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TEMPERATURE INFLUENCE ON SMART STRUCTURES: A FIRST APPROACH

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Abstract. We study a linear coupled thermo-electromagnetoelastic problem and the nondimensionalization of the corresponding linear mixed parabolic-hyperbolic system. We deduce from the quasi-static equations a thin thermo-electromagnetoelastic plate model through the asymptotic expansions method.

1 Introduction

When an elastic structure Ω is subject to a system of external loads, it undergoes a *passive* deformation. In the case of the *smart structures*, the strain state is constantly under control with the help of sensors and actuators, made of piezoelectric and/or piezomagnetic materials, which are integrated within the structure. The goal of this work is to enrich the classical piezoelectric and piezomagnetic models, by adding the energy balance, in order to take into account the influence of the temperature, that in some cases cannot be neglected. For a description of the coupling between the physical quantities and of the multiphysical phenomenologies occurring in such structures, as well as of their applications, the reader can refer, e.g., to [1].

A distinctive feature of the problems encountered in applications is the presence of several parameters, which show the coexistence of different scales when performing a *nondimensionalization* procedure: for instance, the thickness of the piezoelectric layer may be small with respect to the other dimensions of the structure, the temperature influence may be relevant only on certain unknowns or on certain parts of the multistructure, etc. In most situations the superimposition of two wave propagation phenomena characterized by completely different velocities, as is the case with elastodynamic and electromagnetic waves, entails an unworkable numerical treatment of the problem. This issue can be addressed by resorting to a *quasi-static* model, which is justified by means of a nondimensionalization procedure. Such a procedure was performed in [2] without considering the temperature effects; an *a priori* quasi-static assumption on the electric field was made in [3] and [4]. In the general situation considered in Section 2 the state is defined by four unknowns (the displacement field, the electric field, the magnetic field and the temperature), whose corresponding evolution equations are fully coupled in a linear mixed parabolic-hyperbolic system. In Section 3 we carry out a formal nondimensionalization of the equations, so as (*i*) to extend the results by [2] and (*ii*) to justify the quasi-static assumption of [3] and [4]. In Section 4 we rapidly deduce by means of the asymptotic expansions method a model of a thermo-electromagnetoelastic plate that behaves simultaneously as a piezoelectic sensor and piezomagnetic actuator. Others situations (piezoelectric actuator and/or piezomagnetic sensor) can be treated in a similar way.

2 Evolution equations

Let $\Omega \subset \mathbb{R}^3$ denote an open bounded region of the usual three-dimensional euclidean space, occupied by a material body made up of thermo-electromagnetoelastic material in its reference configuration. We denote by $\mathbf{x}$ the typical point of Ω and by t the time. In order to study the evolution of our system, we resort to the point-wise balance equation for three-dimensional continua, to Maxwell's equations and to the linearized version of the energy balance equation:

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma} = \mathbf{f} & \mathbf{x} \in \Omega,\ t > 0, \\ \operatorname{div}\mathbf{D} = \rho_e & \mathbf{x} \in \Omega,\ t > 0, \\ \operatorname{div}\mathbf{B} = \mathbf{0} & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathbf{D}} - \nabla \times \mathbf{H} = -\mathbf{J} & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathbf{B}} + \nabla \times \mathbf{E} = \mathbf{0} & \mathbf{x} \in \Omega,\ t > 0, \\ \dot{\mathcal{S}} + \dfrac{1}{T_0}\operatorname{div}\mathbf{q}(\theta) = r & \mathbf{x} \in \Omega,\ t > 0. \end{cases} \tag{1}$$

where $\rho > 0$ is the mass density, ρ_e the electric charge density, $T_0 > 0$ the (constant) reference temperature, $\boldsymbol{\sigma}$ the Cauchy stress tensor, $\mathbf{D}$ the electric displacement, $\mathbf{B}$ the magnetic induction, $\mathcal{S}$ the entropy per unit volume, θ the temperature variation with respect to T_0, $\mathbf{q} = \mathbf{q}(\theta)$ the heat influx, $\mathbf{f}$ the body force, $\mathbf{J}$ an external current density, r an external entropy supply, $\mathbf{u}$ the displacement field, $\mathbf{E}$ the electric field and $\mathbf{H}$ the magnetic field. In the sequel, we shall assume $\rho_e \equiv 0$ and $\mathbf{J} \equiv \mathbf{0}$.

Let $\mathbf{e}(\mathbf{u}) := \operatorname{sym}\nabla\mathbf{u}$ be the strain tensor, and $\widehat{\mathcal{X}} := (\mathbf{u}, \mathbf{E}, \mathbf{H}, \theta)$ the list of state quantities. We assume the material under study to exhibit a linear behavior, and thus make the following constitutive assumptions:

$$
\begin{aligned}
&\boldsymbol{\sigma}(\widehat{\mathcal{X}}) = \mathbf{C}\mathbf{e}(\mathbf{u}) - \mathbf{P}\mathbf{E} - \mathbf{R}\mathbf{H} - \boldsymbol{\beta}\theta,\\
&\mathbf{D}(\widehat{\mathcal{X}}) = \mathbf{P}^T\mathbf{e}(\mathbf{u}) + \mathbf{X}\mathbf{E} + \boldsymbol{\alpha}\mathbf{H} + \mathbf{p}\theta,\\
&\mathbf{B}(\widehat{\mathcal{X}}) = \mathbf{R}^T\mathbf{e}(\mathbf{u}) + \boldsymbol{\alpha}\mathbf{E} + \mathbf{M}\mathbf{H} + \mathbf{m}\theta,\\
&\mathcal{S}(\widehat{\mathcal{X}}) = \boldsymbol{\beta} : \mathbf{e}(\mathbf{u}) + \mathbf{p}\cdot\mathbf{E} + \mathbf{m}\cdot\mathbf{H} + c_v\theta,\\
&\mathbf{q}(\theta) = -\mathbf{Q}\nabla\theta.
\end{aligned}
\tag{2}
$$

Here, $\mathbf{C}$, $\mathbf{P}$, $\boldsymbol{\beta}$, $\mathbf{X}$, $\mathbf{M}$, $\mathbf{Q}$ and $\mathbf{p}$ denote, respectively, the fourth-order elasticity tensor, the third-order piezoelectric tensor, the second-order thermal expansion tensor, the second-order electric permittivity tensor, the second-order magnetic permittivity tensor, the second-order thermal conductivity tensor and the pyroelectric vector; in addition, we have set $c_v := \lambda/T_0$ with $\lambda > 0$ the specific heat per unit volume. All the usual hypotheses (see, e.g., [4]) on these constitutive parameters remain unmodified here. We have introduced further coupling parameters, namely, $\mathbf{R}$, and $\mathbf{m}$ and $\boldsymbol{\alpha}$. We name $\mathbf{R}$ (a third-order tensor with the same properties as $\mathbf{P}$) the *piezomagnetic tensor*, $\mathbf{m}$ the *pyromagnetic vector* and $\boldsymbol{\alpha}$ the *magnetoelectric tensor*, which is assumed symmetric. By virtue of (2), system (1) takes the form:

$$
\begin{cases}
\rho\ddot{\mathbf{u}} - \operatorname{div}\mathbf{C}\mathbf{e}(\mathbf{u}) + \operatorname{div}\mathbf{P}\mathbf{E} + \operatorname{div}\mathbf{R}\mathbf{H} + \operatorname{div}\boldsymbol{\beta}\theta = \mathbf{f} & \mathbf{x}\in\Omega, \quad t>0,\\
\operatorname{div}\left(\mathbf{P}^T\mathbf{e}(\mathbf{u}) + \mathbf{X}\mathbf{E} + \boldsymbol{\alpha}\mathbf{H} + \mathbf{p}\theta\right) = 0 & \mathbf{x}\in\Omega, \quad t>0,\\
\operatorname{div}\left(\mathbf{R}^T\mathbf{e}(\mathbf{u}) + \boldsymbol{\alpha}\mathbf{E} + \mathbf{M}\mathbf{H} + \mathbf{m}\theta\right) = 0 & \mathbf{x}\in\Omega, \quad t>0,\\
\mathbf{X}\dot{\mathbf{E}} + \mathbf{P}^T\mathbf{e}(\dot{\mathbf{u}}) + \boldsymbol{\alpha}\dot{\mathbf{H}} + \mathbf{p}\dot{\theta} - \nabla\times\mathbf{H} = \mathbf{0} & \mathbf{x}\in\Omega, \quad t>0,\\
\mathbf{M}\dot{\mathbf{H}} + \mathbf{R}^T\mathbf{e}(\dot{\mathbf{u}}) + \boldsymbol{\alpha}\dot{\mathbf{E}} + \mathbf{m}\dot{\theta} + \nabla\times\mathbf{E} = \mathbf{0} & \mathbf{x}\in\Omega, \quad t>0,\\
c_v\dot{\theta} + \boldsymbol{\beta} : \mathbf{e}(\dot{\mathbf{u}}) + \mathbf{p}\cdot\dot{\mathbf{E}} + \mathbf{m}\cdot\dot{\mathbf{H}} - \dfrac{1}{T_0}\operatorname{div}\mathbf{Q}\nabla\theta = r & \mathbf{x}\in\Omega, \quad t>0.
\end{cases}
\tag{3}
$$

This system is equipped with the following initial conditions, for all $\mathbf{x}\in\Omega$,

$$
\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}),\ \mathbf{H}(\mathbf{x},0) = \mathbf{H}_0(\mathbf{x}),\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}),\ \dot{\mathbf{u}}(\mathbf{x},0) = \mathbf{u}_1(\mathbf{x}),\ \theta(\mathbf{x},0) = \theta_0(\mathbf{x}),
$$

and with suitable boundary conditions. In particular, let $t_0 > 0$ and let $\mathbf{n}$ be the outer unit normal vector field on $\partial\Omega$; in addition, for any $\mathbf{x}\in\partial\Omega$, let $\mathbb{T}(\mathbf{x}) := \mathbf{I} - \mathbf{n}(\mathbf{x})\otimes\mathbf{n}(\mathbf{x})$ be the projector on the tangent plane to $\partial\Omega$ in $\mathbf{x}$ (with $\mathbf{I}$ the identity tensor). We decompose the boundary as follows: $\partial\Omega = \partial\Omega_1\cup\partial\Omega_2$, with $\partial\Omega_1\cap\partial\Omega_2 = \emptyset$. We assign boundary values pertaining to the state quantities on $\partial\Omega_2\times(0,t_0)$ and boundary values pertaining to the derived fields on $\partial\Omega_1\times(0,t_0)$, namely:

$$
\begin{cases}
\boldsymbol{\sigma}(\widehat{\mathcal{X}})\mathbf{n} = \mathbf{g} & \text{on } \partial\Omega_1\times(0,t_0), \qquad \mathbf{u} = \overline{\mathbf{u}} & \text{on } \partial\Omega_2\times(0,t_0),\\
\mathbf{D}(\widehat{\mathcal{X}})\cdot\mathbf{n} = d & \text{on } \partial\Omega_1\times(0,t_0), \qquad \mathbb{T}\mathbf{E} = \overline{\mathbf{E}} & \text{on } \partial\Omega_2\times(0,t_0),\\
\mathbf{B}(\widehat{\mathcal{X}})\cdot\mathbf{n} = b & \text{on } \partial\Omega_1\times(0,t_0), \qquad \mathbb{T}\mathbf{H} = \overline{\mathbf{H}} & \text{on } \partial\Omega_2\times(0,t_0),\\
-\mathbf{q}(\theta)\cdot\mathbf{n} = \varrho & \text{on } \partial\Omega_1\times(0,t_0), \qquad \theta = \overline{\theta} & \text{on } \partial\Omega_2\times(0,t_0),
\end{cases}
\tag{4}
$$

with $\overline{\mathbf{E}}\cdot\mathbf{n}=\overline{\mathbf{H}}\cdot\mathbf{n}=0$. Existence and uniqueness of the solution of (3)-(4) can be proven working in the framework of the semigroup theory [5].

3 Nondimensionalization of the equations

With a view toward justifying the quasi-static hypothesis, i.e., $\mathbf{E}=-\nabla\varphi$ and $\mathbf{H}=-\nabla\zeta$, with φ and ζ the electric and magnetic potential respectively, we nondimensionalize system (3) disregarding equations $(3)_2$ and $(3)_3$, which do not involve time derivatives of the state quantities; we shall come back to these equations later on.

Let $\mathcal{L}:=\operatorname{diam}\Omega=\sup_{\mathbf{x},\mathbf{y}\in\Omega}|\mathbf{x}-\mathbf{y}|$ be the characteristic size of Ω, that we choose as a length scale. Proceeding along the lines of [2], we introduce

$$\begin{aligned}
&Q_+ := \sup_{x\in\Omega}||\mathbf{Q}(\mathbf{x})||_2,\quad \beta_+ := \sup_{x\in\Omega}||\boldsymbol{\beta}(\mathbf{x})||_2,\quad c_{v+} := \sup_{x\in\Omega} c_v(\mathbf{x}),\\
&p_+ := \sup_{x\in\Omega}|\mathbf{p}(\mathbf{x})|,\quad m_+ := \sup_{x\in\Omega}|\mathbf{m}(\mathbf{x})|,\quad \alpha_+ := \sup_{x\in\Omega}||\boldsymbol{\alpha}(\mathbf{x})||_2,\quad R_+ := \sup_{x\in\Omega}\left(||\mathbf{R}(\mathbf{x})^T\mathbf{R}(\mathbf{x})||_2\right)^{\frac{1}{2}}
\end{aligned}$$

and rewrite the constitutive parameters as:

$$\begin{aligned}
&\mathbf{Q}(\mathbf{x}) = Q_+\mathbf{Q}_r\left(\tfrac{\mathbf{x}}{\mathcal{L}}\right),\quad \boldsymbol{\beta}(\mathbf{x}) = \beta_+\boldsymbol{\beta}_r\left(\tfrac{\mathbf{x}}{\mathcal{L}}\right),\\
&c_v(\mathbf{x}) = c_{v+}c_{vr}\left(\tfrac{\mathbf{x}}{\mathcal{L}}\right),\quad \mathbf{p}(\mathbf{x}) = p_+\mathbf{p}_r\left(\tfrac{\mathbf{x}}{\mathcal{L}}\right),\\
&\mathbf{m}(\mathbf{x}) = m_+\mathbf{m}_r\left(\tfrac{\mathbf{x}}{\mathcal{L}}\right),\quad \boldsymbol{\alpha}(\mathbf{x}) = \alpha_+\boldsymbol{\alpha}_r\left(\tfrac{\mathbf{x}}{\mathcal{L}}\right),\quad \mathbf{R}(\mathbf{x}) = R_+\mathbf{R}_r\left(\tfrac{\mathbf{x}}{\mathcal{L}}\right).
\end{aligned}$$

A natural approach to nondimensionalize the temperature change θ is to choose, as unit of measurement, the reference temperature T_0:

$$\theta(\mathbf{x},t) = T_0\,\theta_r\left(\frac{\mathbf{x}}{\mathcal{L}},\frac{t}{\mathcal{T}}\right).$$

where $\mathcal{T}$ is the typical time for an elastic wave to travel along distance $\mathcal{L}$, i.e., such that $\mathcal{L}=V_+\mathcal{T}$, $V_+ := \sup_{x\in\Omega}\sup_{|\boldsymbol{\nu}|=1}\max_{1\le j\le 3}V_j(\mathbf{x},\boldsymbol{\nu})$, where $V_j(\mathbf{x},\boldsymbol{\nu})$ denotes the square root of the j-th eigenvalue of the acoustic tensor[1] associated with propagation direction $\boldsymbol{\nu}$, evaluated at x.

As to the other unknowns and constitutive parameters, we make the same choices as

[1] We recall that the acoustic tensor $\mathbf{A}_{\boldsymbol{\nu}}$ associated with unit vector $\boldsymbol{\nu}$ (the propagation direction), a tensor field over Ω, is defined by the following condition:

$$\mathbf{A}_{\boldsymbol{\nu}}\mathbf{a} := \rho^{-1}\mathbf{C}\left[\mathbf{a}\otimes\boldsymbol{\nu}\right]\boldsymbol{\nu},\quad \forall\mathbf{a}\in\mathbb{R}^3,$$

where ρ is the density and $\mathbf{C}$ the elasticity tensor.

in [2], namely,

$$\begin{array}{ll}
\mathbf{E}(\mathbf{x},t) = E^*\mathbf{E}_r\big(\frac{\mathbf{x}}{\mathcal{L}},\frac{t}{T}\big)\,, & \mathbf{H}(\mathbf{x},t) = H^*\mathbf{H}_r\big(\frac{\mathbf{x}}{\mathcal{L}},\frac{t}{T}\big)\,,\\
\rho(\mathbf{x}) = \rho_+\,\rho_r\big(\frac{\mathbf{x}}{\mathcal{L}}\big)\,, & \rho_+ := \sup_{x\in\Omega}\rho(\mathbf{x}),\\
\mathbf{u}(\mathbf{x},t) = \mathcal{L}\,\mathbf{u}_r\big(\frac{\mathbf{x}}{\mathcal{L}},\frac{t}{T}\big) & \mathbf{C}(\mathbf{x}) = \rho_+V_+^2\,\mathbf{C}_r\big(\frac{\mathbf{x}}{\mathcal{L}}\big)\,,\\
\mathbf{X}(\mathbf{x}) = \epsilon_0\,\mathbf{X}_r\big(\frac{\mathbf{x}}{\mathcal{L}}\big)\,, & \mathbf{M}(\mathbf{x}) = \mu_0\,\mathbf{M}_r\big(\frac{\mathbf{x}}{\mathcal{L}}\big)\,,\\
\mathbf{P}(\mathbf{x}) = P_+\,\mathbf{P}_r\big(\frac{\mathbf{x}}{\mathcal{L}}\big)\,, & P_+ := \sup_{x\in\Omega}\big(||\mathbf{P}(\mathbf{x})^T\mathbf{P}(\mathbf{x})||_2\big)^{\frac{1}{2}}\,,
\end{array}$$

where ϵ_0 and μ_0 are, respectively, the electric and magnetic permittivities of the vacuum. We recall that the speed of light is given by $c_0 = (\epsilon_0\mu_0)^{-\frac{1}{2}}$.

All in all, equations $(3)_1$ and $(3)_4$ to $(3)_6$ become (we still denote by $\mathbf{x}$ and t the scaled space and time variables):

$$\begin{cases}
\rho_r\ddot{\mathbf{u}}_r - \operatorname{div}\mathbf{C}_r\mathbf{e}(\mathbf{u}_r) + \left[\dfrac{P_+E^*}{\rho_+V_+^2}\right]\operatorname{div}\mathbf{P}_r\mathbf{E}_r + \left[\dfrac{R_+H^*}{\rho_+V_+^2}\right]\operatorname{div}\mathbf{R}_r\mathbf{H}_r + \left[\dfrac{\beta_+T_0}{\rho_+V_+^2}\right]\operatorname{div}\boldsymbol{\beta}_r\theta_r = \mathbf{f}_r\\
\mathbf{M}_r\dot{\mathbf{H}}_r + \left[\dfrac{R_+}{\mu_0H^*}\right]\mathbf{R}_r^T\mathbf{e}(\dot{\mathbf{u}}_r) + \left[\dfrac{\alpha_+E^*}{\mu_0H^*}\right]\boldsymbol{\alpha}_r\dot{\mathbf{E}}_r + \left[\dfrac{m_+T_0}{\mu_0H^*}\right]\mathbf{m}_r\dot{\theta}_r + \left[\dfrac{\sqrt{\epsilon_0}E^*}{\sqrt{\mu_0}H^*}\right]\left[\dfrac{c_0}{V_+}\right]\nabla\times\mathbf{E}_r = \mathbf{0}\\
\mathbf{X}_r\dot{\mathbf{E}}_r + \left[\dfrac{P_+}{\epsilon_0E^*}\right]\mathbf{P}_r^T\mathbf{e}(\dot{\mathbf{u}}_r) + \left[\dfrac{\alpha_+H^*}{\epsilon_0E^*}\right]\boldsymbol{\alpha}_r\dot{\mathbf{H}}_r + \left[\dfrac{p_+T_0}{\epsilon_0E^*}\right]\mathbf{p}_r\dot{\theta}_r - \left[\dfrac{\sqrt{\mu_0}H^*}{\sqrt{\epsilon_0}E^*}\right]\left[\dfrac{c_0}{V_+}\right]\nabla\times\mathbf{H}_r = \mathbf{0}\\
c_{vr}\dot{\theta}_r + \left[\dfrac{\beta_+}{\lambda_+}\right]\boldsymbol{\beta}_r:\mathbf{e}(\dot{\mathbf{u}}_r) + \left[\dfrac{p_+E^*}{\lambda_+}\right](\mathbf{p}_r\cdot\dot{\mathbf{E}}_r) + \left[\dfrac{m_+H^*}{\lambda_+}\right](\mathbf{m}_r\cdot\dot{\mathbf{H}}_r) - \left[\dfrac{Q_+}{\lambda_+V_+\mathcal{L}}\right]\operatorname{div}\mathbf{Q}_r\nabla\theta_r = r_r,
\end{cases} \tag{5}$$

where we have set $\lambda_+ := c_{v+}T_0$ in the last equation. All the equations hold for $\mathbf{x}\in\widehat{\Omega}$ and $t>0$, with $\widehat{\Omega} := \{\mathbf{x}/\mathcal{L} : \mathbf{x}\in\Omega\}$, and all the coefficients between square parentheses are dimensionless. As in [2], we now choose E^* and H^* such that

$$\sqrt{\epsilon_0}E^* = \sqrt{\mu_0}H^*, \quad \frac{P_+E^*}{\rho_+V_+^2} = \frac{P_+}{\epsilon_0E^*} \quad \text{and} \quad \frac{R_+H^*}{\rho_+V_+^2} = \frac{R_+}{\mu_0H^*}.$$

By setting $\delta := \frac{V_+}{c_0}$, we can rewrite equations $(5)_2$ and $(5)_3$ respectively as follows:

$$\begin{aligned}
\nabla\times\mathbf{E}_r &= -\delta\left(\mathbf{M}_r\dot{\mathbf{H}}_r + \kappa\,\mathbf{R}_r^T\mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+c_0\,\boldsymbol{\alpha}_r\dot{\mathbf{E}}_r + \upsilon\,\mathbf{m}_r\dot{\theta}_r\right),\\
\nabla\times\mathbf{H}_r &= \delta\left(\mathbf{X}_r\dot{\mathbf{E}}_r + \chi\,\mathbf{P}_r^T\mathbf{e}(\dot{\mathbf{u}}_r) + \alpha_+c_0\,\boldsymbol{\alpha}_r\dot{\mathbf{H}}_r + \varsigma\,\mathbf{p}_r\dot{\theta}_r\right),
\end{aligned} \tag{6}$$

with $\kappa := \frac{R_+}{V_+\sqrt{\mu_0\rho_+}}$, $\upsilon := \frac{m_+T_0}{V_+\sqrt{\mu_0\rho_+}}$, $\chi := \frac{P_+}{V_+\sqrt{\epsilon_0\rho_+}}$ and $\varsigma := \frac{p_+T_0}{V_+\sqrt{\epsilon_0\rho_+}}$.

By supposing $T_0 \simeq 293$ K as a typical reference temperature, for a common thermo-electromagnetoelastic material, we have [6] that $\delta \simeq 2\cdot 10^{-5}$, $\alpha_+c_0 \simeq 0.75$, $\kappa \simeq 0.78$, $\upsilon \simeq 3.4\cdot 10^{-3}$, $\chi \simeq 9$ and $\varsigma \simeq 0.03$. Therefore, if the time derivatives on the right-hand side of equations (6) remain bounded for any $t>0$, then, in the limit $\delta\to 0$, we obtain:

$$\nabla\times\mathbf{E}_r = \mathbf{0} \iff \mathbf{E}_r = -\nabla\varphi_r \quad\text{and}\quad \nabla\times\mathbf{H}_r = \mathbf{0} \iff \mathbf{H}_r = -\nabla\zeta_r,$$

i.e., the quasi-static hypothesis (the equivalence holds as we assume Ω simply connected). These statements can be proven rigorously by employing suitable mathematical tools, as in [2]. As a result, we can remove equations $(3)_3$ and $(3)_4$ from system (3), introduce the new list of state quantities $\mathcal{X} := (\mathbf{u}, \varphi, \zeta, \theta)$ and rewrite (2) and (3) as:

$$\begin{cases} \boldsymbol{\sigma}(\mathcal{X}) = \mathbf{C}\mathbf{e}(\mathbf{u}) + \mathbf{P}\nabla\varphi + \mathbf{R}\nabla\zeta + \boldsymbol{\beta}\theta, \\ \mathbf{D}(\mathcal{X}) = \mathbf{P}^T\mathbf{e}(\mathbf{u}) - \mathbf{X}\nabla\varphi - \boldsymbol{\alpha}\nabla\zeta + \mathbf{p}\theta, \\ \mathbf{B}(\mathcal{X}) = \mathbf{R}^T\mathbf{e}(\mathbf{u}) - \boldsymbol{\alpha}\nabla\varphi - \mathbf{M}\nabla\zeta + \mathbf{m}\theta, \\ \mathcal{S}(\mathcal{X}) = \boldsymbol{\beta} : \mathbf{e}(\mathbf{u}) - \mathbf{p}\cdot\nabla\varphi - \mathbf{m}\cdot\nabla\zeta + c_v\theta, \\ \mathbf{q}(\theta) = -\mathbf{Q}\nabla\theta. \end{cases} \tag{7}$$

and

$$\begin{cases} \rho\ddot{\mathbf{u}} - \operatorname{div}\boldsymbol{\sigma}(\mathcal{X}) = \mathbf{f} & \mathbf{x}\in\Omega, \quad t>0, \\ \operatorname{div}\mathbf{D}(\mathcal{X}) = 0 & \mathbf{x}\in\Omega, \quad t>0, \\ \operatorname{div}\mathbf{B}(\mathcal{X}) = 0 & \mathbf{x}\in\Omega, \quad t>0, \\ \mathcal{S}(\dot{\mathcal{X}}) + \dfrac{1}{T_0}\operatorname{div}\mathbf{q}(\theta) = r & \mathbf{x}\in\Omega, \quad t>0. \end{cases} \tag{8}$$

4 Thin thermo-electromagnetoelastic plates: an asymptotic modeling

4.1 Statement of the problem

We now identify Ω with a plate-like region of thickness $2\varepsilon h$. More precisely, let $\omega \subset \mathbb{R}^2$ be a smooth domain in the plane spanned by vectors $\mathbf{i}_\alpha$ $(\alpha = 1, 2)$, let γ_0 be a measurable subset of the boundary γ of the set ω such that length $\gamma_0 > 0$, and let $0 < \varepsilon < 1$ be an dimensionless *small* real parameter which will tend to zero. For each ε, we define

$$\begin{aligned} &\Omega^\varepsilon := \omega \times (-\varepsilon h, \varepsilon h), \\ &\Gamma^\varepsilon := \gamma \times (-\varepsilon h, \varepsilon h), \quad \Gamma^\varepsilon_\pm := \omega \times \{\pm\varepsilon h\}. \end{aligned}$$

Hence the boundary of the set Ω^ε is partitioned into the lateral face Γ^ε and the upper and lower faces Γ^ε_+ and Γ^ε_-, and the lateral face is itself partitioned as $\Gamma^\varepsilon = (\gamma_0 \times (-\varepsilon h, \varepsilon h)) \cup (\gamma_1 \times (-\varepsilon h, \varepsilon h))$, where $\gamma_1 := \gamma\backslash\gamma_0$. We set $\Gamma^\varepsilon_0 := \gamma_0 \times (-\varepsilon h, \varepsilon h)$, $\Gamma^\varepsilon_1 := \gamma_1 \times (-\varepsilon h, \varepsilon h)$ and $\widehat{\Gamma}^\varepsilon := \Gamma^\varepsilon_\pm \cup \Gamma^\varepsilon_1$. Let us remark that $\partial\Omega^\varepsilon = \widehat{\Gamma}^\varepsilon \cup \Gamma^\varepsilon_0 = \Gamma^\varepsilon \cup \Gamma^\varepsilon_\pm$.

We suppose that the thermal, mechanical and electromagnetic coefficients are all independent of ε. However, the thermo-electromagnetoelastic state depends on ε, i.e., we have $\mathcal{X}^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$. Hence the derived tensor and vector fields depend on ε. Furthermore we assume the source terms to depend on ε as well. Therefore the governing equations (8) become:

$$\left\{\begin{array}{ll} \rho\ddot{\mathbf{u}}^\varepsilon - \mathrm{div}^\varepsilon \boldsymbol{\sigma}^\varepsilon(\mathcal{X}^\varepsilon) = \mathbf{f}^\varepsilon & \mathbf{x}\in\Omega^\varepsilon,\ t>0 \\ \mathrm{div}^\varepsilon \mathbf{D}^\varepsilon(\mathcal{X}^\varepsilon) = 0 & \mathbf{x}\in\Omega^\varepsilon,\ t>0, \\ \mathrm{div}^\varepsilon \mathbf{B}^\varepsilon(\mathcal{X}^\varepsilon) = 0 & \mathbf{x}\in\Omega^\varepsilon,\ t>0, \\ \mathcal{S}^\varepsilon(\dot{\mathcal{X}}^\varepsilon) + \dfrac{1}{T_0}\mathrm{div}^\varepsilon \mathbf{q}^\varepsilon(\theta^\varepsilon) = r^\varepsilon & \mathbf{x}\in\Omega^\varepsilon,\ t>0. \end{array}\right. \tag{9}$$

In order to define a well-posed problem the system (9) must be completed *with suitable boundary and initial conditions* on the thermo-electromagnetoelastic state $\mathcal{X}^\varepsilon := (\mathbf{u}^\varepsilon, \varphi^\varepsilon, \zeta^\varepsilon, \theta^\varepsilon)$.

We assume at first that for $t>0$ the following *thermo-mechanical boundary conditions* are satisfied on $\partial\Omega^\varepsilon$:

$$\left\{\begin{array}{llll} \mathbf{u}^\varepsilon = \mathbf{0} & \text{on } \Gamma_0^\varepsilon, & \theta^\varepsilon = 0 & \text{on } \Gamma_0^\varepsilon, \\ \boldsymbol{\sigma}^\varepsilon(\mathcal{X}^\varepsilon)\mathbf{n}^\varepsilon = \mathbf{g}^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon, & -\mathbf{q}^\varepsilon(\theta^\varepsilon)\cdot\mathbf{n}^\varepsilon = \varrho^\varepsilon & \text{on } \widehat{\Gamma}^\varepsilon \end{array}\right. \tag{10}$$

As far as it concerns the *electromagnetic boundary conditions*, in the present paper, we consider *only* those which lead to the so-called *sensor-actuator problem* (see [7, 8]). In this case the plate behaves *simultaneously* as a piezoelectric sensor and a piezomagnetic actuator. More precisely we assume that for $t>0$ the following electromagnetic boundary conditions hold on $\partial\Omega^\varepsilon$:

$$\left\{\begin{array}{llll} \varphi^\varepsilon = 0 & \text{on } \Gamma^\varepsilon, & \zeta^\varepsilon = \zeta^{\pm,\varepsilon} & \text{on } \Gamma_\pm^\varepsilon, \\ \mathbf{D}^\varepsilon(\mathcal{X}^\varepsilon)\cdot\mathbf{n}^\varepsilon = d^\varepsilon & \text{on } \Gamma_\pm^\varepsilon, & \mathbf{B}^\varepsilon(\mathcal{X}^\varepsilon)\cdot\mathbf{n}^\varepsilon = 0 & \text{on } \Gamma^\varepsilon, \end{array}\right. \tag{11}$$

where $\mathbf{n}^\varepsilon = (n_i^\varepsilon)$ is the outward normal unit vector to the boundary $\partial\Omega^\varepsilon$.

The system (9), (10), (11) has to be completed with *the initial conditions for the displacement, the velocity and the temperature* at time $t=0$ on Ω^ε:

$$\left\{\begin{array}{ll} \mathbf{u}^\varepsilon(\mathbf{x}^\varepsilon,0) = \mathbf{u}^\varepsilon(0) = \mathbf{u}_0^\varepsilon & \text{in } \Omega^\varepsilon, \\ \dot{\mathbf{u}}^\varepsilon(\mathbf{x}^\varepsilon,0) = \dot{\mathbf{u}}^\varepsilon(0) = \mathbf{u}_1^\varepsilon & \text{in } \Omega^\varepsilon, \\ \theta^\varepsilon(\mathbf{x}^\varepsilon,0) = \theta^\varepsilon(0) = \theta_0^\varepsilon & \text{in } \Omega^\varepsilon. \end{array}\right. \tag{12}$$

where $(\mathbf{u}_0^\varepsilon, \mathbf{u}_1^\varepsilon, \theta_0^\varepsilon)$ are the given initial displacement, velocity and temperature.
In order to find a model of sensor-actuator thermo-electromagnetoelastic plate *we study the limit for* $\varepsilon\to 0$ *of* $(\mathcal{X}^\varepsilon, \Omega^\varepsilon)$.

Let $\Sigma^\varepsilon \subset \partial\Omega^\varepsilon$, we introduce the following functional spaces

$$\begin{array}{l} V(\Omega^\varepsilon,\Sigma^\varepsilon) := \left\{v^\varepsilon\in H^1(\Omega^\varepsilon);\quad v^\varepsilon = 0 \text{ on } \Sigma^\varepsilon\right\}, \\ \mathbf{V}(\Omega^\varepsilon,\Sigma^\varepsilon) := \left\{\mathbf{v}^\varepsilon = (v_i^\varepsilon)\in H^1(\Omega^\varepsilon;\mathbb{R}^3);\quad \mathbf{v}^\varepsilon = \mathbf{0} \text{ on } \Sigma^\varepsilon\right\}. \end{array}$$

We let $\bar{\zeta}^\varepsilon := \zeta^\varepsilon - \widehat{\zeta}^\varepsilon$, where $\widehat{\zeta}^\varepsilon$ is a trace lifting in $H^1(\Omega^\varepsilon)$ of the boundary potentials $\zeta^{\pm,\varepsilon}$ acting on $\Gamma_\pm^\varepsilon$. The variational formulation $\mathcal{P}^\varepsilon$ of the evolution problem (9), defined over the variable domain Ω^ε, takes the following form

$$\left\{\begin{array}{l}\text{Find } \mathcal{X}^\varepsilon \in \mathbf{V}(\Omega^\varepsilon,\Gamma_0^\varepsilon)\times V(\Omega^\varepsilon,\Gamma^\varepsilon)\times V(\Omega^\varepsilon,\Gamma_\pm^\varepsilon)\times V(\Omega^\varepsilon,\Gamma_0^\varepsilon),\ t\in(0,t_0)\ \text{ such that}\\ A^\varepsilon(\mathcal{X}^\varepsilon(t),\mathcal{Y}^\varepsilon)=L^\varepsilon(\mathcal{Y}^\varepsilon),\end{array}\right. \tag{13}$$

for all $\mathcal{Y}^\varepsilon=(\mathbf{v}^\varepsilon,\psi^\varepsilon,\xi^\varepsilon,\eta^\varepsilon)\in \mathbf{V}(\Omega^\varepsilon,\Gamma_0^\varepsilon)\times V(\Omega^\varepsilon,\Gamma^\varepsilon)\times V(\Omega^\varepsilon,\Gamma_\pm^\varepsilon)\times V(\Omega^\varepsilon,\Gamma_0^\varepsilon)$, with initial conditions $(\mathbf{u}_0^\varepsilon,\mathbf{u}_1^\varepsilon,\theta_0^\varepsilon)$ and

$$\begin{aligned}A^\varepsilon(\mathcal{X}^\varepsilon(t),\mathcal{Y}^\varepsilon) :=\ & \rho^\varepsilon(\ddot{\mathbf{u}}^\varepsilon,\mathbf{v}^\varepsilon)+c(\eta^\varepsilon,\dot{\mathbf{u}}^\varepsilon)+c_v^\varepsilon(\dot{\theta}^\varepsilon,\eta^\varepsilon)-d(\eta^\varepsilon,\dot{\varphi}^\varepsilon)-e(\eta^\varepsilon,\dot{\bar{\zeta}}^\varepsilon)+\\ & +a_u(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon)+b(\varphi^\varepsilon,\mathbf{v}^\varepsilon)-b(\psi^\varepsilon,\mathbf{u}^\varepsilon)+f(\bar{\zeta}^\varepsilon,\mathbf{v}^\varepsilon)-f(\xi^\varepsilon,\mathbf{u}^\varepsilon)-\\ & -c(\theta^\varepsilon,\mathbf{v}^\varepsilon)+a_\varphi(\varphi^\varepsilon,\psi^\varepsilon)+a_\zeta(\bar{\zeta}^\varepsilon,\xi^\varepsilon)+g(\bar{\zeta}^\varepsilon,\psi^\varepsilon)+g(\varphi^\varepsilon,\xi^\varepsilon)-\\ & -d(\theta^\varepsilon,\psi^\varepsilon)-e(\theta^\varepsilon,\xi^\varepsilon)+a_\theta(\theta^\varepsilon,\eta^\varepsilon),\\ L^\varepsilon(\mathcal{Y}^\varepsilon) :=\ & (\mathbf{f}^\varepsilon,\mathbf{v}^\varepsilon)+(\mathbf{g}^\varepsilon,\mathbf{v}^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon;\mathbb{R}^3)}+(r^\varepsilon,\eta^\varepsilon)+(\varrho^\varepsilon,\eta^\varepsilon)_{L^2(\widehat{\Gamma}^\varepsilon)}+(d^\varepsilon,\psi^\varepsilon)_{L^2(\Gamma_\pm^\varepsilon)}-\\ & -a_\zeta(\widehat{\zeta}^\varepsilon,\psi^\varepsilon)-f(\widehat{\zeta}^\varepsilon,\mathbf{v}^\varepsilon)+e(\eta^\varepsilon,\dot{\widehat{\zeta}}^\varepsilon)-g(\widehat{\zeta}^\varepsilon,\psi^\varepsilon),\end{aligned}$$

where $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega^\varepsilon)$ and the bilinear forms $a_u(\cdot,\cdot)$, $a_\varphi(\cdot,\cdot)$, $a_\zeta(\cdot,\cdot)$, $a_\theta(\cdot,\cdot)$, $b(\cdot,\cdot)$, $c(\cdot,\cdot)$, $d(\cdot,\cdot)$, $e(\cdot,\cdot)$, $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are defined as follows:

$$\begin{aligned}
a_u(\mathbf{u}^\varepsilon,\mathbf{v}^\varepsilon) &:= \int_{\Omega^\varepsilon}\mathbf{C}\mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon):\mathbf{e}^\varepsilon(\mathbf{v}^\varepsilon)d\mathbf{x}^\varepsilon, & a_\varphi(\varphi^\varepsilon,\psi^\varepsilon) &:= \int_{\Omega^\varepsilon}\mathbf{X}\nabla^\varepsilon\varphi^\varepsilon\cdot\nabla^\varepsilon\psi^\varepsilon d\mathbf{x}^\varepsilon,\\
a_\zeta(\zeta^\varepsilon,\xi^\varepsilon) &:= \int_{\Omega^\varepsilon}\mathbf{M}\nabla^\varepsilon\zeta^\varepsilon\cdot\nabla^\varepsilon\xi^\varepsilon d\mathbf{x}^\varepsilon, & a_\theta(\theta^\varepsilon,\eta^\varepsilon) &:= \int_{\Omega^\varepsilon}\frac{1}{T_0}\mathbf{Q}\nabla^\varepsilon\theta^\varepsilon\cdot\nabla^\varepsilon\eta^\varepsilon d\mathbf{x}^\varepsilon,\\
b(\psi^\varepsilon,\mathbf{u}^\varepsilon) &:= \int_{\Omega^\varepsilon}\mathbf{P}\nabla^\varepsilon\psi^\varepsilon:\mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon)d\mathbf{x}^\varepsilon, & c(\eta^\varepsilon,\mathbf{u}^\varepsilon) &:= \int_{\Omega^\varepsilon}\eta^\varepsilon\boldsymbol{\beta}:\mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon)d\mathbf{x}^\varepsilon,\\
d(\eta^\varepsilon,\varphi^\varepsilon) &:= \int_{\Omega^\varepsilon}\eta^\varepsilon\mathbf{p}\cdot\nabla^\varepsilon\varphi^\varepsilon d\mathbf{x}^\varepsilon, & e(\eta^\varepsilon,\zeta^\varepsilon) &:= \int_{\Omega^\varepsilon}\eta^\varepsilon\mathbf{m}\cdot\nabla^\varepsilon\zeta^\varepsilon d\mathbf{x}^\varepsilon,\\
f(\xi^\varepsilon,\mathbf{u}^\varepsilon) &:= \int_{\Omega^\varepsilon}\mathbf{R}\nabla^\varepsilon\xi^\varepsilon:\mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon)d\mathbf{x}^\varepsilon, & g(\zeta^\varepsilon,\psi^\varepsilon) &:= \int_{\Omega^\varepsilon}\boldsymbol{\alpha}\nabla^\varepsilon\zeta^\varepsilon\cdot\nabla^\varepsilon\psi^\varepsilon d\mathbf{x}^\varepsilon.
\end{aligned}$$

4.2 Asymptotic expansions and limit model

In order to perform an asymptotic analysis, we need to transform problem (13), posed on a variable domain Ω^ε, onto a problem posed on a fixed domain Ω (independent of ε). We apply the usual change of variables (see [9]), and, thus, we define $\Omega := \omega\times(-h,h)$, $\Gamma_0 := \gamma_0\times(-h,h)$, $\Gamma_1 := \gamma_1\times(-h,h)$, $\Gamma_\pm := \omega\times\{\pm h\}$, $\widehat{\Gamma} := \Gamma_\pm\cup\Gamma_1$.

With the unknown state $\mathcal{X}^\varepsilon := (\mathbf{u}^\varepsilon,\varphi^\varepsilon,\bar{\zeta}^\varepsilon,\theta^\varepsilon)\in \mathbf{V}(\Omega^\varepsilon,\Gamma_0^\varepsilon)\times V(\Omega^\varepsilon,\Gamma^\varepsilon)\times V(\Omega^\varepsilon,\Gamma_\pm^\varepsilon)\times V(\Omega^\varepsilon,\Gamma_0^\varepsilon)$, we associate the rescaled state $\mathcal{X}(\varepsilon) := (\mathbf{u}(\varepsilon),\varphi(\varepsilon),\bar{\zeta}(\varepsilon),\theta(\varepsilon))\in \mathbf{V}(\Omega,\Gamma_0)\times V(\Omega,\Gamma)\times V(\Omega,\Gamma_\pm)\times V(\Omega,\Gamma_0)$, defined by: $u_\alpha^\varepsilon(\mathbf{x}^\varepsilon,t) = u_\alpha(\varepsilon)(\mathbf{x},t)$, $u_3^\varepsilon(\mathbf{x}^\varepsilon,t) = \varepsilon^{-1}u_3(\varepsilon)(\mathbf{x},t)$, $\varphi^\varepsilon(\mathbf{x}^\varepsilon,t) = \varphi(\varepsilon)(\mathbf{x},t)$, $\bar{\zeta}^\varepsilon(\mathbf{x}^\varepsilon,t) = \varepsilon\bar{\zeta}(\varepsilon)(\mathbf{x},t)$ and $\theta^\varepsilon(\mathbf{x}^\varepsilon,t) = \theta(\varepsilon)(\mathbf{x},t)$, for all $\mathbf{x}^\varepsilon = \pi^\varepsilon\mathbf{x}\in\overline{\Omega}^\varepsilon$, $t\in(0,t_0)$, see [4, 10]. Same scalings have been used for the test functions.

We can now reformulate the problem on the fixed domain Ω. It follows that for every $\varepsilon > 0$ the rescaled state $\mathcal{X}(\varepsilon) := (\mathbf{u}(\varepsilon), \varphi(\varepsilon), \bar{\zeta}(\varepsilon), \theta(\varepsilon))$ is the unique solution of the following rescaled problem $\mathcal{P}(\varepsilon)$:

$$\left\{ \begin{array}{l} \text{Find } \mathcal{X}(\varepsilon) \in \mathbf{V}(\Omega, \Gamma_0) \times V(\Omega, \Gamma) \times V(\Omega, \Gamma_\pm) \times V(\Omega, \Gamma_0),\ t \in (0, t_0) \ \text{ such that} \\ A(\varepsilon)(\mathcal{X}(\varepsilon)(t), \mathcal{Y}) = L(\varepsilon)(\mathcal{Y}), \end{array} \right. \tag{14}$$

for all $\mathcal{Y} = (\mathbf{v}, \psi, \xi, \eta) \in V(\Omega, \Gamma_0) \times V(\Omega, \Gamma) \times V(\Omega, \Gamma_\pm) \times V(\Omega, \Gamma_0)$, with initial conditions $(\mathbf{u}_0, \mathbf{u}_1, \theta_0)$.

Since the rescaled problem (14) has a polynomial structure with respect to the small parameter ε, we can look for the solution $\mathcal{X}(\varepsilon) = (\mathbf{u}(\varepsilon), \varphi(\varepsilon), \bar{\zeta}(\varepsilon), \theta(\varepsilon))$ of the problem as a series of powers of ε: $\mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \varepsilon^2 \mathbf{u}^2 + \ldots$, $\varphi(\varepsilon) = \varphi^0 + \varepsilon\varphi^1 + \varepsilon^2\varphi^2 + \ldots$, $\bar{\zeta}(\varepsilon) = \bar{\zeta}^0 + \varepsilon\bar{\zeta}^1 + \varepsilon^2\bar{\zeta}^2 + \ldots$ and $\theta(\varepsilon) = \theta^0 + \varepsilon\theta^1 + \varepsilon^2\theta^2 + \ldots$. By substituting the asymptotic expansions above into the rescaled problem (14), and by identifying the terms with identical power of ε, we obtain, as customary, a set of variational problems, which has to be solved in order to characterize the limit state $\mathcal{X}^0 = (\mathbf{u}^0, \varphi^0, \bar{\zeta}^0, \theta^0)$ and its associated limit evolution problem.

In the sequel we denote with $\widetilde{\mathbf{f}} = (f_\alpha)$, (resp. $\widetilde{\mathbf{F}} = (F_{\alpha\beta})$), $\alpha, \beta = 1, 2$, the physical quantities, such as thermo-electromagnetoelastic coefficients and state variables, which are reduced to the middle plane of the plate ω. Besides, we use the following notations ∇_τ, div_τ and Δ_τ for the two-dimensional gradient, divergence and Laplacian operators, respectively.

By means of the asymptotic analysis we derive a precise characterization of the limit state quantities. Indeed, we obtain that the limit displacement field $\mathbf{u}^0$ satisfies the Kirchhoff-Love kinematical assumptions, so that

$$\widetilde{\mathbf{u}}^0(\tilde{\mathbf{x}}, x_3) = \mathbf{u}_H(\tilde{\mathbf{x}}) - x_3 \nabla_\tau w(\tilde{\mathbf{x}}) \quad \text{and} \quad u_3^0(\tilde{\mathbf{x}}, x_3) = w(\tilde{\mathbf{x}}).$$

Moreover, we obtain that the limit electric potential φ^0 and the limit variation of temperature θ^0 are both independent of x_3, i.e., $\varphi^0(\tilde{\mathbf{x}}, x_3) = \phi(\tilde{\mathbf{x}})$ and $\theta^0(\tilde{\mathbf{x}}, x_3) = \vartheta(\tilde{\mathbf{x}})$. Finally, the limit magnetic potential ζ^0 can be explicitly characterized as a second order polynomial function of x_3, depending on the transversal displacement w of the plate and on the values of the applied magnetic potentials at the upper and lower surfaces $\Gamma_\pm$,

$$\zeta^0(\tilde{\mathbf{x}}, x_3) = \sum_{k=0}^{2} z^k(\tilde{\mathbf{x}}) x_3^k, \tag{15}$$

where $z^0 = \frac{\zeta^+ + \zeta^-}{2} + \frac{h^2}{2} \widetilde{\mathbf{\Lambda}} : \nabla_\tau \nabla_\tau w$, $z^1 = \frac{\zeta^+ - \zeta^-}{2h}$ and $z^2 = -\frac{1}{2} \widetilde{\mathbf{\Lambda}} : \nabla_\tau \nabla_\tau w$, with $\widetilde{\mathbf{\Lambda}} := \frac{\widetilde{\mathbf{R}}_3}{\widetilde{M}_{33}}$.

First we recall that $\boldsymbol{\nu} = (\nu_\alpha)$ and $\boldsymbol{\tau} = (-\nu_2, \nu_1)$ represent, respectively, the unit normal vector and the unit tangent vector to $\partial\omega$. Let us consider the classical functional spaces, commonly used in the theory of linearly elastic plates, $\mathbf{V}_H(\omega, \gamma_0) := \{\mathbf{v}_H = (v_\alpha) \in H^1(\omega; \mathbb{R}^2);\ \mathbf{v}_H = \mathbf{0} \text{ on } \gamma_0\}$ and $V_3(\omega, \gamma_0) := \{v_3 \in H^2(\omega);\ v_3 = \partial_\nu v_3 = 0 \text{ on } \gamma_0\}$, the limit evolution problem decouples into two evolution subproblems, namely, the flexural problem and the two-dimensional thermo-piezoelectric evolution problem. The main results are claimed in Proposition 1 and 2.

Proposition 1 (The flexural problem)**.** *The flexural variational problem reads as follows*

$$\begin{cases} \textit{Find } w(t) \in V_3(\omega,\gamma_0),\ t\in(0,t_0) \textit{ such that} \\ \displaystyle\int_\omega \left\{ \widetilde{\mathbb{M}}(t) : \nabla_\tau\nabla_\tau v_3 + \frac{2h^3}{3}\rho\nabla_\tau\ddot{w}(t)\cdot\nabla_\tau v_3 + 2h\rho\ddot{w}(t)v_3 \right\} d\tilde{\mathbf{x}} = \\ \qquad = \displaystyle\int_\omega \left\{ \widetilde{s}_3 v_3 - \widetilde{\boldsymbol{\ell}}\cdot\nabla_\tau v_3 \right\} d\tilde{\mathbf{x}} + \int_{\gamma_1} \{ \widetilde{r}_3 v_3 - \widetilde{\boldsymbol{\imath}}\cdot\nabla_\tau v_3 \} \, d\gamma,\ \textit{for all } v_3 \in V_3(\omega,\gamma_0). \end{cases}$$

The strong formulation of the flexural problem takes the following form

$$\begin{cases} \mathrm{div}_\tau\mathrm{div}_\tau\widetilde{\mathbb{M}} - \frac{2h^3}{3}\rho\Delta_\tau\ddot{w} + 2h\rho\ddot{w} = \widetilde{f}_3 & \textit{in } \omega\times(0,t_0), \\ w(0) = w_0,\ \dot{w}(0) = w_1 & \textit{in } \omega, \\ \frac{2h^3}{3}\rho\nabla_\tau\ddot{w}\cdot\boldsymbol{\nu} - \mathrm{div}_\tau\widetilde{\mathbb{M}}\cdot\boldsymbol{\nu} - \nabla_\tau(\widetilde{\mathbb{M}}\boldsymbol{\nu}\cdot\boldsymbol{\tau})\cdot\boldsymbol{\tau} = \widetilde{g}_3 & \textit{on } \gamma_1\times(0,t_0), \\ \widetilde{\mathbb{M}}\boldsymbol{\nu}\cdot\boldsymbol{\nu} = 0 & \textit{on } \gamma_1\times(0,t_0), \\ w = \partial_\nu w = 0 & \textit{on } \gamma_0\times(0,t_0), \end{cases}$$

where $\widetilde{\mathbb{M}} := \frac{2h^3}{3}\widetilde{\mathbf{A}}\nabla_\tau\nabla_\tau w$ *represents the moment stress tensor of the plate,* $\widetilde{f}_3 := \widetilde{s}_3 + \mathrm{div}_\tau\widetilde{\boldsymbol{\ell}}$ *and* $\widetilde{g}_3 := \widetilde{r}_3 - \widetilde{\boldsymbol{\ell}}\cdot\boldsymbol{\nu} + \mathrm{div}_\tau\widetilde{\boldsymbol{\imath}}$ *are the reduced transversal loads.*

Proposition 2 (The two-dimensional thermo-piezoelectric evolution problem)**.** *The thermo-piezoelectric evolution problem reads as follows*

$$\begin{cases} \textit{Find } (\mathbf{u}_H(t),\phi(t),\vartheta(t)), \in \mathbf{V}_H(\omega,\gamma_0)\times V(\omega,\gamma_0)\times V(\omega,\gamma_0),\ t\in(0,t_0) \textit{ such that} \\ \displaystyle\int_\omega \left\{ \widetilde{\mathbb{N}}(t) : \widetilde{\mathbf{e}}(\mathbf{v}_H) + 2h\rho\ddot{\mathbf{u}}_H(t)\cdot\mathbf{v}_H - \widetilde{\mathbf{D}}(t)\cdot\nabla_\tau\psi + \dot{\widetilde{\mathcal{S}}}(t)\eta - \widetilde{\mathbf{q}}(t)\cdot\nabla_\tau\eta \right\} d\tilde{\mathbf{x}} = \\ \quad = \displaystyle\int_\omega \left\{ \widetilde{\mathbf{s}}\cdot\mathbf{v}_H - [\![\zeta]\!]\widetilde{\mathbf{R}}_3 : \widetilde{\mathbf{e}}(\mathbf{v}_H) + \widetilde{d}\psi - [\![\zeta]\!]\widetilde{\boldsymbol{\alpha}}_3\cdot\nabla_\tau\psi + (\widetilde{h} + \widetilde{m}_3[\![\dot{\zeta}]\!])\eta \right\} d\tilde{\mathbf{x}} + \\ \quad + \displaystyle\int_{\gamma_1} \{ \widetilde{\mathbf{r}}\cdot\mathbf{v}_H + \widetilde{\varrho}\eta \} \, d\gamma, \quad \textit{for all } (\mathbf{v}_H,\psi,\eta) \in \mathbf{V}_H(\omega,\gamma_0)\times V(\omega,\gamma_0)\times V(\omega,\gamma_0). \end{cases}$$

The strong formulation of the thermo-piezoelectric evolution problem takes the form

$$\begin{cases} 2h\rho\ddot{\mathbf{u}}_H - \mathrm{div}_\tau\widetilde{\mathbb{N}} = \widetilde{\mathbf{s}} + \widetilde{\mathbf{R}}_3[\![\nabla_\tau\zeta]\!] & \text{in } \omega\times(0,t_0),\\ \mathrm{div}_\tau\widetilde{\mathbf{D}} = \widetilde{d} + \widetilde{\boldsymbol{\alpha}}_3\cdot[\![\nabla_\tau\zeta]\!] & \text{in } \omega\times(0,t_0),\\ \dot{\widetilde{\mathcal{S}}} + \mathrm{div}_\tau\widetilde{\mathbf{q}} = \widetilde{h} + \widetilde{m}_3[\![\dot{\zeta}]\!] & \text{in } \omega\times(0,t_0),\\ \mathbf{u}_H(0) = \mathbf{u}_{H,0},\ \dot{\mathbf{u}}_H(0) = \mathbf{u}_{H,1},\ \vartheta(0) = \vartheta_0 & \text{in } \omega,\\ \widetilde{\mathbb{N}}\boldsymbol{\nu} = \widetilde{\mathbf{r}} - [\![\zeta]\!]\widetilde{\mathbf{R}}_3\boldsymbol{\nu} & \text{on } \gamma_1\times(0,t_0),\\ \widetilde{\mathbf{D}}\cdot\boldsymbol{\nu} = [\![\zeta]\!]\widetilde{\boldsymbol{\alpha}}_3\cdot\boldsymbol{\nu} & \text{on } \gamma_1\times(0,t_0),\\ -\widetilde{\mathbf{q}}\cdot\boldsymbol{\nu} = \widetilde{\varrho} & \text{on } \gamma_1\times(0,t_0),\\ \mathbf{u}_H = \mathbf{0},\quad \phi = \vartheta = 0 & \text{on } \gamma_0\times(0,t_0), \end{cases}$$

where

$$\begin{cases} \widetilde{\mathbb{N}} := 2h(\widetilde{\mathbf{C}}\ \widetilde{\mathbf{e}}(\mathbf{u}_H) + \widetilde{\mathbf{P}}\nabla_\tau\phi - \widetilde{\boldsymbol{\beta}}\vartheta),\\ \widetilde{\mathbf{D}} := 2h(\widetilde{\mathbf{P}}^T\ \widetilde{\mathbf{e}}(\mathbf{u}_H) - \widetilde{\mathbf{X}}\nabla_\tau\phi + \widetilde{\mathbf{p}}\vartheta),\\ \widetilde{\mathcal{S}} := 2h(\widetilde{\boldsymbol{\beta}} : \widetilde{\mathbf{e}}(\mathbf{u}_H) - \widetilde{\mathbf{p}}\cdot\nabla_\tau\phi + \widetilde{c}_v\vartheta),\\ \widetilde{\mathbf{q}} := -\frac{2h}{T_0}\widetilde{\mathbf{Q}}\nabla_\tau\vartheta, \end{cases}$$

represent, respectively, the membrane stress tensor, the reduced electric displacement field, the reduced entropy and the reduced heat flow. $[\![\zeta]\!] := \zeta^+ - \zeta^-$ *denotes the jump of the known magnetic potential between the upper and lower faces of the plate and* $\widetilde{\mathbf{s}}$, $\widetilde{d}$, $\widetilde{h}$, $\widetilde{\mathbf{r}}$ *and* $\widetilde{\varrho}$ *are the reduced in-plane thermo-electromechanical charges.*

The reduced thermo-electromagnetoelastic coefficients $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{C}}$, $\widetilde{\mathbf{P}}$, $\widetilde{\boldsymbol{\beta}}$, $\widetilde{\mathbf{X}}$, $\widetilde{\mathbf{p}}$, $\widetilde{c}_v$, $\widetilde{\mathbf{R}}_3$, $\widetilde{\boldsymbol{\alpha}}_3$ $\widetilde{M}_{33}$, $\widetilde{m}_3$ and $\widetilde{\mathbf{Q}}$ are listed in the Appendix.

Remark. It is worthwhile noticing that the flexural behavior of the plate is completely decoupled from its thermo-electromagnetic evolution counterpart. This is a classical result of piezoelectric and piezomagnetic asymptotic plate theories (see [4, 10]).

The three-dimensional thermo-electromagnetic evolution problem reduces to a two-dimensional thermo-piezoelectric evolution problem, defined over the middle plane of the plate. The magnetic behavior enters into the evolution equations as a new *magnetic loading*, depending on the jump of the applied magnetic potentials at the top and bottom of the plate. This result is due to the explicit characterization of the magnetic potential (15), typical of piezomagnetic actuators.

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5 Appendix

The components of the reduced thermo-electromagnetoelastic coefficients $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{C}}$, $\widetilde{\mathbf{P}}$, $\widetilde{\boldsymbol{\beta}}$, $\widetilde{\mathbf{X}}$, $\widetilde{\mathbf{p}}$, $\widetilde{c}_v$, $\widetilde{\mathbf{R}}_3$, $\widetilde{\boldsymbol{\alpha}}_3$ $\widetilde{M}_{33}$, $\widetilde{m}_3$ and $\widetilde{\mathbf{Q}}$ are defined as follows:

$$
\begin{aligned}
&\Delta := \epsilon_{ijk} C_{i313} C_{j323} C_{k333},\\
&\Delta_{1k} = -\tfrac{1}{\Delta}\epsilon_{kij} C_{i323} C_{j333}, \quad \Delta_{2k} = -\tfrac{1}{\Delta}\epsilon_{kij} C_{j313} C_{i333}, \quad \Delta_{3k} = -\tfrac{1}{\Delta}\epsilon_{kij} C_{i313} C_{j323},\\
&C'_{\alpha\beta\sigma\tau} := C_{\alpha\beta\sigma\tau} + 2\Delta_{ik} C_{\alpha\beta i3} C_{\sigma\tau k3}, \quad P'_{i\alpha\beta} := P_{i\alpha\beta} + 2\Delta_{pq} C_{\alpha\beta p3} P_{iq3}\\
&R'_{3\alpha\beta} := R_{3\alpha\beta} + 2\Delta_{ik} C_{\alpha\beta i3} R_{3k3}, \quad \beta'_{\alpha\beta} := \beta_{\alpha\beta} - 2\Delta_{ik} C_{\alpha\beta i3}\beta_{k3},\\
&X'_{ij} := X_{ij} - 2\Delta_{pq} P_{ip3} P_{jq3}, \quad M'_{33} := M_{33} - 2\Delta_{pq} R_{3p3} R_{3q3},\\
&\alpha'_{i3} := \alpha_{i3} - 2\Delta_{pq} P_{ip3} R_{3q3}, \quad p'_i := p_i + 2\Delta_{pq} P_{ip3}\beta_{q3},\\
&m'_3 := m_3 + 2\Delta_{ik} R_{3i3}\beta_{k3}, \quad c'_v := c_v - 2\Delta_{ik}\beta_{i3}\beta_{k3},\\
&\widetilde{Q}_{\alpha\beta} := Q_{\alpha\beta} - \tfrac{Q_{\alpha3}Q_{\beta3}}{Q_{33}}, \quad \widetilde{c}_v := c'_v - \tfrac{p'_3 p'_3}{X'_{33}}, \quad \widetilde{C}_{\alpha\beta\sigma\tau} := C'_{\alpha\beta\sigma\tau} + \tfrac{P'_{3\alpha\beta}P'_{3\sigma\tau}}{X'_{33}},\\
&\widetilde{P}_{\sigma\alpha\beta} := P'_{\sigma\alpha\beta} - \tfrac{P'_{3\alpha\beta}X'_{\sigma3}}{X'_{33}}, \quad \widetilde{R}_{3\alpha\beta} := R'_{3\alpha\beta} - \tfrac{P'_{3\alpha\beta}\alpha'_{33}}{X'_{33}}, \quad \widetilde{\beta}_{\alpha\beta} := \beta'_{\alpha\beta} - \tfrac{P'_{3\alpha\beta}p'_3}{X'_{33}},\\
&\widetilde{X}_{\alpha\beta} := X'_{\alpha\beta} - \tfrac{X'_{\alpha3}X'_{\beta3}}{X'_{33}}, \quad \widetilde{\alpha}_{\alpha3} := \alpha'_{\alpha3} - \tfrac{X'_{\alpha3}p'_3}{X'_{33}}, \quad \widetilde{M}_{33} := M'_{33} - \tfrac{\alpha'_{33}\alpha'_{33}}{X'_{33}},\\
&\widetilde{m}_3 := m'_3 - \tfrac{\alpha'_{33}p'_3}{X'_{33}}, \quad \widetilde{p}_\alpha := p'_\alpha - \tfrac{X'_{\alpha3}p'_3}{X'_{33}}, \quad \widetilde{A}_{\alpha\beta\sigma\tau} := \widetilde{C}_{\alpha\beta\sigma\tau} + \tfrac{\widetilde{R}_{3\alpha\beta}\widetilde{R}_{3\sigma\tau}}{\widetilde{M}_{33}}.
\end{aligned}
$$

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