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A MODAL ANALYSIS METHOD FOR STRUCTURAL MODELS WITH NON-MODAL DAMPING

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Abstract. A general method for the modal decomposition of the equations of motion of damped multi-degree-of-freedom-systems is presented. Two variants of the method are presented, both based on the corresponding eigenvalue problem of the damped structure with symmetric but non-modal damping matrix. The first variant operates with the complex right eigenvectors, normalized relative to the general mass matrix. The second presented variant includes the complex left and right eigenvectors, orthonormal relative to the general stiffness matrix. After initial partitioning of the equations of motion a real modal transformation matrix is built by a combination of two complex transformations, developed analytically be the aid of computer algebra software. For the general case of damped structures with non-diagonalisable symmetric damping matrix a modal analysis can be performed in real arithmetic. Modal damping as a special case is also considered. Two numerical examples with 3 and 10 DOF's demonstrate the accuracy and the advantages of the presented modal solution method.

1 INTRODUCTION

The modal decomposition of the equations of motion is a classical solution method for linear dynamical problems. In the case of structural models (MDOFS) without damping we have to do with fully real free frequencies and eigenvectors. The inclusion of damping in the equations of MDOFS yields a complex (quadratic) eigenvalue problem. The modal decomposition of the equations must be thus performed in complex space.

In this paper are presented two variants of a general method for modal analysis of damped MDOFS with non-modal (non-diagonalisable) symmetric damping matrix. The main goal is to avoid the computation in complex arithmetic. The method is based on a real modal transformation, derived from the complex eigenvalue solution of the MDOFS. The special case of modal damping matrix is also included. The computation of the complex (right resp. left) eigenvectors of the structural models in the presented academic examples (Sec. 2.2 and Sec.3.2) was done by the aid of computer algebra software.

The complex eigenvalue solution for high dimensional problems, which is the base for real applications of both presented procedures, is not the topic of this paper. There are available many literature references and solution strategies for large scaled problems, see [5]- [7].

2 MODAL DECOMPOSITION METHOD BASED ON THE COMPLEX RIGHT EIGENVECTORS

2.1 Theoretical derivation

Equation of motion of a damped SDOFS

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -\omega_0^2 \end{bmatrix}}_{\mathbf{m}} \underbrace{\begin{bmatrix} \dot{w}(t) \\ \dot{v}(t) \end{bmatrix}}_{\dot{\mathbf{q}}} + \underbrace{\begin{bmatrix} 2\eta\omega_0 & \omega_0^2 \\ \omega_0^2 & 0 \end{bmatrix}}_{\mathbf{k}} \underbrace{\begin{bmatrix} w(t) \\ v(t) \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} p(t) \\ 0 \end{bmatrix}}_{\mathbf{p}(t)} \tag{1}$$

where: $w = \dot{v}$, v: displacements, w: velocity = $\dot{v}$, ω_0 : free-vibration frequency and η: Lehr's damping ratio.

The exponential solution $\mathbf{q} = \mathbf{x}e^{\lambda t}$, $\dot{\mathbf{q}} = \lambda\mathbf{x}e^{\lambda t}$, introduced into the homogenous form of the differential equation (1), gives the quadratic eigenvalue problem

$$\left(\lambda \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -\omega_0^2 \end{bmatrix}}_{\mathbf{m}} + \underbrace{\begin{bmatrix} 2\eta\omega_0 & \omega_0^2 \\ \omega_0^2 & 0 \end{bmatrix}}_{\mathbf{k}} \right) \underbrace{\begin{bmatrix} \lambda \\ 1 \end{bmatrix}}_{\mathbf{x}} = \mathbf{0} \;\rightarrow\; (\lambda\mathbf{m} + \mathbf{k})\mathbf{x} = \mathbf{0} \tag{2}$$

where the two complex conjugate eigenvalues ($\eta << 1$) are

$$\lambda_{1/2} = \underbrace{-\eta\omega_0}_{\lambda_r} \pm i\underbrace{\omega_0\sqrt{1-\eta^2}}_{\lambda_i} = \lambda_r \pm i\lambda_i \;\rightarrow \omega_0 = \sqrt{(\lambda_r)^2 + (\lambda_i)^2}\,,\; \eta = -\frac{\lambda_r}{\omega_0} \tag{3}$$

The two complex conjugate eigenvectors $\varphi_{1/2}$, normalized relative to the mass matrix

$$\varphi_{\mathbf{k}} = \frac{\mathbf{x_k}}{\sqrt{\mathbf{x_k^T}\,\mathbf{m}\,\mathbf{x_k}}} = \frac{1}{\sqrt{-\omega_0^2 + (-\lambda_r \pm i\lambda_i)^2}} \begin{bmatrix} \lambda_r \pm i\lambda_i \\ 1 \end{bmatrix}, \qquad (k = 1,2) \tag{4}$$

are combined into a modal matrix $\varphi = [\varphi_1 \quad \varphi_2]$. The orthogonality relationships

$$\varphi^{\mathbf{T}}\mathbf{m}\,\varphi = \varphi^{\mathbf{T}} \begin{bmatrix} 1 & 0 \\ 0 & -\omega_0^2 \end{bmatrix} \varphi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\leftrightarrow\; \varphi^{-\mathbf{T}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \varphi^{-\mathbf{1}} = \begin{bmatrix} 1 & 0 \\ 0 & -\omega_0^2 \end{bmatrix} \tag{5}$$

$$\varphi^{\mathbf{T}}\,\mathbf{k}\,\varphi = \varphi^{\mathbf{T}} \begin{bmatrix} 2\eta\omega_0 & \omega_0^2 \\ \omega_0^2 & 0 \end{bmatrix} \varphi = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \leftrightarrow \varphi^{-\mathbf{T}} \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \varphi^{-\mathbf{1}} = \begin{bmatrix} 2\eta\omega_0 & \omega_0^2 \\ \omega_0^2 & 0 \end{bmatrix} \tag{6}$$

can be derived and by use of computer algebra software can be expressed analytically the inverse of the complex modal matrix $\varphi^{-1}(\omega_0,\eta)$, see [3],[4].

The equations of motion of damped MDOFS (n DOF) are

$$\underbrace{\begin{bmatrix}\mathbf{M} & \\ & -\mathbf{K}\end{bmatrix}}_{\mathbf{M_G}}\underbrace{\begin{bmatrix}\dot{\mathbf{W}}\\ \dot{\mathbf{V}}\end{bmatrix}}_{\dot{\mathbf{Q}}}+\underbrace{\begin{bmatrix}\mathbf{D} & \mathbf{K}\\ \mathbf{K} & \end{bmatrix}}_{\mathbf{K_G}}\underbrace{\begin{bmatrix}\mathbf{W}\\ \mathbf{V}\end{bmatrix}}_{\mathbf{Q}}=\underbrace{\begin{bmatrix}\mathbf{p}(t)\\ \end{bmatrix}}_{\mathbf{P}}\quad,\qquad \dot{\mathbf{V}}=\mathbf{W} \tag{7}$$

M, **D** and **K** are, respectively the (n x n) mass, damping and stiffness matrices, and **V**, $\dot{\mathbf{V}}$ are the (n x 1) displacement and velocity vectors. In structural mechanics problems the **M**, **D** and **K** matrices are considered to be real, symmetric and positive (semi-) definite.

The exponential solution $\mathbf{V}=\mathbf{X}e^{\lambda t}$, $\dot{\mathbf{V}}=\lambda\mathbf{X}e^{\lambda t}$ leads to the (2n x 1) eigenvalue problem

$$(\lambda\mathbf{M_G}+\mathbf{K_G})\begin{bmatrix}\lambda\mathbf{X}^T & \mathbf{X}^T\end{bmatrix}^T=\mathbf{0} \tag{8}$$

The complex conjugate eigenvector-pairs $\mathbf{X}^{(j)}$, $\overline{\mathbf{X}}^{(j)}$ $(j=1,\ldots,n)$ are normalized (index (j) omitted) - by analogy with Eq.(4) – relative to the general mass matrix $\mathbf{M_G}$:

$$\mathbf{\Phi}\left(\overline{\mathbf{\Phi}}\right)=\frac{\mathbf{X}\left(\overline{\mathbf{X}}\right)}{\sqrt{A\pm iB}}=\mathbf{\Phi_r}\pm i\mathbf{\Phi_i},\qquad A\pm iB=\begin{bmatrix}\lambda\mathbf{X}\left(\overline{\mathbf{X}}\right)\\ \mathbf{X}\left(\overline{\mathbf{X}}\right)\end{bmatrix}^T\cdot\mathbf{M_G}\cdot\begin{bmatrix}\lambda\mathbf{X}\left(\overline{\mathbf{X}}\right)\\ \mathbf{X}\left(\overline{\mathbf{X}}\right)\end{bmatrix} \tag{9}$$

The (2n x 2n) complex square modal matrix, denoted by $\mathbf{\Phi_G}$, is made up of the n eigenvector-pairs containing $\mathbf{\Phi}$ resp. $\overline{\mathbf{\Phi}}$, see Eq.(10). The orthogonality relationships can be expressed in terms of the j-th eigenvector-pair – analogous to Eqs.(5), (6):

$$\begin{bmatrix}\lambda\mathbf{\Phi} & \overline{\lambda}\overline{\mathbf{\Phi}}\\ \mathbf{\Phi} & \overline{\mathbf{\Phi}}\end{bmatrix}^T\mathbf{M_G}\begin{bmatrix}\lambda\mathbf{\Phi} & \overline{\lambda}\overline{\mathbf{\Phi}}\\ \mathbf{\Phi} & \overline{\mathbf{\Phi}}\end{bmatrix}=\begin{bmatrix}1 & \\ & 1\end{bmatrix},\ \begin{bmatrix}\lambda\mathbf{\Phi} & \overline{\lambda}\overline{\mathbf{\Phi}}\\ \mathbf{\Phi} & \overline{\mathbf{\Phi}}\end{bmatrix}^T\mathbf{K_G}\begin{bmatrix}\lambda\mathbf{\Phi} & \overline{\lambda}\overline{\mathbf{\Phi}}\\ \mathbf{\Phi} & \overline{\mathbf{\Phi}}\end{bmatrix}=\begin{bmatrix}-\lambda & \\ & -\overline{\lambda}\end{bmatrix} \tag{10}$$

By the aid of Eqs. (10) can be performed a modal decomposition of the equations of motion (7) :

$$\underbrace{\mathbf{\Phi}_G^T\begin{bmatrix}\mathbf{M} & \\ & -\mathbf{K}\end{bmatrix}\mathbf{\Phi_G}}_{\mathbf{E}}\dot{\mathbf{A}}\quad+\quad\underbrace{\mathbf{\Phi}_G^T\begin{bmatrix}\mathbf{D} & \mathbf{K}\\ \mathbf{K} & \end{bmatrix}\mathbf{\Phi_G}}_{diag\{-\lambda^{(j)}\}}\mathbf{A}\quad=\quad\mathbf{\Phi}_G^T\begin{bmatrix}\mathbf{P}(t)\\ \end{bmatrix} \tag{11}$$

where

$$\begin{bmatrix}\mathbf{W}\\ \mathbf{V}\end{bmatrix}=\mathbf{\Phi_G}\cdot\mathbf{A}=\mathbf{\Phi_G}\cdot\begin{bmatrix}a^{(1)}\ b^{(1)} & \cdots & a^{(n)}\ b^{(n)}\end{bmatrix}^T \tag{12}$$

Introducing the real modal coordinates $x^{(j)}$, $y^{(j)}$ through $\begin{bmatrix}a^{(j)} & b^{(j)}\end{bmatrix}^T=\left(\varphi^{(j)}\right)^{-1}\cdot\begin{bmatrix}x^{(j)} & y^{(j)}\end{bmatrix}^T$, the differential equations (11) can be transformed **in pairs** into the real form of SDOFS-equation (index (j) omitted), in regardance of Eqs.(5), (6):

$$\underbrace{\varphi^{-\mathbf{T}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\varphi^{-1}}_{\begin{bmatrix}1 & 0\\ 0 & -(\omega_0)^2\end{bmatrix}}\cdot\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} + \underbrace{\varphi^{-\mathbf{T}}\begin{bmatrix}-\lambda & 0\\ 0 & -\bar{\lambda}\end{bmatrix}\varphi^{-1}}_{\begin{bmatrix}2\eta\omega_0 & (\omega_0)^2\\ (\omega_0)^2 & \end{bmatrix}}\cdot\begin{bmatrix}x\\ y\end{bmatrix} = \underbrace{\varphi^{-\mathbf{T}}\begin{bmatrix}\lambda\,\mathbf{\Phi}^T\,\mathbf{P}(t)\\ \bar{\lambda}\,\bar{\mathbf{\Phi}}^T\,\mathbf{P}(t)\end{bmatrix}}_{\begin{bmatrix}g(t)\\ h(t)\end{bmatrix}} \tag{13}$$

Using the transformations (11) and (13), the equations of motion (7) will be uncoupled into n real SDOFS block equations as follows:

$$\mathbf{Y}^{\mathbf{T}}\cdot\begin{bmatrix}\mathbf{M} & \\ & -\mathbf{K}\end{bmatrix}\cdot\mathbf{Y}\cdot\dot{\mathbf{X}} + \mathbf{Y}^{\mathbf{T}}\cdot\begin{bmatrix}\mathbf{D} & \mathbf{K}\\ \mathbf{K} & \end{bmatrix}\cdot\mathbf{Y}\cdot\mathbf{X} = \mathbf{Y}^{\mathbf{T}}\cdot\begin{bmatrix}\mathbf{P}\\ \end{bmatrix} \tag{14}$$

In Eq.(14) a (2n x 2n) transformation basis $\mathbf{Y}$ have been introduced, defined through:

$$\begin{bmatrix}\mathbf{W}\\ \mathbf{V}\end{bmatrix} = \mathbf{\Phi}_{\mathbf{G}}\cdot\underbrace{\begin{bmatrix}(\varphi^{(1)})^{-1} & & \\ & \dots & \\ & & (\varphi^{(n)})^{-1}\end{bmatrix}}_{\mathbf{\Psi}^{-1}}\cdot\underbrace{\begin{bmatrix}x^{(1)}\\ y^{(1)}\\ \dots\\ x^{(n)}\\ y^{(n)}\end{bmatrix}}_{\mathbf{X}} = \underbrace{\mathbf{\Phi}_{\mathbf{G}}\cdot\mathbf{\Psi}^{-1}}_{\mathbf{Y}}\cdot\mathbf{X} = \mathbf{Y}\cdot\mathbf{X} \tag{15}$$

It can be shown that the $\mathbf{Y}$-matrix and all of the "load vectors" $[g(t) \quad h(t)]^T$ (Eq.(13)) are purely real. After component multiplication of the analytically expressed terms of $\mathbf{\Phi}_{\boldsymbol{G}}$ and of $\mathbf{\Psi}^{-\mathbf{1}}$ all imaginary parts cancel each other, see details in [4].

In each SDOFS block equation from type of Eq.(13) the modal coordinate x can be eliminated to obtain the usual form of the SDOFS equation of motion (index (j) omitted):

$$\ddot{y} + 2\eta\omega_0\,\dot{y} + \omega_0^2\,y = g(t) - \frac{2\eta}{\omega_0}h(t) - \frac{1}{\omega_0^2}\dot{h}(t) = f(t) \tag{16}$$

The dynamic response $y^{(j)}(t)$ can be obtained by step-by-step integration, applied to Eq.(16). The final time response of the original n DOFs is calculated by superposition of the modal coordinates $x^{(j)}$, $y^{(j)}$ in accordance to Eq. (15).

In [4] the same procedure is described in detail. The presented method has been successfully applied in [1]-[3] to analyse fluid-structure-foundation interaction problems. The eigenvalue calculation was based on the Arnoldi/Lanczos method, see [6], [7].

2.2 Numerical example

A numerical example given in Fig. 1 will demonstrate the method. All girders are assumed to be rigid with regard to axial and bending deformation, and the columns – rigid regarding to axial deformation only. The structural system has thus 3 DOF – u_1, u_2, u_3.

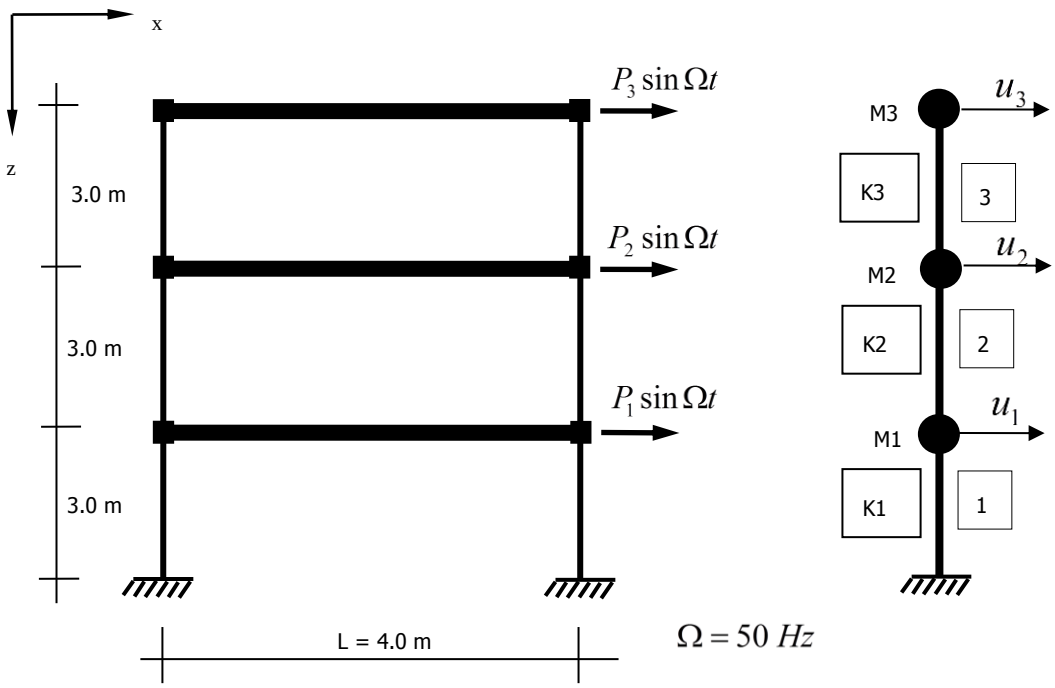

Figure 1: 3-DOF plane frame – geometry, stiffness and loading data

beam stiffness [kN/m] : K1= 8000, K2=5000, K3=3000
mass [t] : M1=1.0, M2=0.7, M3=0.45
loads [kN] : P1=100.0, P2=200.0, P3=300.0

The lumped mass, the stiffness and the non-modal damping matrix of the system are resp.

$$M = \begin{pmatrix} 1.0 & & \\ & 0.7 & \\ & & 0.45 \end{pmatrix} \quad K = \begin{pmatrix} 13000 & -5000 & \\ -5000 & 8000 & -3000 \\ & -3000 & 3000 \end{pmatrix} \quad D_{nmod} = \begin{pmatrix} 1.214 & -0.0479 & \\ -0.0479 & 1.573 & -0.3088 \\ & -0.3088 & 1.248 \end{pmatrix} \tag{17}$$

The non-modal damping matrix $\mathbf{D}_{n\mathrm{mod}}$ is computed from the Rayleigh-damped matrix $\mathbf{D} = c_M \mathbf{M} + c_K \mathbf{K}$ (with an assumed modal damping ratio $\eta = 0.005$) by arbitrary change of four elements.

The three complex conjugate eigenvalue pairs belonging to Eq. (8) are computed to

$$\boldsymbol{\lambda} = \begin{pmatrix} -0.94456 + 44.965\,i \\ -0.94456 - 44.965\,i \\ -1.1105 + 98.708\,i \\ -1.1105 - 98.708\,i \\ -1.0619 + 139.02\,i \\ -1.0619 - 139.02\,i \end{pmatrix} \tag{18}$$

The corresponding real transformation matrix $\mathbf{Y}$ according to Eq. (15) is computed to

$$\mathbf{Y} = \begin{pmatrix} -0.3362 & 0.07295 & 0.67532 & -0.3826 & 0.65651 & 1.1377 \\ -0.73821 & 0.05397 & 0.44007 & 1.1177 & -0.83072 & 0.71571 \\ -1.0599 & -0.10991 & -0.95282 & 0.20018 & 0.43717 & -1.6803 \\ -0.00004 & -0.33627 & 0.00004 & 0.67541 & -0.00006 & 0.65638 \\ -0.00003 & -0.73826 & -0.00011 & 0.43982 & -0.00004 & -0.8308 \\ 0.00005 & -1.0598 & -0.00002 & -0.95286 & 0.00009 & 0.43736 \end{pmatrix} \tag{19}$$

The "load vector" in Eq. (14) ($\sin \Omega t$ is excluded) yields now

$$\mathbf{Y}^{\mathbf{T}} \cdot \begin{bmatrix} \mathbf{P} \\ \\ \end{bmatrix} = \begin{pmatrix} -499.23423 \\ -14.884106 \\ -130.29817 \\ 245.33747 \\ 30.657476 \\ -247.17861 \end{pmatrix} \tag{20}$$

The solution of each SDOFS block equation from type of (13) by step-by-step integration in the time area 0-2.0 sec yields the time responses of the modal coordinates, see Fig 2a. After final modal superposition in accordance of Eq. (15) we obtain the time responses of the original three DOF – in Fig.2b is shown the $u_3(t)$-vibration.

The usual solution of this example by direct step-by-step integration (without modal decomposition) yields exactly the same time responses.

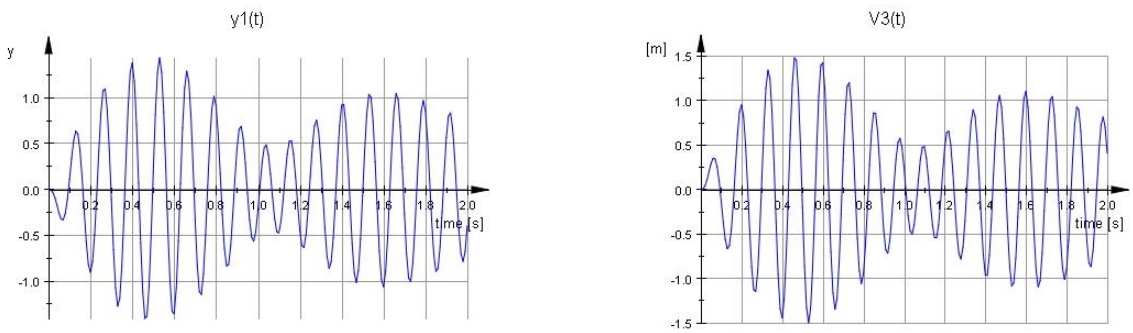

Figure 2: a) Modal coordinates vibration $y_1(t)$ b) system response vibration $u_3(t)$ [m]

The real transformation matrix $\mathbf{Y}$ in the case of modal damping matrix contains the mass-normalized eigenvectors of the undamped eigenvalue problem – Eq.(21)

$$\mathbf{Y} = \begin{pmatrix} -0.33626 & 0 & 0.67537 & 0 & 0.65636 & 0 \\ -0.73825 & 0 & 0.4398 & 0 & -0.83075 & 0 \\ -1.0598 & 0 & -0.95275 & 0 & 0.4374 & 0 \\ 0 & -0.33626 & 0 & 0.67537 & 0 & 0.65636 \\ 0 & -0.73825 & 0 & 0.4398 & 0 & -0.83075 \\ 0 & -1.0598 & 0 & -0.95275 & 0 & 0.4374 \end{pmatrix} \tag{21}$$

3 MODAL PROCEDURE BASED ON THE COMPLEX LEFT AND RIGHT EIGENVECTORS

3.1 Theoretical development of the method

First we express Eq. (2) as a special eigenvalue problem:

$$\left(\underbrace{\mathbf{m}^{-1}\mathbf{k}}_{\mathbf{a}}+\lambda\,\mathbf{e}\right)\mathbf{x}=\mathbf{0} \tag{22}$$

The eigensolution of (22) is given by the two eigenvalues $\lambda_{1/2}$ - Eq.(3), and by the eigenvectors $\mathbf{x_j}=\mathbf{r_j}\ (j=1,2)$ - they are **right eigenvectors** of the matrix $\mathbf{a}=\mathbf{m}^{-1}\mathbf{k}$.

With the substitution

$$\underbrace{\begin{bmatrix} f_w \\ f_v \end{bmatrix}}_{\mathbf{f}}=\underbrace{\begin{bmatrix} 2\eta\omega_0 & \omega_0^2 \\ \omega_0^2 & 0 \end{bmatrix}}_{\mathbf{k}}\underbrace{\begin{bmatrix} w \\ v \end{bmatrix}}_{\mathbf{q}} \quad\rightarrow\quad \mathbf{f}=\mathbf{k}\,\mathbf{q}\,,\qquad \dot{\mathbf{f}}=\mathbf{k}\,\dot{\mathbf{q}} \tag{23}$$

the initial system (1) becomes

$$\mathbf{k}\,\dot{\mathbf{q}}+\mathbf{k}\,(\mathbf{m}^{-1}\mathbf{k})\,\mathbf{q}=\mathbf{k}\,\mathbf{m}^{-1}\,\mathbf{p} \tag{24}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{e}}\underbrace{\begin{bmatrix} \dot{f}_w \\ \dot{f}_v \end{bmatrix}}_{\dot{\mathbf{f}}}+\underbrace{\begin{bmatrix} 2\eta\omega_0 & -1 \\ \omega_0^2 & 0 \end{bmatrix}}_{\mathbf{a}^{\mathbf{T}}=\mathbf{k}\,\mathbf{m}^{-1}}\underbrace{\begin{bmatrix} f_w \\ f_v \end{bmatrix}}_{\mathbf{f}}=\underbrace{\begin{bmatrix} 2\eta\omega_0 & -1 \\ \omega_0^2 & 0 \end{bmatrix}}_{\mathbf{k}\,\mathbf{m}^{-1}}\underbrace{\begin{bmatrix} p(t) \\ 0 \end{bmatrix}}_{\mathbf{p}(t)}$$

The corresponding eigenvalue problem $\left(\mathbf{a}^{\mathbf{T}}+\lambda\,\mathbf{e}\right)\mathbf{x}=\mathbf{0}$ has the same two eigenvalues $\lambda_{1/2}$ given by Eq.(3), but the complex conjugate eigenvectors are

$$\mathbf{x}_j=\mathbf{x}_r\pm i\,\mathbf{x}_i=\begin{bmatrix} \dfrac{\lambda_r\pm i\lambda_i}{(-\omega_0^2)} \\ 1 \end{bmatrix}=\begin{bmatrix} \dfrac{\eta\mp i\sqrt{1-\eta^2}}{\omega_0} \\ 1 \end{bmatrix}=\mathbf{l_j}\qquad (j=1,2) \tag{25}$$

In Eq.(25) $\mathbf{x}_j=\mathbf{l_j}$ represent the **left eigenvectors** of the matrix $\mathbf{a}=\mathbf{m}^{-1}\mathbf{k}$ (resp. the right eigenvectors of the matrix $\mathbf{a}^{\mathbf{T}}=\mathbf{k}\,\mathbf{m}^{-1}$) due to the formulation $\mathbf{x}^{\mathbf{T}}\left(\mathbf{a}+\lambda\,\mathbf{e}\right)=\mathbf{0}$. The eigenvectors in „right“ and „left“ formulation are related as follows, see detailed proof in [4]:

$$\left[\mathbf{l_1}\quad \mathbf{l_2}\right]^T\cdot\left[\mathbf{r_1}\quad \mathbf{r_2}\right]\;=\;\left[\mathbf{r_1}\quad \mathbf{r_2}\right]^T\cdot\mathbf{k}\cdot\left[\mathbf{r_1}\quad \mathbf{r_2}\right]\;=\;\begin{bmatrix} \chi_1 & \\ & \chi_2 \end{bmatrix} \tag{26}$$

The diagonal components χ_j (j=1,2) are used to norm the left eigenvectors

$$\varphi_{\mathbf{j}}^{\mathbf{L}}=\frac{1}{\sqrt{\chi_{jj}}}\mathbf{l_j}\qquad j=(1,2)\qquad\rightarrow\quad \varphi^{\mathbf{L}}=\left[\varphi_{\mathbf{1}}^{\mathbf{L}}\quad \varphi_{\mathbf{2}}^{\mathbf{L}}\right] \tag{27}$$

Using the $\varphi^{\mathbf{L}}$ modal matrix, see Eq.(27), and the spectral matrix $\boldsymbol{\lambda}$ we develop the relations

$$\left(\mathbf{a}^{\mathbf{T}}+\boldsymbol{\lambda}\mathbf{e}\right)\cdot\varphi^{\mathbf{L}}=\mathbf{0}\quad\rightarrow\ \mathbf{a}^{\mathbf{T}}=\varphi^{\mathbf{L}}\cdot\left(-\boldsymbol{\lambda}\right)\cdot\left(\varphi^{\mathbf{L}}\right)^{-1}\quad\leftrightarrow\quad-\boldsymbol{\lambda}=\left(\varphi^{\mathbf{L}}\right)^{-1}\cdot\mathbf{a}^{\mathbf{T}}\cdot\left(\varphi^{\mathbf{L}}\right)\tag{28}$$

The equations of motion of the MDOFS, Eq. (7), are now expressed in the form

$$\dot{\mathbf{Q}}+\underbrace{\mathbf{M}_{\mathbf{G}}^{-1}\cdot\mathbf{K}_{\mathbf{G}}}_{\mathbf{A}}\cdot\mathbf{Q}=\mathbf{M}_{\mathbf{G}}^{-1}\cdot\mathbf{P}\,,\tag{29}$$

The corresponding special eigenvalue problem $\left(\mathbf{A}+\lambda_j\,\mathbf{E}\right)\mathbf{X}_{\mathbf{j}}=\mathbf{0}$ yields the complex conjugate **right eigenvector pairs** $\mathbf{R}_{\mathbf{j}}=\mathbf{X}_{\mathbf{j}},\overline{\mathbf{R}}_{\mathbf{j}}=\overline{\mathbf{X}}_{\mathbf{j}}$, (j=1...n) of the matrix $\mathbf{A}$ .With introduction of the general forces $\mathbf{F}=\mathbf{K}_{\mathbf{G}}\,\mathbf{Q}$ Eq.(29) may be rewritten as:

$$\dot{\mathbf{F}}+\mathbf{K}_{\mathbf{G}}\,\mathbf{M}_{\mathbf{G}}^{-1}\,\mathbf{F}=\mathbf{K}_{\mathbf{G}}\,\mathbf{M}_{\mathbf{G}}^{-1}\,\mathbf{P}\quad\rightarrow\quad\dot{\mathbf{F}}+\mathbf{A}^{\mathbf{T}}\,\mathbf{F}=\mathbf{A}^{\mathbf{T}}\,\mathbf{P}\tag{30}$$

The corresponding special eigenvalue problem $\left(\mathbf{A}^{\mathbf{T}}+\lambda_j\,\mathbf{E}\right)\mathbf{X}_{\mathbf{j}}=\mathbf{0}$ leads to **left eigenvectors** $\mathbf{L}_{\mathbf{j}}=\mathbf{X}_{\mathbf{j}}\,,\overline{\mathbf{L}}_{\mathbf{j}}=\overline{\mathbf{X}}_{\mathbf{j}}$, (j=1...n) of the matrix $\mathbf{A}$ due to the relations $\mathbf{L}_{\mathbf{j}}^{\mathbf{T}}\left(\mathbf{A}+\lambda_j\mathbf{E}\right)=\mathbf{0}$. From the "right" and "left" formulation follows directly the relation $\mathbf{L}_{\mathbf{j}}=\mathbf{K}_{\mathbf{G}}\cdot\mathbf{R}_{\mathbf{j}}$.

Now we compose the right and the left modal matrices $\mathbf{R}$ resp. $\mathbf{L}$ as complete sets of the corresponding n eigenpairs. The orthogonality property $\mathbf{R}^{\mathbf{T}}\cdot\mathbf{L}=\mathbf{R}^{\mathbf{T}}\cdot\mathbf{K}_{\mathbf{G}}\cdot\mathbf{R}=diag\left\{\gamma_{jj}\right\}$ leads with the main diagonal components γ_{kk} to the normalized modal matrices $\boldsymbol{\Phi}^{\mathbf{R}}$, $\boldsymbol{\Phi}^{\mathbf{L}}$:

$$\boldsymbol{\Phi}_{\mathbf{k}}^{\mathbf{L}}=\frac{1}{\sqrt{\gamma_{kk}}}\mathbf{L}_{\mathbf{k}}\qquad\rightarrow\quad\boldsymbol{\Phi}^{\mathbf{L}}=\left[\boldsymbol{\Phi}_{\mathbf{1}}^{\mathbf{L}}\ \ \overline{\boldsymbol{\Phi}}_{\mathbf{1}}^{\mathbf{L}}\ \ \boldsymbol{\Phi}_{\mathbf{2}}^{\mathbf{L}}\ \ \overline{\boldsymbol{\Phi}}_{\mathbf{2}}^{\mathbf{L}}\ \ \ldots\ \ \boldsymbol{\Phi}_{\mathbf{n}}^{\mathbf{L}}\ \ \overline{\boldsymbol{\Phi}}_{\mathbf{n}}^{\mathbf{L}}\right],\tag{31}$$

$\boldsymbol{\Phi}^{\mathbf{R}}$ can be derived analogously to Eq.(31). A consequence of the normalizing relative to the general stiffness matrix $\mathbf{K}_{\mathbf{G}}$ are the following relations, where $\mathbf{E}$ is a (2n x 2n) identity matrix:

$$\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{\mathbf{T}}\cdot\boldsymbol{\Phi}^{\mathbf{R}}=\left(\boldsymbol{\Phi}^{\mathbf{R}}\right)^{\mathbf{T}}\cdot\boldsymbol{\Phi}^{\mathbf{L}}=\mathbf{E}\ \rightarrow\ \boldsymbol{\Phi}^{\mathbf{R}}=\left[\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{\mathbf{T}}\right]^{-1}=\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{-\mathbf{T}},\quad\left(\boldsymbol{\Phi}^{\mathbf{R}}\right)^{\mathbf{T}}=\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{-1}\tag{32}$$

From the new formulated eigenvalue problem $\mathbf{A}^{\mathbf{T}}\cdot\boldsymbol{\Phi}^{\mathbf{L}}+\boldsymbol{\Phi}^{\mathbf{L}}\cdot\boldsymbol{\Lambda}=\mathbf{0}$ we achieve the diagonalization of the $\mathbf{A}^{\mathbf{T}}$ matrix and the associated inverse relation

$$\mathbf{A}^{\mathbf{T}}=\boldsymbol{\Phi}^{\mathbf{L}}\cdot\left(-\boldsymbol{\Lambda}\right)\cdot\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{-1}\quad\leftrightarrow\quad-\boldsymbol{\Lambda}=\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{-1}\cdot\mathbf{A}^{\mathbf{T}}\cdot\boldsymbol{\Phi}^{\mathbf{L}}\tag{33}$$

where $\boldsymbol{\Lambda}=diag\left\{\lambda_j\right\}\quad\left(j=1,\ldots,2n\right)$ is the spectral matrix.

The modal decomposition of the system equations (30) can be done based on $\mathbf{F}=\boldsymbol{\Phi}^{\mathbf{L}}\cdot\left[a_1\quad b_1\quad\ldots\quad a_n\quad b_n\right]^T=\boldsymbol{\Phi}^{\mathbf{L}}\cdot\mathbf{B}^{\mathbf{L}}$ together with the orthogonality relationship (33):

$$\underbrace{\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{-1}\mathbf{E}\,\boldsymbol{\Phi}^{\mathbf{L}}}_{\mathbf{E}}\,\dot{\mathbf{B}}^{\mathbf{L}}\quad+\quad\underbrace{\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{-1}\overbrace{\left(\mathbf{K}_{\mathbf{G}}\,\mathbf{M}_{\mathbf{G}}^{-1}\right)}^{\mathbf{A}^{\mathbf{T}}}\boldsymbol{\Phi}^{\mathbf{L}}}_{-\boldsymbol{\Lambda}}\,\mathbf{B}^{\mathbf{L}}\quad=\quad\underbrace{\left(\boldsymbol{\Phi}^{\mathbf{L}}\right)^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{P}}_{\mathbf{P}^{\mathbf{L}}}\tag{34}$$

where $\mathbf{B}^{\mathbf{L}}=\left[a_1\quad b_1\quad\ldots\quad a_n\quad b_n\right]^T$ are new complex modal coordinates.

Each j-th pair of the n uncoupled equations (34) can now be transformed, premultiplying by $\varphi^{\mathbf{L}(\mathbf{j})}$ and respecting the orthogonality relationship (28), index (j) omitted:

$$\underbrace{\varphi^{\mathbf{L}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\left(\varphi^{\mathbf{L}}\right)^{-1}}_{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}} \cdot \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}^{L} + \underbrace{\varphi^{\mathbf{L}}\begin{bmatrix}-\lambda & 0\\ 0 & -\bar{\lambda}\end{bmatrix}\left(\varphi^{\mathbf{L}}\right)^{-1}}_{\left(\mathbf{a}^{T}\right)\ \begin{bmatrix}2\eta\,\omega_0 & -1\\ (\omega_0)^2 & \end{bmatrix}} \cdot \begin{bmatrix}x\\ y\end{bmatrix}^{L} = \underbrace{\varphi^{\mathbf{L}}\begin{bmatrix}p_a^L\\ p_b^L\end{bmatrix}}_{\begin{bmatrix}g_L\\ h_L\end{bmatrix}} \tag{35}$$

Eq. (35) is similar to Eq.(13) , introducing here the real modal coordinates $x^{(j)L}$, $y^{(j)L}$. Analogously to Eq.(15), a purely real transformation basis $\mathbf{Y_L}$ will be built by combination of two complex transformations

$$\mathbf{F} = \mathbf{\Phi^L} \cdot \underbrace{\begin{bmatrix}\left(\varphi^{\mathbf{L}(\mathbf{1})}\right)^{-1} & & \\ & \dots & \\ & & \left(\varphi^{\mathbf{L}(\mathbf{n})}\right)^{-1}\end{bmatrix}}_{(\mathbf{\Psi_L})^{-1}} \cdot \underbrace{\begin{bmatrix}x_1\\ y_1\\ \dots\\ x_n\\ y_n\end{bmatrix}^{L}}_{\mathbf{X_L}} = \underbrace{\mathbf{\Phi^L} \cdot \left(\mathbf{\Psi_L}\right)^{-1}}_{\mathbf{Y_L}} \cdot \mathbf{X_L} = \mathbf{Y_L} \cdot \mathbf{X_L} \tag{36}$$

In the product of the two complex matrices $\mathbf{\Phi^L} \cdot \left(\mathbf{\Psi_L}\right)^{-1}$ the imaginary parts cancel each other, the same applies to all "load" vectors $\left[g_L \quad h_L\right]^T$ in Eq.(35). Thus the transformed equations of motion (30) can be uncoupled by means of the transformation basis $\mathbf{Y_L}$ into n SDOFS block equations in real arithmetic from type of Eq.(35):

$$\left(\mathbf{Y_L}\right)^{-1} \cdot \mathbf{E} \cdot \mathbf{Y_L} \cdot \dot{\mathbf{X}}_{\mathbf{L}}(t) + \left(\mathbf{Y_L}\right)^{-1} \cdot \mathbf{A^T} \cdot \mathbf{Y_L} \cdot \mathbf{X_L}(t) = \left(\mathbf{Y_L}\right)^{-1} \cdot \mathbf{A^T} \cdot \mathbf{P}(t) \tag{37}$$

Each SDOFS block equation in (37) can be solved similar to Eq.(16) using a step-by-step integration operator (e.g. the Newmark method), thereby obtaining the time response of the modal coordinates $\mathbf{X_L}(t)$. The total response $\mathbf{V}(t)$, $\mathbf{W}(t)$ is then computed by two subsequent back transformations

$$\begin{bmatrix}\mathbf{W}(t)\\ \mathbf{V}(t)\end{bmatrix} = \mathbf{K_G^{-1}} \cdot \mathbf{F}(t) = \begin{bmatrix}\mathbf{D} & \mathbf{K}\\ \mathbf{K} & \end{bmatrix}^{-1} \cdot \mathbf{Y_L} \cdot \mathbf{X_L}(t) \tag{38}$$

3.2 Numerical example with 10 DOF

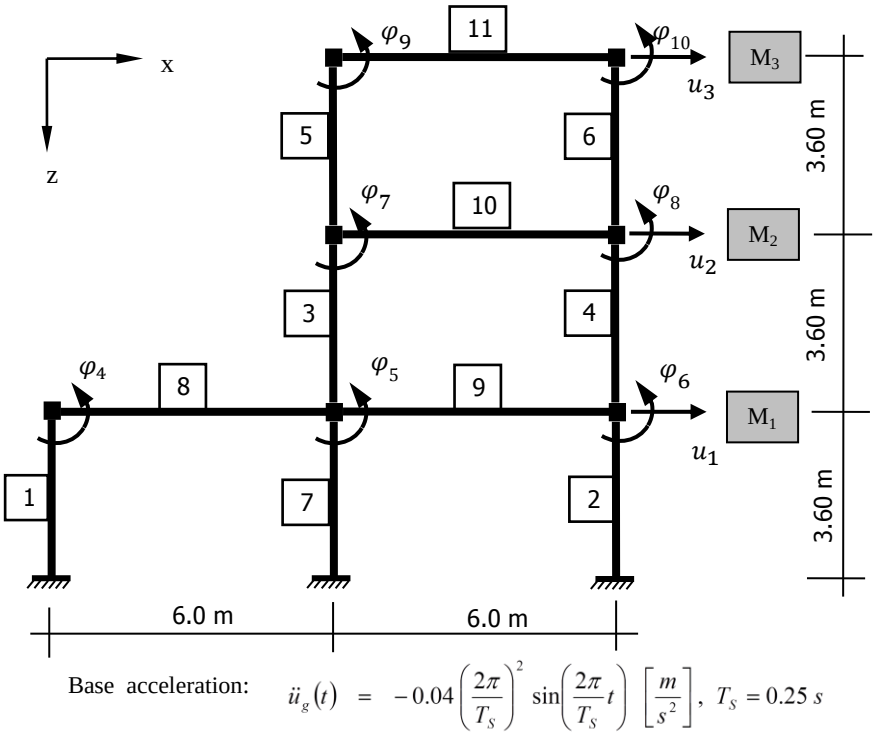

Figure 3: Lumped 10-DOF system – geometry, nodal displacements and rotations

Table 1: Stiffnes and mass data

Beam No.	stiffness EI [kNm^2]	DOF-No.	Mass/rotational inertia M [t] / [kNms^2]
1,2	21950	1	30.0
3-6	16300	2	17.0
7	40470	3	15.0
8-10	48573	4-10	5.0
11	34167		

The stiffness and geometry data of this example are have been taken from [8]. The lumped mass plane frame is subjected to single component translation $\ddot{u}_g(t)$ of the support points. The equations of motion for a system subjected to base acceleration $\ddot{u}_g(t)$ are

$$\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & -\mathbf{K} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{W}} \\ \dot{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{D} & \mathbf{K} \\ \mathbf{K} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{W} \\ \mathbf{V} \end{bmatrix} = \begin{bmatrix} \mathbf{M}\,\mathbf{r}\,\ddot{u}_g(t) \\ \mathbf{0} \end{bmatrix} \tag{39}$$

With an assumed damping ratio $\eta = 5\%$ for the fundamental natural period T_1 we generate a non-modal damping matrix from $\mathbf{D} = c_K\mathbf{K}$ in similar manner like in Sec.2.2.

The number of modes considered in the modal transformation are limited to only the first two eigenvector pairs. The real transformation matrix $\mathbf{Y_L}$ is in this case a (20 x 4) matrix. We have determined the vibration-response in the time 0.0 - 2.0 s; the time step length for the applied Newmark integration method to integrate the modal equations is 0.002 s.

The time response of the modal coordinates $x_1(t), y_1(t)$ is shown in figure 4.

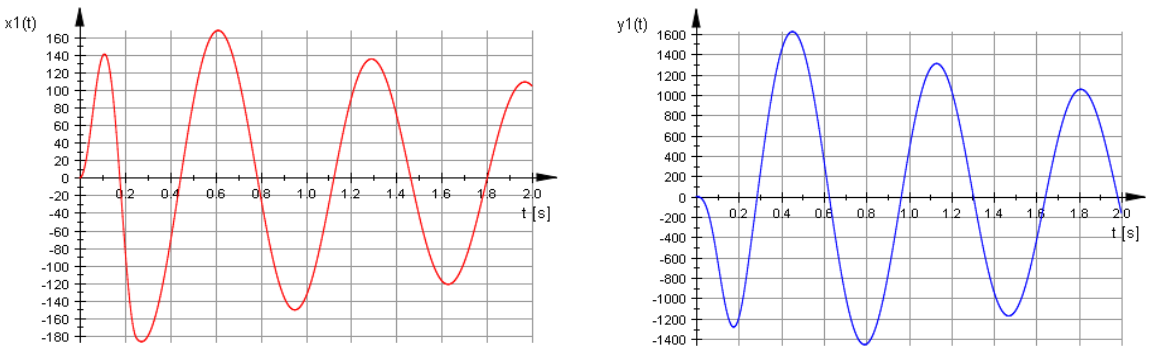

Figure 4: Modal coordinates vibration $x_1(t),\ y_1(t)$

By a back transformation according to Eq.(38) the total system response $\mathbf{V}(t)$ is obtain - see Fig. 5:

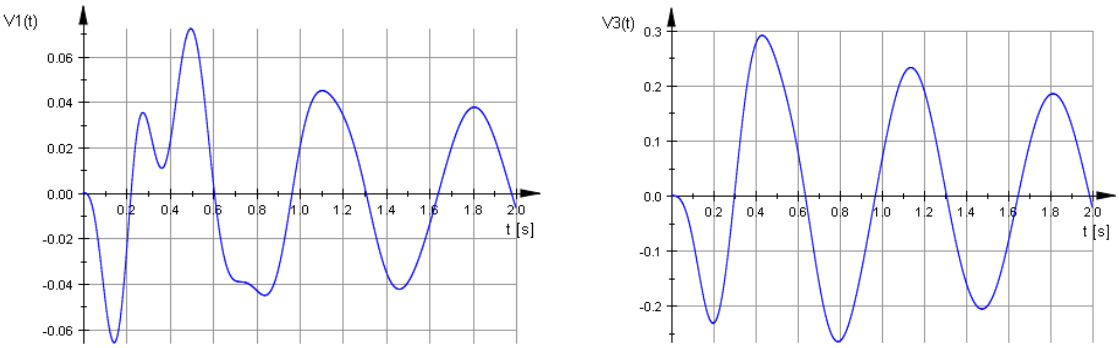

Figure 5: Total vibration $u_1(t)\ [m]$ at story 1, $u_3(t)\ [m]$ at story 3

The presented solution is compared to a solution obtained by the Newmark direct integration method (without modal transformation). The time series $u_1(t), u_2(t), u_3(t)$ in both cases are practically identical. Only in the time response of the rotational degrees of freedom $\varphi_j(t)$ minor deviations between the two solutions can be observed. More details of this solution can be seen in [4].

4 CONCLUSIONS

Both procedures, introduced in Section 2 and 3, are modal decomposition methods, based on the complex eigenvalue solution of the damped structural model. In both methods a new real transformation matrix has been built to perform a modal decomposition of the equations of motion in real arithmetic. In the procedure presented in Section 1 we use the complex right eigenvectors, normalized relative to the general mass matrix, whereas in the method in Section 2 both right and left (complex) eigenvectors - orthonormal relative to the general stiffness matrix - are employed. The suggested new approach has besides the modal transformation matrix in real space two major benefits:

1) The equations of motion are uncoupled into SDOFS block equations, and
2) An uncompleted transformation according to Eq. (15) resp. Eq. (36), employing only a few (k) right and left eigenvector pairs in the $\mathbf{Y}$ -basis (k<<n), leads with sufficient numerical accuracy to the total time response of all n DOF after the final back coordinate transformation.

The suggested modal analysis method can be applied in structural systems containing different damping and energy-loss mechanism in various parts of the structure and also in structure-environment interaction problems, where a non-modal damping matrix is occurring.

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