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DISCRETIZATIONS AND REGULARIZATION MODELS FOR COMPRESSIBLE FLOW THAT PRESERVE THE SKEW-SYMMETRY OF CONVECTIVE TRANSPORT

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Abstract. Convective transport conserves mass, momentum, and total energy, but also kinetic energy and internal energy separately. Preserving kinetic energy conservation of convective transport at the discrete level is known to improve the stability of a simulation method, and also ensures that the modeled sub-grid-scale dissipation is not overwhelmed by numerical dissipation in a large-eddy simulation. By rewriting the compressible Navier-Stokes equations to square-root variables, the many conservation properties of convective transport can be related to the skew-symmetry of the convective terms. By preserving this skew-symmetry at the discrete level, the conservation properties of convective transport can be transferred naturally to a simulation method. This paper demonstrates that discrete skew-symmetric operators can be related to the skew-symmetry of discretization matrices on both collocated and staggered computational grids. Also a skew-symmetry-preserving regularization model for compressible flow is proposed. The proposed symmetry-preserving methods are used to perform simulations of turbulent compressible channel flow. It is observed that preserving symmetries allows for channel flow simulations without artificial dissipation, and that simulations with a symmetry-preserving regularization model can produce better results for channel flow at low Reynolds numbers than simulations with an advanced eddy-viscosity model.

1 INTRODUCTION

Flow structures small compared to the mesh spacing can trigger a non-linear convective instability of compressible flow simulations. Therefore, simulation methods for compressible flow often need explicit filtering or artificial dissipation to attain numerical stability. An unpleasant side-effect of these ad hoc stabilization techniques is that they can suppress interesting flow phenomena such as transition to turbulence.

For incompressible flow an alternative road to numerical stability exists; the symmetry-preserving discretization [1]. The symmetry-preserving discretization preserves the skew-symmetry of the convective terms at the discrete level. This skew-symmetry prevents non-physical creation of discrete kinetic energy through convective transport, and thereby eliminates the corresponding numerical instability. A large-eddy-simulation model for incompressible flow that follows the same line of thought is the symmetry-preserving regularization model [2]. Symmetry-preserving regularization filters the convective terms in order to stop the creation of smaller scales near the grid cut-off, but preserves the skew-symmetry of convection so that good numerical stability is preserved upon regularization.

This paper proposes a possible generalization of the symmetry-preserving discretization and regularization models to compressible flow. In the literature, some conservative discretizations for compressible flow have already been proposed [3, 4]. These discretizations start from the conservative form of the compressible Navier-Stokes equations, identify the mathematical equalities needed to demonstrate conservation of kinetic energy by convective transport, and preserve these equalities at the discrete level. This procedure suffices for the derivation of a highly stable symmetry-preserving discretization, but the mathematical framework is not concise enough to facilitate the derivation of symmetry-preserving regularizations for compressible flow. In this paper the many conservation properties of convective transport are expressed in terms of unusual square-root variables, inner products, and skew-symmetric differential operators. This yields a concise mathematical explanation of conservation by convective transport, which allows for straightforward derivation of symmetry-preserving discretizations and regularization models for compressible flow.

2 THE SKEW-SYMMETRIC NATURE OF CONVECTIVE TRANSPORT

The standard conservative notation of the compressible Navier-Stokes equations is

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \vec{u}) &= 0 \\ \partial_t \rho \vec{u} + \nabla \cdot (\rho \vec{u} \otimes \vec{u}) &= -\nabla p + \nabla \cdot \sigma \\ \partial_t \rho E + \nabla \cdot (\rho \vec{u} E) &= -\nabla \cdot (p \vec{u}) + \nabla \cdot (\sigma \cdot \vec{u}) - \nabla \cdot \vec{q} \end{aligned} \tag{1}$$

with ρ is the mass density, $\vec{u}$ the flow velocity vector, p the pressure, $E = \frac{1}{2}\vec{u} \cdot \vec{u} + e$ the total energy, e the internal energy, σ the stress tensor, and $\vec{q}$ the diffusive heat flux. The second terms at the left-hand-side of the above equations model how mass, momentum, and total energy are transported with the flow velocity. This *going with the flow* is called

convective transport. Because convective transport just moves things around, convective transport conserves many interesting physical quantities. The conservative form of the compressible Navier-Stokes equations expresses that convective transport conserves mass, momentum, and total energy, but through mathematical manipulations it can be shown that convective transport also conserves kinetic energy and internal energy separately.

The many conservation properties of convective transport can be expressed in a concise mathematical notation by rewriting the convective terms to square-root variables $\sqrt{\rho}$, $\sqrt{\rho}\vec{u}/\sqrt{2}$, and $\sqrt{\rho e}$. Because mass and total energy are conserved, the square-root variables are square-integrable and live in the mathematical Hilbert space $L^2(\Omega)$. Rewriting the left-hand-side of the compressible Navier-Stokes equations (1) to square-root variables gives

$$\begin{aligned} \partial_t \sqrt{\rho} + \frac{1}{2}\nabla\cdot(\vec{u}\sqrt{\rho}) + \frac{1}{2}\vec{u}\cdot\nabla\sqrt{\rho} &= \dots \\ \partial_t \frac{\sqrt{\rho}u_j}{\sqrt{2}} + \frac{1}{2}\nabla\cdot\left(\vec{u}\left(\frac{\sqrt{\rho}u_j}{\sqrt{2}}\right)\right) + \frac{1}{2}\vec{u}\cdot\nabla\left(\frac{\sqrt{\rho}u_j}{\sqrt{2}}\right) &= \dots \\ \partial_t \sqrt{\rho e} + \frac{1}{2}\nabla\cdot(\vec{u}\sqrt{\rho e}) + \frac{1}{2}\vec{u}\cdot\nabla\sqrt{\rho e} &= \dots \end{aligned} \tag{2}$$

where the dots denote non-convective terms. Thus in square-root variables, the convective terms attain the same form in each equation

$$\partial_t \phi + c(\vec{u})\phi = \dots \tag{3}$$

where

$$c(\vec{u})\phi = \frac{1}{2}\nabla\cdot(\vec{u}\phi) + \frac{1}{2}\vec{u}\cdot\nabla\phi \tag{4}$$

denotes the convective operator.

In square-root variables, the many conservation properties of convective transport can be explained from skew-symmetry of the convective operator. To see this, define the standard inner product on the Hilbert space $L^2(\Omega)$ by

$$\langle\phi,\psi\rangle = \int_\Omega \phi\psi \,\mathrm{d}x \tag{5}$$

with Ω a periodic domain. A differential operator K is skew-symmetric (anti self-adjoint) with respect to this inner product if $\langle K\phi,\psi\rangle + \langle\phi,K\psi\rangle = 0$. It can be shown by application of the product rule for differentiation that the convective operator satisfies

$$\phi c(\vec{u})\psi + \psi c(\vec{u})\phi = \nabla\cdot(\vec{u}\phi\psi)\ . \tag{6}$$

Integration of this equality over the periodic domain gives

$$\langle\phi, c(\vec{u})\psi\rangle + \langle c(\vec{u})\phi,\psi\rangle = \int_\Omega \nabla\cdot(\vec{u}\phi\psi)\ \mathrm{d}x = \int_{\partial\Omega} \phi\psi\vec{u}\cdot\vec{n}\,\mathrm{d}S = 0 \tag{7}$$

which vanishes by the periodicity of the domain Ω. Thus the convective operator $c(\vec{u})$ is skew-symmetric with respect the standard $L^2(\Omega)$ inner product (5). This skew-symmetry is directly related to conservation; the evolution equation for the product of the square-root variables ϕ and ψ is

$$\partial_t \phi\psi = \phi(\partial_t \psi) + (\partial_t \phi)\psi = -\phi c(\vec{u})\psi - \psi c(\vec{u})\phi = -\nabla \cdot (\vec{u}\phi\psi) \tag{8}$$

and upon integration of this equality over the periodic domain Ω

$$\partial_t \langle \phi, \psi \rangle = \langle \phi, \partial_t \psi \rangle + \langle \partial_t \phi, \psi \rangle = -\langle \phi, c(\vec{u})\psi \rangle - \langle c(\vec{u})\phi, \psi \rangle = 0 \tag{9}$$

which vanishes by the skew-symmetry of $c(\vec{u})$. Thus, products of square-root variables are conserved because the convective operator is skew-symmetric. Many interesting physical quantities are inner products of square-root variables, for example mass $\langle \sqrt{\rho}, \sqrt{\rho} \rangle$, momentum $\sqrt{2}\langle \sqrt{\rho}, \sqrt{\rho}u_j/\sqrt{2} \rangle$, kinetic energy $\langle \sqrt{\rho}u_j/\sqrt{2}, \sqrt{\rho}u_j/\sqrt{2} \rangle$, and internal energy $\langle \sqrt{\rho e}, \sqrt{\rho e} \rangle$. Thus, convective transport conserves mass, momentum, kinetic energy, internal energy, and total energy because the convective operator is skew-symmetric.

In the sequel this concise mathematical explanation of conservation by convective transport will be used to derive highly conservative discretizations of the convective terms on both collocated and staggered computational grids, and to generalize incompressible symmetry-preserving regularization models [2] to compressible flow.

3 SYMMETRY-PRESERVING DISCRETIZATION OF THE CONVECTIVE TERMS

A symmetry-preserving discretization of the convective terms preserves the skew-symmetric nature of convective transport at the discrete level. Because the skew-symmetry of the convective terms induces the many conservation properties of convective transport, a symmetry-preserving discretizations preserves all these conservation properties at the discrete level. There can be many motivations for preserving the skew-symmetry of convective transport at the discrete level. Our motivation for preserving symmetries is twofold. First, preserving symmetries eliminates the convective instability related to spurious transfer of kinetic energy to internal energy on stretched grids, and allows for simulations at reduced levels of artificial dissipation. Secondly, preserving symmetries ensures that the numerical dissipation of resolved kinetic energy by the convective terms does not overwhelm the modeled sub-grid-scale dissipation in a large-eddy simulation.

3.1 Collocated curvilinear computational grids

The skew-symmetry of convective transport can be transferred to a discretization on a collocated curvilinear grid by repeating the continuous argument from the previous section at the discrete level. Thus, the numerical solution is expressed in grid functions of the square-root variables $\sqrt{\rho}$, $\sqrt{\rho}\vec{u}/\sqrt{2}$, and $\sqrt{\rho e}$ and a discrete $L^2(\Omega)$ inner product on the curvilinear grid is defined as

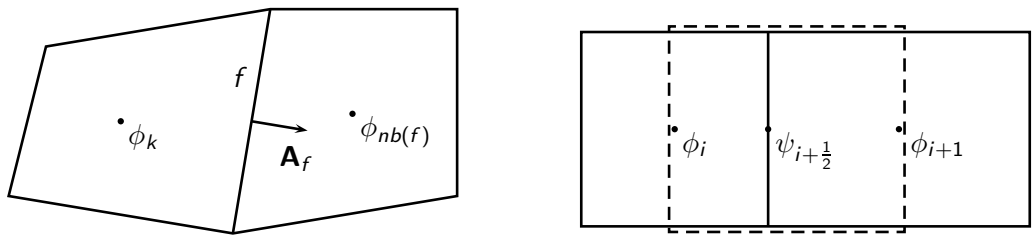

Figure 1: Two cells of a curvilinear collocated grid (left) and a rectilinear staggered grid (right) with the cell-center variable ϕ and the cell-face variable ψ. The dashed line shows a cell-face grid cell.

$$\langle \phi, \psi \rangle = \sum_i \Omega_i \phi_i \psi_i \tag{10}$$

where ϕ and ψ denote grid functions and Ω_i denotes the volume of grid cell i. A discretization of the convective operator $c(\vec{u})$ conserves inner products of square-root variables if it is skew-symmetric with respect to the discrete inner product. An example of a second-order accurate skew-symmetric discretization is

$$(c(\vec{u})\phi)_i = \frac{1}{\Omega_i} \sum_{f \in F_i} \frac{1}{2} \vec{u}_f \cdot \vec{A}_f \phi_{nb(f)} \tag{11}$$

where F_i is the set of faces of grid cell i, $\vec{u}_f$ denotes some second-order accurate interpolation of $\vec{u}$ to the face f, $\vec{A}_f$ denotes a discretization of the area vector at face f, and $nb(f)$ denotes the neighboring grid cell at face f (see Figure 1). Here, the interpolation law $\vec{u}_f = (\vec{u}_i + \vec{u}_{nb(f)})/2$ is used, because this leads to a natural implementation of the regularizations that will be proposed in the sequel. Other interpolations correspond to symmetry-preserving methods from the literature; the discretization by Kok [3] is obtained for the interpolation $\vec{u}_f = \frac{1}{2}((\rho\vec{u})_i + (\rho\vec{u})_{nb(f)})/\sqrt{\rho}_i\sqrt{\rho}_{nb(f)}$, and the discretization by Morinishi [4] is obtained for $\vec{u}_f = \frac{1}{4}(\rho_i + \rho_{nb(f)})(\vec{u}_i + \vec{u}_{nb(f)})/\sqrt{\rho}_i\sqrt{\rho}_{nb(f)}$. The proposed discretization (11) is skew-symmetric because

$$\langle c(\vec{u})\phi, \psi \rangle + \langle \phi, c(\vec{u})\psi \rangle = \sum_i \sum_{f \in F_i} \vec{u}_f \cdot \vec{A}_f \frac{1}{2} \left(\phi_{nb(f)}\psi_i + \phi_i \psi_{nb(f)} \right) = 0 \tag{12}$$

which vanishes because $\phi_{nb(f)}\psi_i + \phi_i\psi_{nb(f)}$ is a flux function, so that terms of the sum corresponding to the same face f cancel. A higher-order accurate skew-symmetric discretization of the convective term can be created through Richardson extrapolation. Here, Richardson extrapolation is applied as in [3] to create a fourth-order accurate dispersion-relation-preserving skew-symmetric discretization of the convective terms.

The skew-symmetry of a discretization is related to skew-symmetric matrices. This can be seen by writing the discrete inner product as

$$\langle \phi, \psi \rangle = \phi^T \Omega_c \psi \tag{13}$$

where Ω_c is a matrix with the grid cell volumes of the collocated grid on the diagonal. If the discretization of the convective operator $c(\vec{u})\phi$ is denoted $C\phi$ with C a discretization matrix, then skew-symmetry of the discrete convective transport requires

$$\langle C\phi, \psi \rangle + \langle \phi, C\psi \rangle = \phi^T C^T \Omega_c \psi + \phi^T \Omega_c C \psi = \phi^T \left((\Omega_c C)^T + \Omega_c C \right) \psi = 0 \tag{14}$$

which is satisfied if $\Omega_c C$ is a skew-symmetric matrix. This suggests to discretize the convective operator as $C = \Omega_c^{-1} K$ with K a skew-symmetric matrix $K^T = -K$. Indeed, the second-order accurate skew-symmetric discretization (11) is such a discretization; on a one-dimensional domain with a cell-centered grid function $\phi = (\phi_0, \phi_1, \ldots)$ the discretization matrix corresponding to (11) is

$$C = \Omega_c^{-1} \begin{pmatrix} 0 & \frac{1}{2}u_{\frac{1}{2}} & & \\ -\frac{1}{2}u_{\frac{1}{2}} & 0 & \frac{1}{2}u_{\frac{3}{2}} & \\ & -\frac{1}{2}u_{\frac{3}{2}} & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} . \tag{15}$$

If the simple interpolation law $u_{i+\frac{1}{2}} = \frac{1}{2}(u_i + u_{i+1})$ is used, U denotes the matrix with the cell-centered velocities on the diagonal, and D denotes the simple skew-symmetric differencing matrix

$$D = \begin{pmatrix} 0 & \frac{1}{2} & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & \\ & -\frac{1}{2} & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix} , \tag{16}$$

then the discretization matrix (15) can be written as $C = \Omega_c^{-1}(\frac{1}{2}DU + \frac{1}{2}UD)$, which is a direct matrix-vector translation of the continuous convective operator (4).

3.2 Staggered rectilinear computational grids

A discretization on staggered rectilinear grids that preserves the conservation of mass, momentum, kinetic energy, internal energy and total energy by convective transport has been proposed by Morinishi [4]. This section demonstrates that this conservative staggered discretization is a special case of a more general class of staggered symmetry-preserving discretizations.

Assume a staggered rectilinear grid (see Figure 1). The square-root variables $\sqrt{\rho}$ and $\sqrt{\rho e}$ are located at cell centers, and the components of $\sqrt{\rho}\vec{u}/\sqrt{2}$ are located at cell faces.

Assume that a consistent skew-symmetric discretization has been defined on the cell-centered grid

$$\langle \phi_c, \psi_c \rangle_c = \phi_c^T \Omega_c \psi_c \quad , \quad \partial_t \phi_c + C_c \phi_c = \dots \quad , \quad (\Omega_c C_c)^T = -\Omega_c C_c \,. \tag{17}$$

The formulation of a staggered symmetry-preserving discretization should define an inner product $\langle \phi_s, \psi_s \rangle_s$ on the cell-face grid, prescribe how mixed cell-center and cell-face inner products $\langle \phi_c, \psi_s \rangle$ are discretized, and specify a cell-face discretization matrix C_s which is skew-symmetric with respect to both the cell-face inner product and the mixed cell-center cell-face inner product.

Here the inner product on the cell-face grid is defined independently of the inner product on the cell-center grid

$$\langle \phi_s, \psi_s \rangle_s = \phi_s^T \Omega_s \psi_s \quad , \quad \partial_t \phi_s + C_s \phi_s = \dots \quad , \quad (\Omega_s C_s)^T = -\Omega_s C_s \,. \tag{18}$$

where Ω_s is a matrix with the volumes of cell-face grid cells on the diagonal, and C_s is a yet to be determined skew-symmetric discretization. A discretization matrix C_s which is skew-symmetric with respect to the cell-face inner product can be derived straightforwardly with Hicken's shift-transformation procedure [5]. However, the discretization matrix C_s required here should also be skew-symmetric with respect to discrete mixed inner products $\langle \phi_c, \psi_s \rangle$ of cell-center variables and cell-face variables. Therefore, a shift-transformation procedure can only be applied under suitable cell-center to cell-face interpolations of the square-root variables.

Skew-symmetry is inherently related to conservation, and therefore a conservative cell-center to the cell-face interpolation is used

$$\tilde{\phi}_s = \sqrt{\sum_l \alpha_l \Omega_s^{-1} \Pi_l \Phi_c \Omega_c \phi_c} \tag{19}$$

where the square-root is a component-wise operator, the coefficients α_l satisfy $\sum_l \alpha_l = 1$, Φ_c is a matrix with the components of ϕ_c on the diagonal, and each Π_l is a cell-center to cell-face shift matrix with column and row sums of one, so $\Pi_l \mathbf{1} = \Pi_l^T \mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is a column vector with ones. This interpolation is conservative in the sense that

$$\tilde{\phi}_s^T \Omega_s \tilde{\phi}_s = \sum_l \alpha_l \mathbf{1}^T \Pi_l \Phi_c \Omega_c \phi_c = \sum_l \alpha_l \mathbf{1}^T \Phi_c \Omega_c \phi_c = \left(\sum_l \alpha_l \right) \phi_c^T \Omega_c \phi_c = \phi_c^T \Omega_c \phi_c \tag{20}$$

so that $\left\langle \tilde{\phi}_s, \tilde{\phi}_s \right\rangle_s = \langle \phi_c, \phi_c \rangle_c$. The evolution equation for the conservative interpolation $\tilde{\phi}_s$ can be derived straightforwardly from (17)

$$\begin{aligned}
\sum_l \alpha_l \Pi_l \Phi_c \Omega_c \left(\partial_t \phi_c + C_c \phi_c\right) &= \sum_l \alpha_l \Pi_l \Phi_c \Omega_c \partial_t \phi_c + \sum_l \alpha_l \Pi_l \Phi_c \Omega_c C_c \phi_c \\
&= \frac{1}{2}\partial_t \left(\sum_l \alpha_l \Pi_l \Phi_c \Omega_c \phi_c\right) + \sum_l \alpha_l \Pi_l \Phi_c \Omega_c C_c \phi_c \\
&= \frac{1}{2}\partial_t \tilde{\Phi}_s \Omega_s \tilde{\phi}_s + \sum_l \alpha_l \Pi_l \Phi_c \Omega_c C_c \phi_c \\
&= \tilde{\Phi}_s \Omega_s \partial_t \tilde{\phi}_s + \sum_l \alpha_l \Pi_l \Phi_c \Omega_c C_c \phi_c \,.
\end{aligned} \tag{21}$$

If ϕ_c and $\tilde{\phi}_s$ do not vanish, then this form of the induced cell-face evolution equation can be related to a skew-symmetric discretization matrix by rewriting the evolution equation to

$$\begin{aligned}
\partial_t \tilde{\phi}_s + \sum_l \alpha_l \Omega_s^{-1} \tilde{\Phi}_s^{-1} \Pi_l \Phi_c \Omega_c C_c \phi_c &= \partial_t \tilde{\phi}_s + \sum_l \alpha_l \Omega_s^{-1} \tilde{\Phi}_s^{-1} \Pi_l \Phi_c \Omega_c C_c \Phi_c \Pi_l^T \tilde{\Phi}_s^{-1} \tilde{\phi}_s \\
&= \partial_t \tilde{\phi}_s + \left(\sum_l \alpha_l \Omega_s^{-1} \tilde{\Phi}_s^{-1} \Pi_l \Phi_c \Omega_c C_c \Phi_c \Pi_l^T \tilde{\Phi}_s^{-1}\right) \tilde{\phi}_s \\
&= \partial_t \tilde{\phi}_s + C_s \tilde{\phi}_s
\end{aligned} \tag{22}$$

where it was used that $\phi_c = \Phi_c \mathbf{1} = \Phi_c \Pi_l^T \mathbf{1} = \Phi_c \Pi_l^T \tilde{\Phi}_s^{-1} \tilde{\phi}_s$ holds, and the cell-face discretization matrix is defined as $C_s = \sum_l \alpha_l \Omega_s^{-1} \tilde{\Phi}_s^{-1} \Pi_l \Phi_c \Omega_c C_c \Phi_c \Pi_l^T \tilde{\Phi}_s^{-1}$. It can be verified straightforwardly that the matrix $\Omega_s C_s$ is skew-symmetric because the matrix $\Omega_c C_c$ is skew-symmetric. Thus, if the interpolation of ϕ_c to cell faces is defined as in (19), then the cell-centered skew-symmetric operator induces a cell-face skew-symmetric operator.

As an example consider the one-dimensional discretization from the previous section with the cell-centered convective discretization C_c defined by (15). Assume that cell-face variables are denoted $\tilde{\phi}_f = (\tilde{\phi}_{\frac{1}{2}}, \tilde{\phi}_{\frac{3}{2}}, \ldots)$. A simple skew-symmetric cell-face discretization can be obtained by defining the cell-center to cell-face shift matrices

$$\Pi_1 = I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \tag{23}$$

and setting $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{2}$. Substitution of these shift matrices in (22) yields a skew-symmetric cell-face discretization matrix C_s with elements

$$\left(\Omega_s C_s\right)_{i,i+1} = -\left(\Omega_s C_s\right)_{i+1,i} = \frac{u_{i-\frac{1}{2}} \phi_{i-1} \phi_i + u_{i+\frac{1}{2}} \phi_i \phi_{i+1}}{4 \tilde{\phi}_{i-\frac{1}{2}} \tilde{\phi}_{i+\frac{1}{2}}} \,. \tag{24}$$

The corresponding discrete evolution of the mixed product of a cell-face variable ψ_s with the conservative interpolation $\tilde{\phi}_s$ in finite-difference notation is

$$\begin{aligned}\Omega_{i+\frac{1}{2}}\partial_t\tilde{\phi}_{i+\frac{1}{2}}\psi_{i+\frac{1}{2}} \quad &+ \quad \frac{1}{2}\left(u_{i+\frac{1}{2}}\phi_i\phi_{i+1} + u_{i+\frac{3}{2}}\phi_{i+1}\phi_{i+2}\right)\frac{1}{2}\left(\frac{\psi_{i+\frac{1}{2}}}{\tilde{\phi}_{i+\frac{1}{2}}} + \frac{\psi_{i+\frac{3}{2}}}{\tilde{\phi}_{i+\frac{3}{2}}}\right) \\ &- \quad \frac{1}{2}\left(u_{i-\frac{1}{2}}\phi_{i-1}\phi_i + u_{i+\frac{1}{2}}\phi_i\phi_{i+1}\right)\frac{1}{2}\left(\frac{\psi_{i-\frac{1}{2}}}{\tilde{\phi}_{i-\frac{1}{2}}} + \frac{\psi_{i+\frac{1}{2}}}{\tilde{\phi}_{i+\frac{1}{2}}}\right) = \dots .\end{aligned} \tag{25}$$

A possible staggered symmetry-preserving discretization of the momentum equation is obtained by setting $\phi = \sqrt{\rho}$, $\psi = \sqrt{\rho}u/\sqrt{2}$, defining $u^*_{i+\frac{1}{2}} \equiv \sqrt{2}\psi_{i+\frac{1}{2}}/\tilde{\phi}_{i+\frac{1}{2}}$ and $\tilde{\rho}_{i+\frac{1}{2}} \equiv \widetilde{\sqrt{\rho}}^2_{i+\frac{1}{2}}$, and setting the interpolation $u_{i+\frac{1}{2}} = \frac{1}{2}(\phi_i^2 + \phi_{i+1}^2)u^*_{i+\frac{1}{2}}/\phi_i\phi_{i+1}$, which gives

$$\begin{aligned}\partial_t\Omega_{i+\frac{1}{2}}\tilde{\rho}_{i+\frac{1}{2}}u^*_{i+\frac{1}{2}} \quad &+ \quad \frac{1}{2}\left(\frac{1}{2}\left(\rho_i + \rho_{i+1}\right)u^*_{i+\frac{1}{2}} + \frac{1}{2}\left(\rho_{i+1} + \rho_{i+2}\right)u^*_{i+\frac{3}{2}}\right)\frac{1}{2}\left(u^*_{i+\frac{1}{2}} + u^*_{i+\frac{3}{2}}\right) \\ &- \quad \frac{1}{2}\left(\frac{1}{2}\left(\rho_{i-1} + \rho_i\right)u^*_{i-\frac{1}{2}} + \frac{1}{2}\left(\rho_i + \rho_{i+1}\right)u^*_{i+\frac{1}{2}}\right)\frac{1}{2}\left(u^*_{i-\frac{1}{2}} + u^*_{i+\frac{1}{2}}\right) = \dots\end{aligned} \tag{26}$$

where $\Omega_{i+\frac{1}{2}}\tilde{\rho}_{i+\frac{1}{2}} = \frac{1}{2}\Omega_i\rho_i + \frac{1}{2}\Omega_{i+1}\rho_{i+1}$. This is exactly the conservative divergence form proposed by Morinishi [4, equation 172]. Thus Morinishi's staggered discretization is a special case of the proposed staggered symmetry-preserving discretization (25); it falls within the proposed framework of staggered square-root variables, skew-symmetric matrices, conservative interpolations, and shift transformations.

4 SYMMETRY-PRESERVING REGULARIZATION

A large-eddy-simulation model that follows the same line of thought as the symmetry-preserving discretization is symmetry-preserving regularization of the convective terms. The non-linear convective terms drive the turbulent energy cascade, and are an important source of smaller flow structures in a turbulent flow. A regularization explicitly filters the convective terms in order to sooth the creation of smaller flow structures near the grid cut-off. A regularization is symmetry-preserving if the filtering is applied so that the skew-symmetric nature of convective transport is preserved. By construction, a symmetry-preserving regularization does not dissipate the kinetic energy of sub-grid scales, which makes it a gentle large-eddy-simulation model.

The proposed convective operator $c(\vec{u})$ for compressible flow is similar to the skew-symmetric form of the convective operator for incompressible flow, and therefore symmetry-preserving regularizations for incompressible flow [2] can be applied to the compressible convective operator directly. Here two simple symmetry-preserving regularizations are considered; Leray-α regularization and c_2 regularization. These regularizations can be obtained by replacing the convective operator $c(\vec{u})$ by one of its filtered counterparts

$$c_\alpha(\vec{u})\phi = c(\overline{\vec{u}})\phi \qquad \text{and} \qquad c_2(\vec{u})\phi = \overline{c(\overline{\vec{u}})\overline{\phi}} \tag{27}$$

where the bar is a filter with a residual $\phi' = \phi - \overline{\phi}$. The Leray-$\alpha$ regularization $c_\alpha(\vec{u})$ is skew-symmetric, and the c_2 regularization $c_2(\vec{u})$ is skew-symmetric if the filter is self-adjoint $\langle \overline{\phi}, \psi \rangle = \langle \phi, \overline{\psi} \rangle$ with respect to the $L^2(\Omega)$ inner product. In this paper a simple discrete self-adjoint filter with filter length $\Delta = r\Delta x$ is used

$$\overline{\phi}_i = \phi_i + \frac{1}{\Omega_i} \sum_{f \in F_i} \frac{r^2}{24} \vec{A}_f \cdot (\vec{x}_{nb(f)} - \vec{x}_i)(\phi_{nb(f)} - \phi_i) \,. \tag{28}$$

For more information on regularization models for compressible flow, see [6].

5 LARGE-EDDY SIMULATION OF TURBULENT CHANNEL FLOW

The fourth-order accurate symmetry-preserving discretization on the collocated grid is used to perform simulations of weakly compressible channel flow at bulk Mach number $\mathrm{M_b} = 0.2$. The channel flow is driven by a uniform body force which fixes the bulk Reynolds number $\mathrm{Re_b}$ at 2800 or 6875, corresponding to friction Reynolds numbers Re_τ of approximately 180 or 395 [7]. The used computational grids follow a hyperbolic sine distribution in the wall-normal direction [1]. The grids do not resolve the turbulent channel flow; the height of the smallest grid cell is $\Delta y^+_{\min} = 3.4$ for the channel flow at $\mathrm{Re}_\tau \approx 180$ and $\Delta y^+_{\min} = 2.6$ for the channel flow at $\mathrm{Re}_\tau \approx 395$ [6]. Time integration is performed with a low-storage four-stage explicit Runge-Kutta method. To confine the conservation errors due to time stepping, the Courant number is kept smaller than one.

The channel flow simulations are performed without model, with the proposed symmetry-preserving regularization models, and with the Vreman [8] and singular value [9] eddy-viscosity large-eddy-simulation models. Whereas general finite-volume methods typically need artificial dissipation to attain numerical stability, channel flow simulations with the fourth-order accurate symmetry-preserving discretization are stable without artificial dissipation. Table 1 lists the recorded friction Reynolds numbers Re_τ and Figure 2 shows the recorded normalized mean stream-wise velocity profiles.

Table 1: The friction Reynolds numbers recorded in the present channel flow simulations at $Re_\tau \approx 180$ and $Re_\tau \approx 395$, and the actual (DNS) friction Reynolds numbers [7].

no mod.	c_α $(r = \frac{1}{2})$	c_α $(r = 1)$	c_2 $(r = \frac{1}{2})$	c_2 $(r = 1)$	Vreman	sing. val.	DNS
182.3	180.7	168.8	182.0	175.2	150.0	161.3	178.1
413.6	406.8	386.0	411.5	413.3	373.4	382.9	392.3

At the lower friction Reynolds number of $\mathrm{Re}_\tau \approx 180$ the no-model simulation predicts a friction Reynolds number that slightly exceeds the actual value, and a normalized stream-wise velocity profile that almost coincides with the DNS [7]. The results obtained with the symmetry-preserving regularization models at $r = \frac{1}{2}$ $(\Delta = \Delta x/2)$ practically coincide

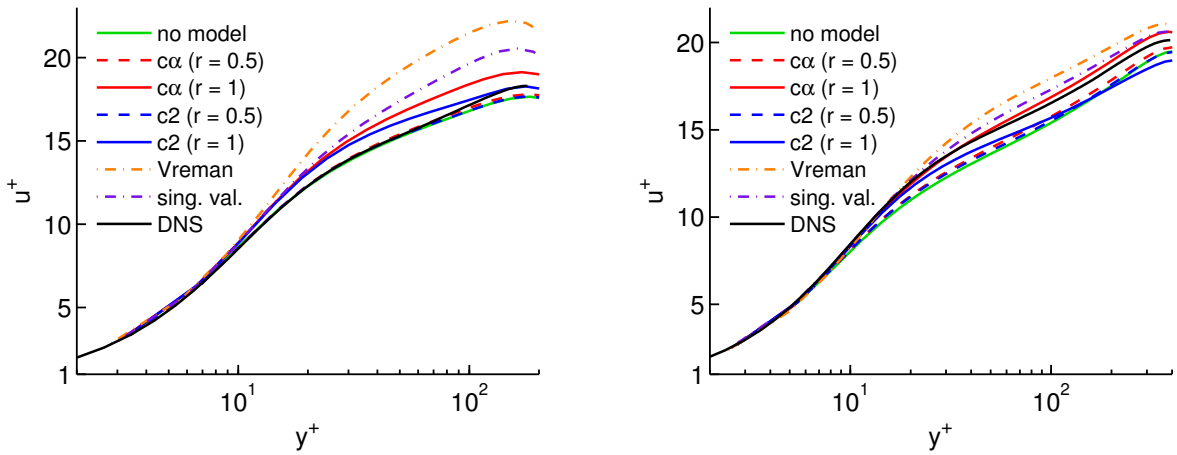

Figure 2: The normalized mean stream-wise flow velocity profiles recorded in channel flow simulations at $Re_\tau \approx 180$ (left) and $Re_\tau \approx 395$ (right).

with results of a no-model simulation. Simulations with the symmetry-preserving regularization models at $r = 1$ ($\Delta = \Delta x$) predict lower friction Reynolds numbers than the DNS. The c_2 regularization decreases the slope of the stream-wise velocity profile in the log layer compared to a no-model simulation. The considered advanced eddy-viscosity models are too dissipative; they under-predict the friction Reynolds number considerably, and produce less accurate results compared to the considered regularization models.

At the higher friction Reynolds number of $\mathrm{Re}_\tau \approx 395$, the no-model simulation over-predicts the friction Reynolds number considerably. Once again, results obtained with the symmetry-preserving regularizations at $r = \frac{1}{2}$ are practically identical to the no-model results. Results obtained with the Leray-α regularization at $r = 1$ accurately coincide with the DNS. Once again, the c_2 regularization at $r = 1$ decreases the slope of the stream-wise velocity profile in the log layer compared to a no-model simulation. The eddy-viscosity models again predict lower friction Reynolds numbers than regularization models, but at the current Reynolds number results obtained with the singular value eddy-viscosity model are almost as accurate as the Leray-α results.

At $\mathrm{Re}_\tau \approx 180$ the most accurate results are obtained without model and with the c_2 regularization. At $\mathrm{Re}_\tau \approx 395$ the most accurate results are obtained with the Leray-α regularization and with the singular value eddy-viscosity model. None of the considered models is perfectly accurate at both the Reynolds numbers. Possibly, a hybrid model that combines the dispersion of regularization models with the dissipation of eddy-viscosity models can produce accurate results at both Reynolds numbers.

6 CONCLUSIONS

By rewriting the compressible Navier-Stokes equations to square-root variables, it was shown that convective transport conserves many physical quantities because the con-

vective operator is skew-symmetric. Discretizations that preserve the skew-symmetry of convective transport were proposed on both collocated and staggered computational grids. Preserving symmetries of the convective terms improves the stability of a simulation method, and allows for simulations of turbulent compressible channel flow without artificial dissipation. For turbulent channel flow simulations at low Reynolds numbers, symmetry-preserving regularization models can produce more accurate results than advanced eddy-viscosity large-eddy-simulation models.

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