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DISCRETE EQUIVALENCE PRINCIPLE FOR MODEL ORDER REDUCTION OF DYNAMIC STRUCTURES

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Abstract. *In dynamic structural analyses, in order to improve the analyses performance, model order reduction methods are applied to mathematical models obtained by computational techniques to reduce the size of the associated problems. There are various techniques for model order reduction with different performance levels applying to complex problems. In this study, model order reduction approaches based on discrete equivalent principle are compared with some methods given in literature.*

Newly developed discrete equivalent model order reduction approach is based on that the response of the original and reduced order model is equal to each other at discrete time steps. In the proposed model order reduction approach, the degrees-of-freedom (DOFs) that give responses conforming with the original system are the main criteria at the selection of the DOFs which are transferred into the reduced system as the active DOFs.

The developed model order reduction method was applied to create some sample structures by using some Matlab codes and the results of the model order reduction analyses are compared with the original system response. In the same Matlab codes, responses of some other model order reduction methods were also compared with the original system responses. Damping characteristics of the systems are taken into account in all these comparisons and the results of analyses with different model orders are evaluated in terms of calculation time and performance.

To develop computational models, finite element methods are employed. It is observed that as the damping of the original system decreases, the accuracy of the model order reduction process decreases. In simulation studies, the inputs for the sample systems are selected to be external forces applied to some nodes. If the input force is smooth enough then the performance of model order reduction process based on discrete equivalent methods improves substantially.

1 INTRODUCTION

Model order reduction methods have been the subject of significant number of recent studies, e.g., see [1] for an overview and discussion. Many techniques have been proposed to obtain reduced order finite element models by reducing the orders of mass and stiffness matrices. One class of techniques such as [2, 3, 4, 5, 6, 7, 8, 9, 10] uses a direct reduction of physical model coordinates by omitting some selected degrees-of-freedom (DOF), whose accuracy is very sensitive to the selection of active DOF. The second class of methods such as the works [11] and [12] is the modal methods having better accuracy, yielding decoupled equations and giving a better insight into the problems; however, they have the following disadvantages: computation of the modes may be costly if too many modes are needed in the analyses and their results may be unacceptable due to the effects of truncated modes. In order to overcome this drawback, quasi-static compensation (QSC) method studied in [13, 14, 15, 16, 17, 18] was developed. The third class of methods is the Ritz vector methods (e.g., see [1, 19, 20, 21, 22, 23, 24, 25, 27, 28]) developed to eliminate the need to compute costly eigenvalue problems and to improve the accuracy in cases where eigenvectors are not the best choice; these methods can be regarded as a generalized approach replacing the eigenvectors by more generally defined Ritz vectors. Juanmin Gu et. al. in [29] used the data recovery techniques in dynamic analyses to recover the physical response from the modal response obtained by using the mode-displacement method. In [30], the frequency and time domain responses of some popular model order reduction methods such as QSC, quasi-static mode synthesis (QSM) method [19], subspace based identification [31] and nonlinear least square (NLS) fit in frequency domain [32, 33] are compared.

In this paper, a family of model order reduction techniques is derived in time domain based on the principle that the exact solution of the semi-discrete equation of a system and the solution of the reduced order model match at discrete time steps. It is sufficient to pursue exact match at discrete time steps, i.e., discrete equivalence, since the solutions of semi-discrete equations are obtained only at the time steps. Numerical examples are presented to compare the performance of the proposed approach with that of exact solution. It is shown that proposed methods have certain advantages over conventional methods.

2 DEFINITIONS

A linear time-invariant discrete-time system is represented by the transfer function obtained by the ratio of the Z-transform of input to the Z-transform of output as follows

$$G_z(z) = \frac{D_z(z)}{F_z(z)} = \frac{num(z)}{den(z)} \tag{1}$$

where $D_z(z)$ and $F_z(z)$ are respectively Z-transforms of output and input, $num(z) = n_k z^k + n_{k-1} z^{k-1} + \ldots + n_1 z + n_0$, $den(z) = d_l z^l + d_{l-1} z^{l-1} + \ldots + d_1 z + d_0$ and z is the complex Z-transform variable [34]. For a signal having the discrete values $u_0, u_1, u_2, .. u_k$, the Z-transform is defined as follows

$$U(z) = Z\{u(k)\} = \sum_{k=-\infty}^{\infty} u_k z^{-k} \tag{2}$$

On the other hand, a linear time-invariant continuous-time system is represented by the transfer function in terms of the complex Laplace transform variable s as follows

$$G_s(s) = \frac{D_s(s)}{F_s(s)} = \frac{num(s)}{den(s)} \tag{3}$$

where $D_s(s)$ and $F_s(s)$ are respectively Laplace transforms of the output and input, $num(s) = n_k s^k + n_{k-1}s^{k-1} + ... + n_1 s + n_0$ and $den(s) = d_l s^l + d_{l-1}s^{l-1} + ... + d_1 s + d_0$. For a time varying signal $u(t)$, the Laplace transform is defined as follows

$$U(s) = L\{u(t)\} = \int_0^\infty u(t)e^{-st}dt \tag{4}$$

3 DISCRETE EQUIVALENCE PRINCIPLE

It is common to study a time integration algorithm by considering the single modal equation. Accordingly, the following continuous-time transfer function $G_s(s)$ of the first-order single modal equation of a parabolic problem is considered

$$G_s(s) = \frac{1}{s+\lambda} \tag{5}$$

where λ is the modal parameter. For the second-order structural systems, $G_s(s)$ is equal to

$$G_s(s) = \frac{1}{s^2 + 2\xi\Omega s + \Omega^2} \tag{6}$$

where Ω is the undamped natural frequency and ξ the damping ratio. Note that Ω and λ belong to the finite element discretization, and they are usually denoted by the superscript h in literature that is omitted in this article for brevity. Let $G_z(z)$ denote the discrete-time transfer function of a time integration method which is unknown to us. Suppose that we have a forcing term f(t) that is applied to the semi-discrete equation and we want to integrate the semi-discrete equation via the time integration method. Let $F_z(z) = \{f(k)\}$ and $F_s(s) = \{f(t)\}$ denote the *Z*-transform and Laplace transform of the forcing term *f(t)*, respectively. Then, if the solution of the time integration method is equal to the exact solution of the semi-discrete equation at discrete time steps, the following equation should hold:

$$Z^{-1}\{G_z(z)F_z(z)\} = L^{-1}\{G_s(s)F_s(s)\}_{t=k\Delta t} \tag{7}$$

Equivalently,

$$G_z(z) = \frac{1}{F_z(z)} Z\{L^{-1}\{G_s(s)F_s(s)\}_{t=k\Delta t}\} \tag{8}$$

where $Z^{-1}\{\cdot\}$ and $L^{-1}\{\cdot\}$ are, respectively, the inverse Z-transform and inverse Laplace transform operators, Δt time step and *k* an integer denoting the time step number. Note that (8) guarantees that the solution of the time integration method matches with the exact solution at discrete time steps no matter what the time step is; however, the frequency response of this time integration method cannot match with that of the original semi-discrete equation. Therefore, time integration methods obtained by using (8) are called discrete equivalent methods. Now, (8) can be used to find the unknown discrete-time transfer function $G_z(z)$ of the time integration method, e.g., see [30] for details.

4 DESIGN OF TIME INTEGRATION METHODS BY USING DISCRETE EQUIVALENCE PRINCIPLE

Consider two systems having the following semi-discrete equations

$$M\ddot{d} + C\dot{d} + Kd = f \tag{9}$$

$$\hat{M}\ddot{\hat{d}} + \hat{C}\dot{\hat{d}} + \hat{K}\hat{d} = \hat{f} \tag{10}$$

where (9) represents the original system and (10) represents the system obtained by a model order reduction method. It is noteworthy that in general the solutions *d(t)* and $\hat{d}(t)$ are different [35]. Suppose that even though (9) and (10) represent two different systems, certain outputs of these systems obtained by discrete equivalence principle (7) are equal to each other at discrete time steps. That is, at a discrete time step *t=kΔt*, we have

$$\hat{d}_k = Ed_k \quad \text{and} \quad \hat{f}_k = Ef_k \tag{11}$$

where E is the transformation matrix defined by

$$E = \begin{bmatrix} e_i^T \\ e_j^T \\ \vdots \\ e_k^T \end{bmatrix} \tag{12}$$

and e_i is the unit vector in the ith direction whose elements are zero except for the *i*th element which is 1 as follows

$$e_i = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 & 0 & \dots\dots & 0 \end{bmatrix}^T \tag{13}$$

where superposed T denotes transpose. Thus, the outputs of systems represented by (9) and (10) that correspond to the unit vectors e_i in (12) are equal to each other at discrete time steps. Note that the transformation matrix is in general rectangular and by definition $EE^T = I$ due to $e_i e_j^T = \delta_{ij}$ is equal to the Kronecker delta function.

Consider the following semi-discrete equations of a parabolic problem

$$M\dot{d} + Kd = f \tag{14}$$

where M is the mass matrix, K stiffness matrix, f load vector, d displacement vector and $\dot{d}$ time derivative of d. Note that M and K are symmetric, M positive definite and K positive se midefinite. The load vector f is the function of time denoted by $f=f(t)$ for $t \in [0, T]$. The purpose of the initial value problem defined by (14) is to find the solution $d=d(t)$ satisfying the initial condition $d(0)=d_0$. Next, we will apply "response invariant" time integration methods to the system represented by (14).

Suppose that we would like to have that the impulse response of the original system represented by (9) obtained by IRI method [6] is equal to the exact solution of (9) at discrete time steps. Consider the following single DOF modal equation of (9)

$$\dot{d} + \lambda d = f \tag{15}$$

The discrete-time solution of the time integration method is equal to $f_d(k\Delta t) = Z^{-1}\{G_z(z)\}$, and the corresponding continuous-time impulse response of the semi-discrete equation is equal to $f_c(t) = L^{-1}\{G_s(s)\}$; then, $f_d(k\Delta t) = \Delta t f_c(t)\big|_{t=k\Delta t}$ [6]. On the other hand, since

$$G_z(z) = Z\{f_d(k\Delta t)\} = Z\{f_c(t)\}\Delta t = Z\{L^{-1}\{G_s(s)\}\}\Delta t \tag{16}$$

where $G_s(s)$ is given by (5); then, (16) yields

$$G_z(z) = \frac{D_z(z)}{F_z(z)} = Z\left\{L^{-1}\left\{\frac{1}{s+\lambda}\right\}\right\}\Delta t = \frac{z}{z - e^{-\lambda\Delta t}}\Delta t \tag{17}$$

where z is the time shift operator such that $z^n d_k = d_{k+n}$ where d_k is the output value at the kth time step, i.e., $t = k\Delta t$, then (12) yields that

$$d_{k+1} = e^{-\lambda\Delta t} d_k + \Delta t f_{k+1} \tag{18}$$

5 TOTAL ENERGY EQUIVALENCE METHOD

The main criteria in newly proposed model order reduction method is based on principal that "The total energy level of the model order reduced system should be equal to the total energy level of the original system.". In this approach, the main method accuracy criteria comes from the comparison of total energy levels of both systems. In order to reach the total energy, the kinetic and potential energy of the original system can firstly be given as,

$$\text{Kinetic energy: } \boldsymbol{H}_k = \frac{1}{2} \dot{\boldsymbol{d}}_k{}^T \boldsymbol{M} \dot{\boldsymbol{d}}_k \tag{19}$$

$$\text{Potential energy: } \boldsymbol{P}_k = \frac{1}{2} \boldsymbol{d}_k{}^T \boldsymbol{K} \boldsymbol{d}_k \tag{20}$$

By using (19) and (20), total energy is obtained as

$$\text{Total energy: } \boldsymbol{T}_k = \boldsymbol{H}_k + \boldsymbol{P}_k \tag{21}$$

To obtain the lowest deviation on the total energy error of both systems, system transformation formulation is developed and then system mass M and system stiffness K matrices by finite element methods are produced.

To calculate the total energy, forward difference integration method is applied to (21) by $\boldsymbol{d}_k$ and $\boldsymbol{d}_{k+1}$.

By using $\dot{\boldsymbol{d}}_k \cong \dfrac{\boldsymbol{d}_{k+1} - \boldsymbol{d}_k}{\Delta t}$ statement, kinetic energy of original model can be written,

$$H_k = \frac{1}{2\Delta t^2} \left(d_{k+1}^T - d_k^T \right) M \left(d_{k+1} - d_k \right) \tag{22}$$

and (21) equation can be orientated again by (20), original model total energy (22) yields,

$$T_k = H_k + P_k = \frac{1}{2} d_k^T K d_k + \frac{1}{2\Delta t^2} \left[d_{k+1}^T M d_{k+1} - d_{k+1}^T M d_k - d_k^T M d_{k+1} + d_k^T M d_k \right] \tag{23}$$

The matrix form of (23) is given by,

$$T_k = \frac{1}{2} \begin{bmatrix} d_k^T & d_{k+1}^T \end{bmatrix} \begin{bmatrix} K + \dfrac{M}{\Delta t^2} & -\dfrac{M}{\Delta t^2} \\ -\dfrac{M}{\Delta t^2} & \dfrac{M}{\Delta t^2} \end{bmatrix} \begin{Bmatrix} d_k \\ d_{k+1} \end{Bmatrix} \tag{24}$$

The total energy of the model order reduced system is similarly formed as,

$$\hat{T}_k = \frac{1}{2} \begin{bmatrix} \hat{d}_k^T & \hat{d}_{k+1}^T \end{bmatrix} \begin{bmatrix} \hat{K} + \dfrac{\hat{M}}{\Delta t^2} & -\dfrac{\hat{M}}{\Delta t^2} \\ -\dfrac{\hat{M}}{\Delta t^2} & \dfrac{\hat{M}}{\Delta t^2} \end{bmatrix} \begin{Bmatrix} \hat{d}_k \\ \hat{d}_{k+1} \end{Bmatrix} \tag{25}$$

If (25) is modified, below equation is obtained,

$$\hat{T}_k = \frac{1}{2} \begin{bmatrix} d_k^T & d_{k+1}^T \end{bmatrix} E^T \begin{bmatrix} \hat{K} + \dfrac{\hat{M}}{\Delta t^2} & -\dfrac{\hat{M}}{\Delta t^2} \\ -\dfrac{\hat{M}}{\Delta t^2} & \dfrac{\hat{M}}{\Delta t^2} \end{bmatrix} E \begin{Bmatrix} d_k \\ d_{k+1} \end{Bmatrix} \tag{26}$$

When the active DOFs of original and model order reduced systems is taken into consideraiton, because total energy equivalence $\boldsymbol{T}_k \equiv \hat{\boldsymbol{T}}_k$ is desired, the identical parameters of the matrices should be equal (energy saving algorithm). Thus,

$$\boldsymbol{E}^T \begin{bmatrix} \hat{\boldsymbol{K}} + \dfrac{\hat{\boldsymbol{M}}}{\Delta t^2} & -\dfrac{\hat{\boldsymbol{M}}}{\Delta t^2} \\ -\dfrac{\hat{\boldsymbol{M}}}{\boldsymbol{\Delta t^2}} & \dfrac{\hat{\boldsymbol{M}}}{\Delta t^2} \end{bmatrix} \boldsymbol{E} = \begin{bmatrix} \boldsymbol{K} + \dfrac{\boldsymbol{M}}{\Delta t^2} & -\dfrac{\boldsymbol{M}}{\Delta t^2} \\ -\dfrac{\boldsymbol{M}}{\Delta t^2} & \dfrac{\boldsymbol{M}}{\Delta t^2} \end{bmatrix} \tag{27}$$

and, the first elements of the matrices are considered equal as below,

$$\boldsymbol{E}^T \hat{\boldsymbol{K}} \boldsymbol{E} + \boldsymbol{E}^T \frac{\hat{\boldsymbol{M}}}{\Delta t^2} \boldsymbol{E} = \boldsymbol{K} + \frac{\boldsymbol{M}}{\Delta t^2} \tag{28}$$

the second parameters should be equal.

$$\boldsymbol{E}^T \frac{\hat{\boldsymbol{M}}}{\Delta t^2} \boldsymbol{E} = \frac{\boldsymbol{M}}{\Delta t^2} \tag{29}$$

On the other hand, if (29) is premultiplied by the transformation matrix E, and then also postmultiplied by the transpose of transformation matrix $\boldsymbol{E}^T$, and using the fact that $EE^T = I$, we obtain

$$\hat{\boldsymbol{M}} = \boldsymbol{EME}^T \tag{30}$$

Similarly from the equality of the first parameters of (28), and by using pre and post multiplying, we get

$$\hat{\boldsymbol{K}} = \boldsymbol{EKE}^T \tag{31}$$

Note that (30) and (31) determine respectively the mass and stiffness matrices for the reduced order system whose certain outputs are equal to those of the exact solution at discrete time steps for which the Total Energy Equivalence Method is used as the time integration method; namely, their impulse responses should be identical at discrete time steps for certain outputs defined by the transformation matrix *E*. Note that the second order systems could be cast in the state space form by defining the state-space matrices appropriately.

6 NUMERICAL EXAMPLE

The results of the example system will be presented in this section such as a cantilever beam. More examples and argument can be found in [35]. By considering the cantilever planar beam, the model order reduction method is applied to the beam given in Figure 1 by using the programs developed in Matlab environment.

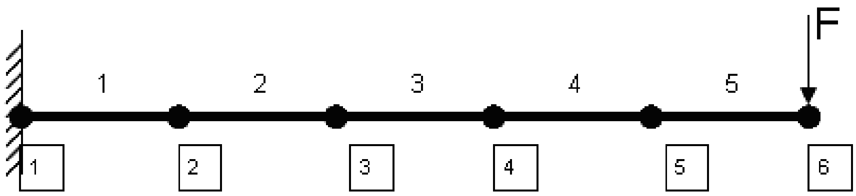

Figure 1: The Euler-Bernoulli beam system on which model order reduction is applied.

In the planar beam element, each node has 1 axial, 1 vertical and 1 rotational DOF. The length of the beam is 0.5 m, cross-sectional area is 4x10^{-4} m^2, beam height is 2x10^{-2} m, area moment of the cross-section is 1.33x10^{-8} m^4, density of the material of the beam is 7850 kg/m^3 and material elasticity modulus is 210 GPa (steel material). The loading is applied to the beam tip downward having the magnitude of 100 N as shown in Figure 1. By considering the proportional Rayleigh damping $C = \beta K$, the reduced model order system is created and then the analyses are repeated for different Rayleigh damping ratios as β=0, 0.00008, 0.001, 0.01 and 0.1.

Different element number combinations were studied as seen at Table 1. Through these studies, impulse and sinusoidal forces were separately loaded. The results of the beam having 5 elements are presented here. The impulse responses and forced responses for the input of *f(t)*=100*sin(2*time)+50*sin(5*time) are presented in Figures 2 and 3.

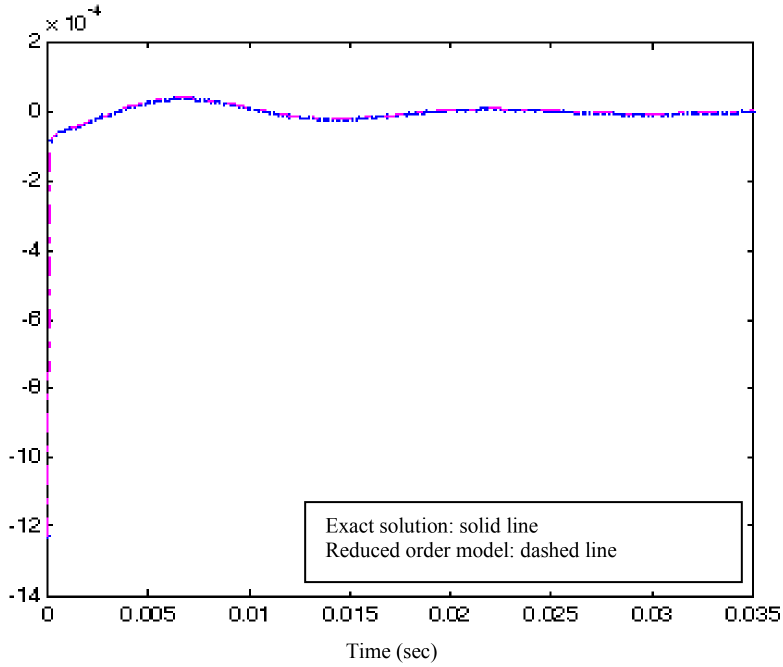

Figure 2: The beam tip displacement (in meter) for the impulse response where β =0.001.

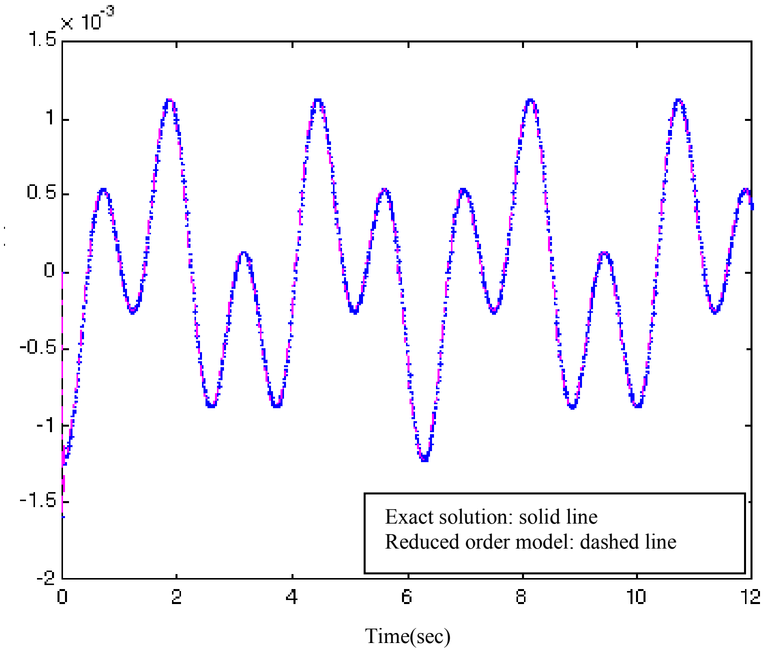

Figure 3: The beam tip displacement (in meter) for the sinusoidal input response where β =0.001.

Note that other simulation studies were also carried out whose results were not presented here for brevity. In sum, as the damping coefficient β reduces, the accuracy also reduces. Moreover, the accuracy of the response for the sinusoidal input is always superior to that of impulse response.

In order to compare the performance of the proposed model order reduction method, the CPU times of the original system and other systems obtained by using the existing model order reduction methods are listed in Table 1, where the damping coefficient β and number of elements are changing. In sum, creating the reduced order model is always beneficial by considerable time.

Number of Elements	Original DOF/Reduced DOF	Rayleigh stiffness proportional damping co-efficient	CPU time used by reduced order model by Total Energy Equivalence MOR method [sec.]	CPU time used by reduced order model by Forward difference MOR method [sec.]	CPU time used by reduced order model by Newmark MOR method [sec.]	CPU time for Original System [sec.]
5	15/10	0	0.001	0.001	0.001	0.037
		0.001	0.001	0.001	0.001	0.031
		0.01	0.001	0.001	0.001	0.032
		0.1	0.001	0.001	0.001	0.032
15	45/12	0	0.015	0.016	0.015	0.156
		0.001	0.015	0.032	0.016	0.141
		0.01	0.015	0.015	0.016	0.141
		0.1	0.015	0.015	0.016	0.141
100	300/148	0	0.175	0.527	0.175	1.531
		0.001	0.172	0.507	0.235	1.498
		0.01	0.172	0.515	0.235	1.525
		0.1	0.175	0.527	0.235	1.484
1000	3000/1000	0	2.579	16.515	3.906	1106.969
		0.001	2.578	15.671	3.813	1105.484
		0.01	2.515	16.375	4.093	1082.531
		0.1	2.408	16.453	3.906	1092.031

Table 1: CPU times while the number elements and damping coefficient β are changing.

7 CONCLUSIONS

A model order reduction techniques –Total Energy Equivalence Method- is derived in time domain based on the principle that the exact solution of the semi-discrete equation of a system and the solution of the reduced order model match at discrete time steps. It is sufficient to pursue exact match at discrete time steps, i.e., discrete equivalence, since the solutions of semi-discrete equations are obtained only at the time steps. A numerical example is presented to show the advantages of the pro-posed method and to compare the performance of them with that of some popular methods. The proposed method is applied to a sample problem and the results are compared with exact solutions. It is shown that proposed methods have certain advantages. It is concluded that characteristics of the responses of discrete equivalent reduced order models are similar as well no matter what the magnitude of the damping is. In order to improve the convergence, the DOFs neighboring the DOFs to which forces are applied and outputs are computed should be selected as active DOFs. It was determined that especially, in case of the sudden input force changes like impulse or step functions, performance of the model order reduction process based on discrete equivalent methods decreases. More examples and discussion about the presented approach can be found in [35]. In the future, this method would be applied to other kinds of problems and will be tested further.

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