar{u}_2^1}{\partial x}\right)\frac{\partial^3 h}{\partial t^2\partial y}\right]$$
$$+\frac{1}{h}\left(\bar{f}_{W_1}^2+\bar{f}_{R_1}^2\right)+2\phi(\cos\varphi)\left[\frac{\partial h}{\partial x}\bar{u}_1^0+h\frac{\partial\bar{u}_1^0}{\partial x}+\frac{h}{2}\frac{\partial\bar{u}_2^0}{\partial y}-\bar{u}_2^0\frac{\partial b}{\partial y}\right]+O(\eta^4) \tag{34}$$

Equations (33) and (34) are used to obtain:

$$\frac{\partial\tilde{u}_1^\varepsilon}{\partial t}+\tilde{u}_1^\varepsilon\frac{\partial\tilde{u}_1^\varepsilon}{\partial x}+\tilde{u}_2^\varepsilon\frac{\partial\tilde{u}_1^\varepsilon}{\partial y}+\frac{\eta^2}{3}\left(\frac{\partial\tilde{u}_1^\varepsilon}{\partial t}\frac{\partial^2\tilde{u}_1^\varepsilon}{\partial t\partial x}+\frac{\partial\tilde{u}_2^\varepsilon}{\partial t}\frac{\partial^2\tilde{u}_1^\varepsilon}{\partial t\partial y}\right)$$
$$=\frac{\partial\bar{u}^0}{\partial t}+\varepsilon\frac{\partial\bar{u}^1}{\partial t}+(\bar{u}^0+\varepsilon\bar{u}^1)\left(\frac{\partial\bar{u}^0}{\partial x}+\varepsilon\frac{\partial\bar{u}^1}{\partial x}\right)+(\bar{v}^0+\varepsilon\bar{v}^1)\left(\frac{\partial\bar{u}^0}{\partial y}+\varepsilon\frac{\partial\bar{u}^1}{\partial y}\right)$$
$$+\frac{\eta^2}{3}\left[\left(\frac{\partial\bar{u}^0}{\partial t}+\varepsilon\frac{\partial\bar{u}^1}{\partial t}\right)\left(\frac{\partial^2\bar{u}^0}{\partial t\partial x}+\varepsilon\frac{\partial^2\bar{u}^1}{\partial t\partial x}\right)+\left(\frac{\partial\bar{v}^0}{\partial t}+\varepsilon\frac{\partial\bar{v}^1}{\partial t}\right)\left(\frac{\partial^2\bar{u}^0}{\partial t\partial y}+\varepsilon\frac{\partial^2\bar{u}^1}{\partial t\partial y}\right)\right]$$
$$=-\frac{\partial\bar{p}_s}{\partial x}-\varepsilon\frac{\partial s}{\partial x}g+2\phi\left(\sin\varphi\right)\tilde{u}_2^\varepsilon+\frac{1}{Re}\left(4\frac{\partial^2\tilde{u}_1^\varepsilon}{\partial x^2}+\frac{\partial^2\tilde{u}_1^\varepsilon}{\partial y^2}+3\frac{\partial^2\tilde{u}_2^\varepsilon}{\partial x\partial y}\right)$$
$$+\frac{1}{hRe}\left\{2\left(2\frac{\partial\tilde{u}_1^\varepsilon}{\partial x}+\frac{\partial\tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial h}{\partial x}+\left(\frac{\partial\tilde{u}_1^\varepsilon}{\partial y}+\frac{\partial\tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial h}{\partial y}\right.$$
$$+\frac{\eta^2}{6}\left[4\frac{\partial}{\partial t}\left(2\frac{\partial\tilde{u}_1^\varepsilon}{\partial x}+\frac{\partial\tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial^2 h}{\partial t\partial x}+2\left(2\frac{\partial\tilde{u}_1^\varepsilon}{\partial x}+\frac{\partial\tilde{u}_2^\varepsilon}{\partial y}\right)\frac{\partial^3 h}{\partial t^2\partial x}\right.$$
$$\left.\left.+2\frac{\partial}{\partial t}\left(\frac{\partial\tilde{u}_1^\varepsilon}{\partial y}+\frac{\partial\tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial^2 h}{\partial t\partial y}+\left(\frac{\partial\tilde{u}_1^\varepsilon}{\partial y}+\frac{\partial\tilde{u}_2^\varepsilon}{\partial x}\right)\frac{\partial^3 h}{\partial t^2\partial y}\right]\right\}$$
$$+(2\phi(\cos\varphi)\tilde{u}_1^\varepsilon-g)\varepsilon\left[\frac{\eta^2}{3h}\frac{\partial h}{\partial t}\frac{\partial^2 h}{\partial t\partial x}\right]+2\phi(\cos\varphi)\varepsilon\left[\frac{\partial h}{\partial x}\tilde{u}_1^\varepsilon+h\frac{\partial\tilde{u}_1^\varepsilon}{\partial x}+\frac{h}{2}\frac{\partial\tilde{u}_2^\varepsilon}{\partial y}-\tilde{u}_2^\varepsilon\frac{\partial b}{\partial y}\right]$$
$$+\frac{1}{\varepsilon h}\left(\tilde{f}_{W_1}^\varepsilon+\tilde{f}_{R_1}^\varepsilon\right)+O(\eta^4)+O(\varepsilon^2) \tag{35}$$

The system of equations for $\tilde{u}_i (i=1,2)$ is coupled with the following equation for the water depth:

$$\frac{\partial h}{\partial t}+\frac{\partial(h\tilde{u}^\varepsilon)}{\partial x}+\frac{\partial(h\tilde{v}^\varepsilon)}{\partial y}-\frac{\eta^2}{6}\left[\frac{\partial}{\partial x}\left(h\frac{\partial^2\tilde{u}^\varepsilon}{\partial t^2}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial^2\tilde{v}^\varepsilon}{\partial t^2}\right)\right]=O(\eta^4) \tag{36}$$

4 THE NEW SHALLOW WATER SYSTEM

In this section, we present the model that we have derived dropping the $O(\varepsilon^2)$ and $O(\eta^4)$ terms in the above equations. We write the model in the original variables (not the non-dimensional ones). For notational convenience, we henceforth drop the $\tilde{}$.

$$\frac{\partial H}{\partial t} + \nabla \cdot \left[H \left(\vec{\bar{\mathbf{U}}} - \frac{\eta^2 T_C^2}{6} \frac{\partial^2 \vec{\bar{\mathbf{U}}}}{\partial t^2} \right) \right] = 0 \tag{37}$$

$$\begin{aligned}
&\frac{\partial \vec{\bar{\mathbf{U}}}}{\partial t} + \nabla \vec{\bar{\mathbf{U}}} \cdot \vec{\bar{\mathbf{U}}} + \frac{\eta^2 T_C^2}{3} \nabla \left(\frac{\partial \vec{\bar{\mathbf{U}}}}{\partial t} \right) \cdot \frac{\partial \vec{\bar{\mathbf{U}}}}{\partial t} - \nu \Delta \vec{\bar{\mathbf{U}}} - 3\nu \nabla \left(\nabla \cdot \vec{\bar{\mathbf{U}}} \right) \\
&\quad - \frac{\nu}{H} \left\{ \bar{\mathbf{R}} \cdot \nabla H + \frac{\eta^2 T_C^2}{6} \left[2 \frac{\partial \bar{\mathbf{R}}}{\partial t} \cdot \nabla \left(\frac{\partial H}{\partial t} \right) + \bar{\mathbf{R}} \cdot \nabla \left(\frac{\partial^2 H}{\partial t^2} \right) \right] \right\} \\
&\quad = -\frac{1}{\rho_0} \nabla \bar{P}_s + 2\Phi \left(\sin \varphi \right) \begin{pmatrix} \bar{U}_2 \\ -\bar{U}_1 \end{pmatrix} + \frac{1}{\rho_0 H} \left(\vec{\bar{\mathbf{F}}}_W + \vec{\bar{\mathbf{F}}}_R \right) \\
&\quad + \left(2\Phi (\cos \varphi) \bar{U}_1 - g \right) \left[\nabla S + \frac{\eta^2 T_C^2}{3H} \frac{\partial H}{\partial t} \nabla \left(\frac{\partial H}{\partial t} \right) \right] \\
&\quad + 2\Phi (\cos \varphi) \left[\frac{1}{2} H \nabla (\bar{U}_1) + \begin{pmatrix} -\vec{\bar{\mathbf{U}}} \cdot \nabla B + \frac{1}{2} H \nabla \cdot \vec{\bar{\mathbf{U}}} \\ 0 \end{pmatrix} \right]
\end{aligned} \tag{38}$$

where $\vec{\bar{\mathbf{U}}} = (\bar{U}_1, \bar{U}_2)$ is the time-averaged horizontal velocity, $\bar{P}_s$ is the time-averaged atmospheric pressure at the surface, $\vec{\bar{\mathbf{F}}}_W$, $\vec{\bar{\mathbf{F}}}_R$ are the wind and friction forces and

$$\bar{\mathbf{R}} = \begin{pmatrix} 4\dfrac{\partial \bar{U}_1}{\partial x} + 2\dfrac{\partial \bar{U}_2}{\partial y} & \dfrac{\partial \bar{U}_1}{\partial y} + \dfrac{\partial \bar{U}_2}{\partial x} \\ \dfrac{\partial \bar{U}_1}{\partial y} + \dfrac{\partial \bar{U}_2}{\partial x} & 2\dfrac{\partial \bar{U}_1}{\partial x} + 4\dfrac{\partial \bar{U}_2}{\partial y} \end{pmatrix}$$

$$\bar{P} = \bar{P}_s + \rho_0 (S - z) \left(g - 2\Phi (\cos \varphi) \bar{U}_1 \right) - 2\mu \nabla \cdot \vec{\bar{\mathbf{U}}} \tag{39}$$

$$\bar{U}_3^\varepsilon = (B - z) \nabla \cdot \vec{\bar{\mathbf{U}}} + \vec{\bar{\mathbf{U}}} \cdot \nabla B \tag{40}$$

5 NUMERICAL RESULTS

We have opted for MacCormack scheme ([10]) due to its good stability properties and to the fact that has been applied successfully to the resolution of similar problems. It has been implemented for the numerical resolution of all the examples presented here. We have approximated different analytical solutions of Navier-Stokes equations to compare the results the new model provides with the results that the classical shallow water model (with the new viscosity terms) (see [11]) can obtain.

We consider the family of exact solutions of Navier-Stokes equations:

$$
\begin{aligned}
U_1^\varepsilon &= (A_1 + A_2 x^\varepsilon)\sin\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right) + (B_1 + B_2 x^\varepsilon)\cos\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right)\\
&+ (C_1 + C_2 x^\varepsilon)\sin\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right) + (D_1 + D_2 x^\varepsilon)\cos\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)\\
U_2^\varepsilon &= 0\\
U_3^\varepsilon &= -z^\varepsilon\left[A_2\sin\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right) + B_2\cos\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right) + C_2\sin\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)\right.\\
&+ \left. D_2\cos\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)\right]\\
B^\varepsilon &= 0\\
H^\varepsilon &= Ee^{\frac{T_p}{2\pi}\left[\frac{A_2}{n_1}\cos\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right) - \frac{B_2}{n_1}\sin\left(\frac{2\pi n_1}{T_p}t^\varepsilon\right) + \frac{C_2}{n_2}\cos\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right) - \frac{D_2}{n_2}\sin\left(\frac{2\pi n_2}{T_p}t^\varepsilon\right)\right]}\\
P_s^\varepsilon &= P^\varepsilon(z^\varepsilon = H^\varepsilon) - 2\mu\frac{\partial U_3^\varepsilon}{\partial z^\varepsilon} + F_{W_3}^\varepsilon\\
F_{W_1}^\varepsilon &= F_{W_2}^\varepsilon = 0\\
\vec{F}_R^\varepsilon &= \vec{0}
\end{aligned}
\tag{41}
$$

where $A_i, B_i, C_i, D_i, n_i (i = 1, 2)$ and T_p are any real value, and we are able to calculate an analytical expression for P^ε, but it is too long to include it here

We consider that D is a rectangular basin of length 10 meters and width 2 meters with a 100×20 points grid. We choose the values of the parameters so the maximum depth is always smaller than 1 meter, and thus the aspect ratio is always smaller than 10^{-1}. The discretization step used is $\Delta x^\varepsilon = \Delta y^\varepsilon = 0.1$.

We introduce these three sets of values for the constants:

$$A_1 = B_1 = 2, C_1 = D_1 = 0.5, A_2 = B_2 = C_2 = D_2 = 0, E = 1, \eta = 0.25 \tag{42}$$

$$A_1 = B_1 = 1, C_1 = D_1 = 0.5, A_2 = B_2 = 0.5, C_2 = D_2 = 0, E = 0.75, \eta = 5 \times 10^{-4} \tag{43}$$

$$
\begin{aligned}
&A_1 = B_1 = 1, C_1 = D_1 = 0.1, A_2 = B_2 = 0.5, C_2 = D_2 = 10^{-2}, E = 0.4,\\
&\qquad \eta = 1.75 \times 10^{-3}
\end{aligned}
\tag{44}
$$

and we present in tables 1–3 the errors (in 2-norm) obtained when we approximate solution (41) with these choices of constants. We have computed the depth and the average horizontal velocity using the shallow water model presented in [11] (that we shall denote by **SW**) and then applying model (37)-(38) (**NM**).

For all the examples presented: $\Phi = 0$, $g = 9.8$, $\rho = 998.2$, $\nu = 1.02 \times 10^{-6}$, $T_p = 1$, $n_1 = 0.1$, $n_2 = 100$ and $T_C = 10$.

Table 1: Error bounds for example (41) with data (42)

Δt	Error bound for H **SW**	Error bound for H **NM**	Error bound for U_1 **SW**	Error bound for $\bar{U}_1$ **NM**
0.01	0	0	$1.5e4$	$4.5e-2$
0.005	0	0	$1.2e1$	$2.2e-2$
0.001	0	0	$1.8e0$	$4.4e-3$
0.0001	0	0	$1.8e-1$	$4.4e-4$

In Table 1 we can see that the errors of the velocity obtained by using the classical model with a time step $\Delta t = 10^{-4}$ are larger than the errors obtained by using the new model with a time step $\Delta t = 10^{-2}$ (100 times bigger).

Table 2: Error bounds for example (41) with data (43)

Δt	Error bound for H **SW**	Error bound for H **NM**	Error bound for U_1 **SW**	Error bound for $\bar{U}_1$ **NM**
0.005	$4.5e-2$	$4.5e-2$	$1.4e1$	$3.9e-2$
0.001	$8.8e-3$	$8.8e-3$	$2.1e0$	$7.9e-3$
0.0001	$8.8e-4$	$8.8e-4$	$2.0e-1$	$7.9e-4$

Using data (43) larger values for η are possible except for $\Delta t = 0.0001$.

Table 3: Error bounds for example (41) with data (44)

Δt	Error bound for H **SW**	Error bound for H **NM**	Error bound for U_1 **SW**	Error bound for $\bar{U}_1$ **NM**
0.005	$2.4e-1$	$1.4e-1$	$3.6e0$	$5.7e-1$

In Tables 2 and 3 we observe that, when velocity depends on the spatial variable x, the results achieved with the new model are quite better.

6 CONCLUSIONS

Numerical experiments confirm that this new model is able to obtain, in most of the cases, a given accuracy using time steps larger than the time step needed by classical shallow water models. In some cases the time step can be even a hundred times larger.

Although the numerical results achieved improve those of the model without time filtering, allowing larger time steps for the same accuracy, we think that it would be convenient to use a combination of spacial and time filters, as suggested in [6], because the use of spacial "projective" filters is necessary to reduce the number of degrees of freedom of the problem.

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