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A NOVEL GALERKIN PROJECTION APPROACH FOR DAMPED STOCHASTIC DYNAMIC SYSTEMS

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Abstract. *This article provides the theoretical development and simulation results of a novel Galerkin subspace projection scheme for damped dynamic systems with stochastic coefficients. The fundamental idea involved here is to solve the stochastic dynamic system in the frequency domain by projecting the solution into a reduced finite dimensional spatio-random vector basis to approximate the response. A Nueumann expansion type of approach is used to generate the complex stochastic basis functions. The proposed method is applicable to linear dynamic systems with Gaussian and non-Gaussian random fields. Galerkin weighting coefficients have been employed to minimize the error induced due to the reduced basis and finite order spectral functions and hence to explicitly evaluate the stochastic system response. The statistical moments of the solution have been evaluated at all frequencies to illustrate and compare the stochastic system response with the deterministic case. The results have been compared to direct Monte-Carlo simulation for different correlation lengths and variability of randomness.*

1 INTRODUCTION

Due to the significant development in computational hardware it is now possible to solve very high resolution models in various computational physics problems, ranging from fluid mechanics to nano-bio mechanics. However, the spatial resolution is not enough to determine the credibility of the numerical model, the physical model as well its parameters are also crucial. Since neither of these may not be exactly known, there has been increasing research activities over the past three decades to model the governing partial differential equations within the framework of stochastic equations. We refer to few recent review papers [1–3]. Consider a bounded domain $\mathcal{D} \in \mathbb{R}^d$ with piecewise Lipschitz boundary $\partial\mathcal{D}$, where $d \leq 3$ is the spatial dimension and $t \in \mathbb{R}^+$ is the time. Further, consider that $(\Theta, \mathcal{F}, P)$ is a probability space where $\theta \in \Theta$ is a sample point from the sampling space Θ, $\mathcal{F}$ is the complete σ-algebra over the subsets of Θ and P is the probability measure. We consider a linear stochastic partial differential equation (PDE) of the form

$$\rho(\mathbf{r},\theta)\frac{\partial^2 U(\mathbf{r},t,\theta)}{\partial t^2} + \mathfrak{L}_\alpha \frac{\partial U(\mathbf{r},t,\theta)}{\partial t} + \mathfrak{L}_\beta U(\mathbf{r},t,\theta) = p(\mathbf{r},t); \quad \mathbf{r} \in \mathcal{D}, t \in [0,T] \tag{1}$$

with the associated Dirichlet condition

$$U(\mathbf{r},t,\theta) = 0; \quad \mathbf{r} \text{ on } \partial\mathcal{D}. \tag{2}$$

In the above equation $\rho(\mathbf{r},\theta)$ is the random mass distribution of the system, $p(\mathbf{r},t)$ is the distributed time-varying forcing function, $\mathfrak{L}_\beta$ is the spatial self-adjoint stochastic stiffness operator, $\mathfrak{L}_\alpha$ is the spatial self-adjoint stochastic damping operator and $U(\mathbf{r},t,\theta)$ in the stochastic response quantity to be obtained. Note that $\mathfrak{L}_\alpha(\bullet)$ and $\mathfrak{L}_\beta(\bullet)$ involve linear stochastic differential operators with coefficients $\alpha(\mathbf{r},\theta)$ and $\beta(\mathbf{r},\theta)$ as the second order random fields such that $\alpha, \beta : \mathbb{R}^d \times \Theta \to \mathbb{R}$. We assume the random fields to be stationary and square integrable. The purpose of this paper is to investigate a new solution approach for Eqn. (1) after the discretization of the spatio-random fields using the well established techniques of stochastic finite element method (SFEM) as can be found in references [4–6].

The random fields in Eqn. (1) can be discretized to represent them as spectral functions using a finite number of random variables using one of the established techniques available in literatures [4, 7]. Hence the stochastic PDE along with the boundary conditions would result in an equation of the form

$$\mathbf{M}(\theta)\ddot{\mathbf{u}}(\theta,t) + \mathbf{C}(\theta)\dot{\mathbf{u}}(\theta,t) + \mathbf{K}(\theta)\mathbf{u}(\theta,t) = \mathbf{f}_0(t) \tag{3}$$

where $\mathbf{u}(\theta,t)$ is the discretized response vector, $\mathbf{M}(\theta) = \mathbf{M}_0 + \sum_{i=1}^{p} \mu_i(\theta_i)\mathbf{M}_i \in \mathbb{R}^{n\times n}$ is the random mass matrix, $\mathbf{K}(\theta) = \mathbf{K}_0 + \sum_{i=1}^{p} \nu_i(\theta_i)\mathbf{K}_i \in \mathbb{R}^{n\times n}$ is the random stiffness matrix along with $\mathbf{C}(\theta) \in \mathbb{R}^{n\times n}$ as the random damping matrix. Here the mass and stiffness matrices have been expressed in terms of their deterministic components ($\mathbf{M}_0$ and $\mathbf{K}_0$) and the corresponding random contributions ($\mathbf{M}_i$ and $\mathbf{K}_i$) obtained from discretizing the stochastic field with a finite number of random variables ($\mu_i(\theta_i)$ and $\nu_i(\theta_i)$) and their corresponding spatial basis functions. This has been elaborated in subsection 2.1. In the present work proportional damping is considered for which $\mathbf{C}(\theta) = \zeta_1\mathbf{M}(\theta) + \zeta_2\mathbf{K}(\theta)$, where ζ_1 and ζ_2 are deterministic scalars. For the harmonic analysis of the structural system considered in Eqn. (3), it is represented in the frequency domain as

$$\left[-\omega^2\mathbf{M}(\theta) + i\omega\mathbf{C}(\theta) + \mathbf{K}(\theta)\right]\widetilde{\mathbf{u}}(\theta,\omega) = \widetilde{\mathbf{f}}_0(\omega) \tag{4}$$

where $\widetilde{\mathbf{u}}(\theta,\omega)$ is the complex frequency domain system response amplitude, $\widetilde{\mathbf{f}}_0(\omega)$ is the amplitude of the harmonic force.

Now we group the random variables associated with the mass and damping matrices of Eqn. (3) as

$$\xi_i(\theta) = \mu_i(\theta) \quad \text{and} \quad \xi_{i+p}(\theta) = \nu_i(\theta) \quad \text{for} \quad i = 1, 2, \ldots, p$$

following which the expression for the linear structural system in Eqn. (4) can be expressed as

$$\left(\mathbf{A}_0(\omega) + \sum_{i=1}^{2p} \xi_i(\theta_i)\mathbf{A}_i(\omega)\right) \widetilde{\mathbf{u}}(\omega,\theta) = \widetilde{\mathbf{f}}_0(\omega) \tag{5}$$

where $\mathbf{A}_0$ and $\mathbf{A}_i \in \mathbb{C}^{n\times n}$ represent the complex deterministic and stochastic parts respectively of the mass, the stiffness and the damping matrices ensemble. For the case of proportional damping the matrices $\mathbf{A}_0$ and $\mathbf{A}_i$ can be written as

$$\mathbf{A}_0(\omega) = \left[-\omega^2 + i\omega\zeta_1\right]\mathbf{M}_0 + \left[i\omega\zeta_2 + 1\right]\mathbf{K}_0 \tag{6}$$

$$\mathbf{A}_i(\omega) = \left[-\omega^2 + i\omega\zeta_1\right]\mathbf{M}_i \quad \text{for} \quad i = 1, 2, \ldots, p \tag{7}$$

$$\text{and} \quad \mathbf{A}_j(\omega) = \left[i\omega\zeta_2 + 1\right]\mathbf{K}_j \quad \text{for} \quad j = p+1, p+2, \ldots, 2p\,.$$

The paper has been arranged as follows. In section 2 a brief overview of spectral stochastic finite element method is presented. The projection theory in the vector space is developed in section 3. In section 4 an error minimization approach in the Hilbert space is proposed. The idea of the reduced orthonormal vector basis is introduced in subsection section 5. The post processing of the results to obtain the response moments are discussed in section 6. Based on the theoretical results, a simple computational approach is shown in section 7 where the proposed method of reduced orthonormal basis is applied to the stochastic mechanics of an Euler-Bernoulli beam. From the theoretical developments and numerical results, some conclusions are drawn in section 8.

2 Overview of the spectral stochastic finite element method

2.1 Discretization of the stochastic PDE

First consider $a(\mathbf{r},\theta)$ is a Gaussian random field with a covariance function $C_a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ defined in the domain $\mathcal{D}$. Since the covariance function is square bounded, symmetric and positive definite, it can be represented by a spectral decomposition in an infinite dimensional Hilbert space. Using this spectral decomposition, the random process $a(\mathbf{r},\theta)$ can be expressed [see for example, [4, 8]] in a generalized Fourier type of series known as the Karhunen-Loève (KL) expansion

$$a(\mathbf{r},\theta) = a_0(\mathbf{r}) + \sum_{i=1}^{\infty} \sqrt{\nu_i}\widetilde{\xi}_i(\theta)\varphi_i(\mathbf{r}) \tag{8}$$

Here $a_0(\mathbf{r})$ is the mean function, $\widetilde{\xi}_i(\theta)$ are uncorrelated standard Gaussian random variables, ν_i and $\varphi_i(\mathbf{r})$ are eigenvalues and eigenfunctions satisfying the integral equation

$$\int_{\mathcal{D}} C_a(\mathbf{r}_1,\mathbf{r}_2)\varphi_j(\mathbf{r}_1)\mathrm{d}\mathbf{r}_1 = \nu_j\varphi_j(\mathbf{r}_2), \quad \forall\ j = 1, 2, \cdots \tag{9}$$

The Gaussian random field model is not applicable for strictly positive quantities arising in many practical problems. Eqn. (8) could also represent the Karhunen-Loève expansion of a non-Gaussian random field, which is also well defined. Alternatively, when $a(\mathbf{r}, \theta)$ is a general non-Gaussian random field, it can be expressed in a mean-square convergent series in random variables and spatial functions using the polynomial chaos expansion. For example Ghanem [9] expanded log-normal random fields in a polynomial chaos expansion. In general, non Gaussian random fields can be expressed in a series like

$$a(\mathbf{r}, \theta) = a_0(\mathbf{r}) + \sum_{i=1}^{\infty} \xi_i(\theta) a_i(\mathbf{r}) \tag{10}$$

using Wiener-Askey chaos expansion [10–12]. Here $\xi_i(\theta)$ are in general non-Gaussian and correlated random variables and $a_i(\mathbf{r})$ are deterministic functions. In this paper we use this general form of the decomposition of the random field.

Truncating the series in Eqn. (10) upto the M-th term and using the same approach for the governing PDE (1) with boundary conditions, the discretized system equation in the frequency domain (3) can be represented by Eqn. (5), with $M = 2p$ as

$$\left[\mathbf{A}_0 + \sum_{i=1}^{M} \xi_i(\theta) \mathbf{A}_i \right] \mathbf{u}(\theta, \omega) = \mathbf{f}_0(\omega). \tag{11}$$

The 'tilde' sign has been omitted from the notations of the frequency domain quantities of $\mathbf{u}(\theta, \omega)$ and $\mathbf{f}_0(\omega)$ for the sake of notational convenience and this shall be followed henceforth. The necessary technical details to obtain the discrete stochastic algebraic equations from the stochastic partial differential equation (1) have become standard in the literature. Excellent references, for example [4, 13–15] are available on this topic. In Eqn. (11), $\mathbf{A}_0$ is a complex symmetric positive definite matrix, $\mathbf{A}_i \in \mathbb{C}^{n \times n}; i = 1, 2, \ldots, M$ are complex symmetric matrices, $\mathbf{u}(\theta, \omega) \in \mathbb{C}^n$ is the solution vector and $\mathbf{f}_0(\omega) \in \mathbb{C}^n$ is the input vector. For most practical applications, uncertainties are small compared to the deterministic values. Therefore, we normally have

$$\|\mathbf{A}_0\| \geq \left\| \sum_{i=1}^{M} \xi_i(\theta) \mathbf{A}_i \right\| ; \forall\ \theta \in \Theta \tag{12}$$

Here by $\|\bullet\|$ we imply the Frobenius matrix norm [16], defined as $\|\mathbf{A}\| = \mathrm{Trace}\left(\mathbf{A}\mathbf{A}^T\right)$ for any $\mathbf{A} \in \mathbb{R}^{n \times n}$. The number of terms M in Eqn. (11) can be selected based on the accuracy desired for the representation of the underlying random field. One of the main aim of a stochastic dynamic analysis is to obtain $\mathbf{u}(\theta, \omega)$ for $\theta \in \Theta$ and for all frequency value ω from Eqn. (11) in an efficient manner and is the main topic of this paper. We propose a solution technique for Eqn. (11) when $\xi_i(\theta)$ are in general non-Gaussian and correlated random variables.

2.2 Brief review of the solution techniques

The solution of the set of stochastic linear algebraic equations (11) is a key step in the stochastic finite element analysis. As a result, several methods have been proposed. These methods include, first- and second-order perturbation methods [5, 17], Neumann expansion method [18, 19], Galerkin approach [20], linear algebra based methods [21–23] and simulation methods [24]. More recently efficient collocation methods have been proposed [25, 26]. Another class of methods which have been used widely in the literature is known as the spectral methods (see [1] for a recent review). These methods include the polynomial chaos (PC)

expansion [4], stochastic reduced basis method [27–29] and Wiener−Askey chaos expansion [10–12]. According to the polynomial chaos expansion, second-order random variables $u_j(\theta)$ can be represented by the mean-square convergent expansion

$$\begin{aligned} u_j(\theta) = u_{i_0} h_0 + \sum_{i_1=1}^{\infty} u_{i_1} h_1(\xi_{i_1}(\theta)) \\ + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} u_{i_1,i_2} h_2(\xi_{i_1}(\theta),\xi_{i_2}(\theta)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} u_{i_1 i_2 i_3} h_3(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta)) \\ + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2}\sum_{i_4=1}^{i_3} u_{i_1 i_2 i_3 i_4}\, h_4(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta),\xi_{i_4}(\theta)) + \dots, \end{aligned} \tag{13}$$

where $u_{i_1,\dots,i_r}$ are deterministic constants to be determined and $h_r(\ \xi_{i_1}(\theta),\dots,\xi_{i_r}(\theta)\ )$ is the r^{th} order homogeneous Chaos. When $\xi_i(\theta)$ are Gaussian random variables, the functions $h_r(\xi_{i_1}(\theta),\dots,\xi_{i_r}(\theta)\ )$ are the r^{th} order Hermite polynomial so that it becomes orthonormal with respect to the Gaussian probability density function. The same idea can be extended to non-Gaussian random variables, provided more generalized functional basis are used [10–12] so that the orthonormality with respect to the probability density functions can be retained. When we have a random vector, as in the case of the solution of Eqn. (11), then it is natural to replace the constants $u_{i_1,\dots i_r}$ by vectors $\mathbf{u}_{i_1,\dots i_r} \in \mathbb{R}^n$. Suppose the series is truncated after P number of terms. The value of P depends on the number of basic random variables M and the order of the PC expansion r as

$$P = \sum_{j=0}^{r} \frac{(M+j-1)!}{j!(M-1)!} = \binom{M+r}{r} \tag{14}$$

After the truncation, there are P number of unknown vectors of dimension n. Then a mean-square error minimization approach can be applied and the unknown vectors can be solved using the Galerkin approach [4]. Since P increases very rapidly with the order of the chaos r and the number of random variables M, the final number of unknown constants Pn becomes very large. As a result several methods have been developed (see for example [27–31]) to reduce the computational cost. In the polynomial chaos based solution approach, the *only* information used to construct the basis is the probability density function of the random variables. In the context of the discretized Eqn. (11), more information such as the matrices $\mathbf{A}_i, i = 0, 1, 2 \dots M$ are available. It may be possible to construct alternative basis using these matrices. Here we investigate such an approach, where instead of projecting the solution in the space of orthonormal polynomials, the solution is projected in an orthonormal vector basis generated from the coefficient matrices.

3 Spectral decomposition in the vector space

3.1 Derivation of the spectral functions

Following the spectral stochastic finite element method, or otherwise, an approximation to the solution of Eqn. (11) can be expressed as a linear combination of functions of random variables and deterministic vectors. Recently Nouy [32, 33] discussed the possibility of an optimal spectral decomposition. The aim is to use small number of terms to reduce the computation without loosing the accuracy. Here an orthonormal vector basis is considered. Fixing a

value of θ, say $\theta = \theta_1$, the solution of Eqn. (11) $\mathbf{u}(\theta_1)$ can be expanded in a complete basis as $\mathbf{u}(\theta_1) = \alpha_1^{(1)}\boldsymbol{\phi}_1 + \alpha_2^{(1)}\boldsymbol{\phi}_2 + \ldots \alpha_n^{(1)}\boldsymbol{\phi}_n$. Repeating this for $\theta_1, \theta_2, \ldots$ eventually the whole sample-space can be covered and it would be possible to expand $\mathbf{u}(\theta), \forall \theta \in \Theta$ as a linear combination of $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n$.

We use the eigenvectors $\boldsymbol{\phi}_k \in \mathbb{R}^n$ of the generalized eigenvalue problem

$$\mathbf{K}_0\boldsymbol{\phi}_k = \lambda_k \mathbf{M}_0\boldsymbol{\phi}_k; \quad k = 1, 2, \ldots n \tag{15}$$

Since the matrices $\mathbf{K}_0$ and $\mathbf{M}_0$ are symmetric and generally non-negative definite, the eigenvectors $\boldsymbol{\phi}_k$ for $k = 1, 2, \ldots n$ form a complete basis. Note that in principle any complete basis can be used. This choice is selected due to the analytical simplicity as will be seen later. For notational convenience, define the matrix of eigenvalues and eigenvectors

$$\boldsymbol{\lambda}_0 = \mathrm{diag}\left[\lambda_1, \lambda_2, \ldots, \lambda_n\right] \in \mathbb{R}^{n\times n} \quad \text{and} \quad \boldsymbol{\Phi} = \left[\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_n\right] \in \mathbb{R}^{n\times n} \tag{16}$$

Eigenvalues are ordered in the ascending order so that $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. We use the orthogonality property of the modal matrix $\boldsymbol{\Phi}$ as

$$\boldsymbol{\Phi}^T\mathbf{K}_0\boldsymbol{\Phi} = \boldsymbol{\lambda}_0, \quad \text{and} \quad \boldsymbol{\Phi}^T\mathbf{M}_0\boldsymbol{\Phi} = \mathbf{I} \tag{17}$$

Using these we have

$$\begin{aligned}\boldsymbol{\Phi}^T\mathbf{A}_0\boldsymbol{\Phi} &= \boldsymbol{\Phi}^T\left([-\omega^2 + i\omega\zeta_1]\mathbf{M}_0 + [i\omega\zeta_2 + 1]\mathbf{K}_0\right)\boldsymbol{\Phi} \\ &= \left(-\omega^2 + i\omega\zeta_1\right)\mathbf{I} + \left(i\omega\zeta_2 + 1\right)\boldsymbol{\lambda}_0 \qquad (18)\\ \text{which gives,} \quad \boldsymbol{\Phi}^T\mathbf{A}_0\boldsymbol{\Phi} &= \boldsymbol{\Lambda}_0 \quad \text{and} \quad \mathbf{A}_0 = \boldsymbol{\Phi}^{-T}\boldsymbol{\Lambda}_0\boldsymbol{\Phi}^{-1} \qquad (19)\end{aligned}$$

where $\boldsymbol{\Lambda}_0 = \left(-\omega^2 + i\omega\zeta_1\right)\mathbf{I} + \left(i\omega\zeta_2 + 1\right)\boldsymbol{\lambda}_0$ and $\mathbf{I}$ is the identity matrix. Hence, $\boldsymbol{\Lambda}_0$ can also be written as

$$\boldsymbol{\Lambda}_0 = \mathrm{diag}\left[\lambda_{0_1}, \lambda_{0_2}, \ldots, \lambda_{0_n}\right] \in \mathbb{C}^{n\times n} \tag{20}$$

where $\lambda_{0_j} = \left(-\omega^2 + i\omega\zeta_1\right) + \left(i\omega\zeta_2 + 1\right)\lambda_j$ and λ_j is as defined in Eqn. (16). We also introduce the transformations

$$\widetilde{\mathbf{A}}_i = \boldsymbol{\Phi}^T\mathbf{A}_i\boldsymbol{\Phi} \in \mathbb{C}^{n\times n}; i = 0, 1, 2, \ldots, M. \tag{21}$$

Note that $\widetilde{\mathbf{A}}_0 = \boldsymbol{\Lambda}_0$ is a diagonal matrix and

$$\mathbf{A}_i = \boldsymbol{\Phi}^{-T}\widetilde{\mathbf{A}}_i\boldsymbol{\Phi}^{-1} \in \mathbb{C}^{n\times n}; i = 1, 2, \ldots, M. \tag{22}$$

Suppose the solution of Eqn. (11) is given by

$$\hat{\mathbf{u}}(\omega, \theta) = \left[\mathbf{A}_0(\omega) + \sum_{i=1}^{M}\xi_i(\theta)\mathbf{A}_i(\omega)\right]^{-1}\mathbf{f}_0(\omega) \tag{23}$$

Using Eqs. (16)–(22) and the orthonormality of $\boldsymbol{\Phi}$ one has

$$\hat{\mathbf{u}}(\omega, \theta) = \left[\boldsymbol{\Phi}^{-T}\boldsymbol{\Lambda}_0(\omega)\boldsymbol{\Phi}^{-1} + \sum_{i=1}^{M}\xi_i(\theta)\boldsymbol{\Phi}^{-T}\widetilde{\mathbf{A}}_i\boldsymbol{\Phi}^{-1}\right]^{-1}\mathbf{f}_0(\omega) = \boldsymbol{\Phi}\boldsymbol{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right)\boldsymbol{\Phi}^T\mathbf{f}_0(\omega) \tag{24}$$

where

$$\boldsymbol{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \left[\boldsymbol{\Lambda}_0(\omega) + \sum_{i=1}^{M}\xi_i(\theta)\widetilde{\mathbf{A}}_i(\omega)\right]^{-1} \tag{25}$$

and the M-dimensional random vector

$$\boldsymbol{\xi}(\theta) = \{\xi_1(\theta), \xi_2(\theta), \ldots, \xi_M(\theta)\}^T \tag{26}$$

Now we separate the diagonal and off-diagonal terms of the $\widetilde{\mathbf{A}}_i$ matrices as

$$\widetilde{\mathbf{A}}_i = \boldsymbol{\Lambda}_i + \boldsymbol{\Delta}_i, \quad i = 1, 2, \ldots, M \tag{27}$$

Here the diagonal matrix

$$\boldsymbol{\Lambda}_i = \text{diag}\left[\widetilde{\mathbf{A}}_i\right] = \text{diag}\left[\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}\right] \in \mathbb{C}^{n\times n} \tag{28}$$

and the matrix containing only the off-diagonal elements $\boldsymbol{\Delta}_i = \widetilde{\mathbf{A}}_i - \boldsymbol{\Lambda}_i$ is such that $\text{Trace}\left(\boldsymbol{\Delta}_i\right) = 0$. Using these, from Eqn. (25) one has

$$\boldsymbol{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) = \left[\underbrace{\boldsymbol{\Lambda}_0(\omega) + \sum_{i=1}^{M} \xi_i(\theta)\boldsymbol{\Lambda}_i(\omega)}_{\boldsymbol{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right)} + \underbrace{\sum_{i=1}^{M} \xi_i(\theta)\boldsymbol{\Delta}_i(\omega)}_{\boldsymbol{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)}\right]^{-1} \tag{29}$$

where $\boldsymbol{\Lambda}\left(\omega, \boldsymbol{\xi}(\theta)\right) \in \mathbb{C}^{n\times n}$ is a diagonal matrix and $\boldsymbol{\Delta}\left(\omega, \boldsymbol{\xi}(\theta)\right)$ is an off-diagonal only matrix. In the subsequent expressions we choose to omit the inclusion of frequency dependence of the individual matrices for the sake of notational simplicity, so that $\boldsymbol{\Psi}\left(\omega, \boldsymbol{\xi}(\theta)\right) \equiv \boldsymbol{\Psi}\left(\boldsymbol{\xi}(\theta)\right)$ and so on. Hence, we rewrite Eqn. (29) as

$$\boldsymbol{\Psi}\left(\boldsymbol{\xi}(\theta)\right) = \left[\boldsymbol{\Lambda}\left(\boldsymbol{\xi}(\theta)\right)\left[\mathbf{I}_n + \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)\right]\right]^{-1} \tag{30}$$

The above expression can be represented using a Neumann type of matrix series [18] as

$$\boldsymbol{\Psi}\left(\boldsymbol{\xi}(\theta)\right) = \sum_{s=0}^{\infty}(-1)^s \left[\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)\right]^s \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right) \tag{31}$$

Taking an arbitrary r-th element of $\hat{\mathbf{u}}(\theta)$, Eqn. (24) can be rearranged to have

$$\hat{u}_r(\theta) = \sum_{k=1}^{n} \Phi_{rk}\left(\sum_{j=1}^{n} \Psi_{kj}\left(\boldsymbol{\xi}(\theta)\right)\left(\boldsymbol{\phi}_j^T \mathbf{f}_0\right)\right) \tag{32}$$

Defining

$$\Gamma_k\left(\boldsymbol{\xi}(\theta)\right) = \sum_{j=1}^{n} \Psi_{kj}\left(\boldsymbol{\xi}(\theta)\right)\left(\boldsymbol{\phi}_j^T \mathbf{f}_0\right) \tag{33}$$

and collecting all the elements in Eqn. (32) for $r = 1, 2, \ldots, n$ one has

$$\hat{\mathbf{u}}(\theta) = \sum_{k=1}^{n} \Gamma_k\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\phi}_k \tag{34}$$

This shows that the solution vector $\hat{\mathbf{u}}(\theta)$ can be projected in the space spanned by $\boldsymbol{\phi}_k$.

Now assume the series in Eqn. (31) is truncated after m-th term. We define the truncated function

$$\boldsymbol{\Psi}^{(m)}\left(\boldsymbol{\xi}(\theta)\right) = \sum_{s=0}^{m}(-1)^s \left[\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)\right]^s \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right) \tag{35}$$

From this one can obtain a sequence for different m

$$\hat{\mathbf{u}}^{(m)}(\theta) = \sum_{k=1}^{n} \Gamma_k^{(m)}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\phi}_k; \quad m = 1, 2, 3, \ldots \tag{36}$$

Since $\theta \in \Theta$ is arbitrary, comparing (11) and (23) we observe that $\hat{\mathbf{u}}^{(m)}(\theta)$ is the solution of Eqn. (11) for every θ when $m \to \infty$. This implies that

$$\text{Prob}\left\{\theta \in \Theta : \lim_{m\to\infty} \hat{\mathbf{u}}^{(m)}(\theta) = \hat{\mathbf{u}}(\theta)\right\} = 1 \tag{37}$$

Therefore, $\hat{\mathbf{u}}(\theta)$ is the solution of Eqn. (11) in probability. In this derivation, the probability density function of the random variables has not been used. Therefore, the random variables can be general as long as the solution exists.

Remark 1. The matrix power series in (31) is different from the classical Neumann series [18]. The classical Neumann series is a power series in $\mathbf{A}_0^{-1}\left[\boldsymbol{\Delta}\mathbf{A}\left(\boldsymbol{\xi}(\theta)\right)\right]$, where the first term is deterministic and the second term is random. The elements of this matrix series are polynomials in $\xi_i(\theta)$. In contrast, the series in (31) is in terms of $\left[\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\right]\left[\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)\right]$, where both terms are random. The elements of this matrix series are not simple polynomials in $\xi_i(\theta)$, but are in terms of a ratio of polynomials as seen in Eqn. (39). The convergence of this series depends of the spectral radius of

$$\mathbf{R} = \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right) \tag{38}$$

A generic term of this matrix can be obtained as

$$R_{rs} = \frac{\Delta_{rs}}{\Lambda_{rr}} = \frac{\sum_{i=1}^{M}\xi_i(\theta)\Delta_{i_{rs}}}{\lambda_{0_r} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_r}} = \frac{\sum_{i=1}^{M}\xi_i(\theta)\widetilde{A}_{i_{rs}}}{\lambda_{0_r} + \sum_{i=1}^{M}\xi_i(\theta)\widetilde{A}_{i_{rr}}}; r \neq s \tag{39}$$

Since $\mathbf{A}_0$ is positive definite, $\lambda_{0_r} > 0$ for all r. It can be seen from Eqn. (39) that the spectral radius of $\mathbf{R}$ is also controlled by the diagonal dominance of the $\widetilde{\mathbf{A}}_i$ matrices. If the diagonal terms are relatively larger than the off-diagonal terms, the series will converge faster even if the relative magnitude of λ_{0_r} is not large.

The series in (36) approaches to the exact solution of the governing Eqn. (11) for every $\theta \in \Theta$ for $m \to \infty$. For this reason it converges in probability 1. The convergence in probability 1 is a stronger convergence than, for example, the mean-square convergence often used in the stochastic finite element analysis. Since the convergence in probability 1 automatically implies the mean-square convergence, the series in Eqn. (34) is also a mean-square convergent series.

Definition 1. The functions $\Gamma_k\left(\boldsymbol{\xi}(\theta)\right), k = 1, 2, \ldots n$ are called the spectral functions as they are expressed in terms of the spectral properties of the coefficient matrix $\mathbf{K}_0$ and $\mathbf{M}_0$ arising in the discretized equation.

For certain class of problems the series in Eqn. (34) can give useful physical insights into the uncertainty propagation. For structural mechanics problems, the eigenvectors $\boldsymbol{\phi}_k$ are vibrational modes [34]. Eqn. (34) says that the response of a stochastic system is a linear combination of fundamental vibrational modes weighted by the random variables Γ_k.

3.2 Properties of the spectral functions

In this section we discuss some important properties of these functions. From the series expansion in Eqn. (31) we have

$$\boldsymbol{\Psi}\left(\boldsymbol{\xi}(\theta)\right) = \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right) - \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right) + \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right) + \dots \quad (40)$$

Since $\boldsymbol{\Lambda}\left(\boldsymbol{\xi}(\theta)\right)$ is a diagonal matrix, its inverse is simply a diagonal matrix containing the inverse of each of the diagonal elements. Also recall that the diagonal of $\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)$ contains only zeros. Different terms of the series in (40) can be obtained using a simple recursive relationship [18]. The numerical computation of the series is therefore computationally very efficient. For further analytical results, truncating the series upto different terms, we define spectral functions of different order.

Definition 2. The first-order spectral functions $\Gamma_k^{(1)}(\boldsymbol{\xi}(\theta)), k = 1, 2, \dots, n$ are obtained by retaining one term in the series (40).

Retaining one term in (40) we have

$$\boldsymbol{\Psi}^{(1)}\left(\boldsymbol{\xi}(\theta)\right) = \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right) \quad \text{or} \quad \Psi_{kj}^{(1)}\left(\boldsymbol{\xi}(\theta)\right) = \frac{\delta_{kj}}{\lambda_{0_k} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_k}} \quad (41)$$

Using the definition of the spectral function in Eqn. (33), the first-order spectral functions can be explicitly obtained as

$$\Gamma_k^{(1)}\left(\boldsymbol{\xi}(\theta)\right) = \sum_{j=1}^{n}\Psi_{kj}^{(1)}\left(\boldsymbol{\xi}(\theta)\right)\left(\boldsymbol{\phi}_j^T\mathbf{f}_0\right) = \frac{\boldsymbol{\phi}_k^T\mathbf{f}_0}{\lambda_{0_k} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_k}} \quad (42)$$

From this expression it is clear that $\Gamma_k^{(1)}\left(\boldsymbol{\xi}(\theta)\right)$ are correlated non-Gaussian random variables. Since we assumed that all eigenvalues λ_{0_k} are distinct, every $\Gamma_k^{(1)}\left(\boldsymbol{\xi}(\theta)\right)$ in Eqn. (42) are different for different values of k.

Definition 3. The second-order spectral functions $\Gamma_k^{(2)}(\boldsymbol{\xi}(\theta)), k = 1, 2, \dots, n$ are obtained by retaining two terms in the series (40).

Retaining two terms in (40) we have

$$\boldsymbol{\Psi}^{(2)}\left(\boldsymbol{\xi}(\theta)\right) = \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right) - \boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Delta}\left(\boldsymbol{\xi}(\theta)\right)\boldsymbol{\Lambda}^{-1}\left(\boldsymbol{\xi}(\theta)\right) \quad (43)$$

$$\text{or} \quad \Psi_{kj}^{(2)}\left(\boldsymbol{\xi}(\theta)\right) = \frac{\delta_{kj}}{\lambda_{0_k} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_k}} - \frac{\sum_{i=1}^{M}\xi_i(\theta)\Delta_{i_{kj}}}{\left(\lambda_{0_k} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_k}\right)\left(\lambda_{0_j} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_j}\right)} \quad (44)$$

Using the definition of the spectral function in Eqn. (33), the second-order spectral functions can be obtained in closed-form as

$$\Gamma_k^{(2)}\left(\boldsymbol{\xi}(\theta)\right) = \frac{\boldsymbol{\phi}_k^T\mathbf{f}_0}{\lambda_{0_k} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_k}} - \sum_{\substack{j=1\\ j\neq k}}^{n}\frac{\left(\boldsymbol{\phi}_j^T\mathbf{f}_0\right)\sum_{i=1}^{M}\xi_i(\theta)\Delta_{i_{kj}}}{\left(\lambda_{0_k} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_k}\right)\left(\lambda_{0_j} + \sum_{i=1}^{M}\xi_i(\theta)\lambda_{i_j}\right)} \quad (45)$$

The second-order function can be viewed as adding corrections to the first-order expression derived in Eqn. (42).

Definition 4. The vector of spectral functions of order s can be obtained by retaining s terms in the series (40) and can be expressed as

$$\boldsymbol{\Gamma}^{(s)}(\boldsymbol{\xi}(\theta)) = \left[\mathbf{I}_n - \mathbf{R}(\boldsymbol{\xi}(\theta)) + \mathbf{R}(\boldsymbol{\xi}(\theta))^2 - \mathbf{R}(\boldsymbol{\xi}(\theta))^3 \ldots s^{\text{th term}}\right] \boldsymbol{\Gamma}^{(1)}(\boldsymbol{\xi}(\theta)) \tag{46}$$

where $\mathbf{I}_n$ is the n-dimensional identity matrix and $\mathbf{R}$ is defined in Eqn. (38) as $\mathbf{R}(\boldsymbol{\xi}(\theta)) = [\boldsymbol{\Lambda}^{-1}(\boldsymbol{\xi}(\theta))][\boldsymbol{\Delta}(\boldsymbol{\xi}(\theta))]$. Different terms of this series can be obtained recursively from the previous term [18].

4 Error minimization using the Galerkin approach

In subsection 3.1 we derived the spectral functions such that a projection in an orthonormal basis converges to the exact solution in probability 1. The spectral functions are expressed in terms of a convergent infinite series. First, second and higher order spectral functions obtained by truncating the infinite series have been derived. We have also showed that they have the same functional form as the exact solution of Eqn. (11). This motivates us to use these functions as 'trial functions' to construct the solution. The idea is to minimize the error arising due to the truncation. A Galerkin approach is proposed where the error is made orthogonal to the spectral functions.

We express the solution vector by the series representation

$$\hat{\mathbf{u}}(\theta) = \sum_{k=1}^{n} c_k \widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k \tag{47}$$

Here the functions $\widehat{\Gamma}_k : \mathbb{C}^M \to \mathbb{C}$ are the spectral functions and the constants $c_k \in \mathbb{C}$ need to be obtained using the Galerkin approach. The functions $\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))$ can be the first-order (42), second-order (45) or any higher-order spectral functions (46) and $\boldsymbol{\phi}_k$ are the eigenvectors introduced earlier in Eqn. (15). Substituting the expansion of $\hat{\mathbf{u}}(\theta)$ in the governing equation (11), the error vector can be obtained as

$$\boldsymbol{\varepsilon}(\theta) = \left(\sum_{i=0}^{M} \mathbf{A}_i \xi_i(\theta)\right)\left(\sum_{k=1}^{n} c_k \widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k\right) - \mathbf{f}_0 \in \mathbb{C}^n \tag{48}$$

where $\xi_0 = 1$ is used to simplify the first summation expression. The expression (47) is viewed as a projection where $\left\{\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k\right\} \in \mathbb{C}^n$ are the basis functions and c_k are the unknown constants to be determined. We wish to obtain the coefficients c_k using the Galerkin approach so that the error is made orthogonal to the basis functions, that is, mathematically

$$\boldsymbol{\varepsilon}(\theta) \perp \left(\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_j\right) \quad \text{or} \quad \left\langle \widehat{\Gamma}_j(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_j, \boldsymbol{\varepsilon}(\theta)\right\rangle = 0 \quad \forall\ j = 1, 2, \ldots, n \tag{49}$$

Here $\langle \mathbf{u}(\theta), \mathbf{v}(\theta)\rangle = \int_\Theta P(\mathrm{d}\theta)\mathbf{u}(\theta)\mathbf{v}(\theta)$ defines the inner product norm. Imposing this condition and using the expression of $\boldsymbol{\varepsilon}(\theta)$ from Eqn. (48) one has

$$\mathrm{E}\left[\left(\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_j\right)^T \left(\sum_{i=0}^{M} \mathbf{A}_i \xi_i(\theta)\right)\left(\sum_{k=1}^{n} c_k \widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k\right) - \left(\widehat{\Gamma}_j(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_j\right)^T \mathbf{f}_0\right] = 0\ \forall\ j \tag{50}$$

Interchanging the $\mathrm{E}[\bullet]$ and summation operations, this can be simplified to

$$\sum_{k=1}^{n}\left(\sum_{i=0}^{M}\left(\boldsymbol{\phi}_j^T\mathbf{A}_i\boldsymbol{\phi}_k\right)\mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_j^T(\boldsymbol{\xi}(\theta))\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right]\right)c_k=\mathrm{E}\left[\widehat{\Gamma}_j^T(\boldsymbol{\xi}(\theta))\right]\left(\boldsymbol{\phi}_j^T\mathbf{f}_0\right) \quad (51)$$

$$\text{or} \quad \sum_{k=1}^{n}\left(\sum_{i=0}^{M}\widetilde{A}_{i_{jk}}D_{ijk}\right)c_k=b_j \quad (52)$$

Defining the vector $\mathbf{c}=\{c_1,c_2,\ldots,c_n\}^T$, these equations can be expressed in a matrix form as

$$\mathbf{S}\,\mathbf{c}=\mathbf{b} \quad (53)$$

with

$$S_{jk}=\sum_{i=0}^{M}\widetilde{A}_{i_{jk}}D_{ijk};\quad \forall\ j,k=1,2,\ldots,n \quad (54)$$

where

$$\widetilde{A}_{i_{jk}}=\boldsymbol{\phi}_j^T\mathbf{A}_i\boldsymbol{\phi}_k, \quad (55)$$

$$D_{ijk}=\mathrm{E}\left[\xi_i(\theta)\widehat{\Gamma}_j^T(\boldsymbol{\xi}(\theta))\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right] \quad (56)$$

$$\text{and}\quad b_j=\mathrm{E}\left[\widehat{\Gamma}_j^T(\boldsymbol{\xi}(\theta))\right]\left(\boldsymbol{\phi}_j^T\mathbf{f}_0\right). \quad (57)$$

Higher order spectral functions can be used to improve the accuracy and convergence of the series (47). This will be demonstrated in the numerical examples later in the paper.

Remark 2. (Comparison with the classical spectral SFEM) We compare this Galerkin approach with the classical spectral stochastic finite element approach for further insight. The number of equations to be solved for the unknown coefficients in Eqn. (53) is n, the same dimension as the original governing equation (11). There are only n unknown constants, as opposed to nP unknown constants arising in the polynomial chaos expansion. The coefficient matrix **S** and the vector **b** in Eqn. (53) should be obtained numerically using the Monte Carlo simulation or other numerical integration technique. In the classical PC expansion, however, the coefficient matrix and the associated vector are obtained exactly in closed-form. In addition, the coefficient matrix is a sparse matrix whereas the matrix **S** in Eqn. (53) is in general a fully populated matrix.

It can be observed that the matrix **S** in Eqn. (53) is symmetric. Therefore, one need to determine $n(n+1)/2$ number of coefficients by numerical methods. Any numerical integration method, such as the Gaussian quadrature method, can be used to obtain the elements of D_{ijk} and b_j in Eqn. (55). In this paper Monte Carlo simulation is used. The samples of the spectral functions $\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))$ can be simulated from Eqn. (42), (45) or (46) depending on the order. These can be used to compute D_{ijk} and b_j from Eqn. (55). The simulated spectral functions can also be 'recycled' to obtain the statistics and probability density function (pdf) of the solution. In summary, compared to the classical spectral stochastic finite element method, the proposed Galerkin approach results in a smaller size matrix but requires numerical integration techniques to obtain its entries. The numerical method proposed here therefore can be considered as a hybrid analytical-simulation approach.

5 Model reduction using a reduced number of basis

The Galerkin approach proposed in the previous section requires the solution of $n \times n$ algebraic equations. Although in general this is smaller compared to the polynomial chaos approach, the computational cost can still be high for large n as the coefficient matrix is in general a dense matrix. The aim of this section is to reduce it further so that, in addition to large number of random variables, problems with large degrees of freedom can also be solved efficiently.

Suppose the eigenvalues of $\mathbf{A}_0$ are arranged in an increasing order such that

$$\lambda_{0_1} < \lambda_{0_2} < \ldots < \lambda_{0_n} \tag{58}$$

From the expression of the spectral functions observe that the eigenvalues appear in the denominator:

$$\Gamma_k^{(1)}\left(\boldsymbol{\xi}(\theta)\right) = \frac{\boldsymbol{\phi}_k^T \mathbf{f}_0}{\lambda_{0_k} + \sum_{i=1}^{M} \xi_i(\theta)\lambda_{i_k}} \tag{59}$$

The numerator $(\boldsymbol{\phi}_k^T \mathbf{f}_0)$ is the projection of the force on the deformation mode. Since the eigenvalues are arranged in an increasing order, the denominator of $|\Gamma_{k+r}^{(1)}\left(\boldsymbol{\xi}(\theta)\right)|$ is larger than the denominator of $|\Gamma_k^{(1)}\left(\boldsymbol{\xi}(\theta)\right)|$ according a suitable measure. The numerator $(\boldsymbol{\phi}_k^T \mathbf{f}_0)$ depends on the nature of forcing and the eigenvectors. Although this quantity is deterministic, in general an ordering cannot be easily established for different values of k. Because all the eigenvectors are normalized to unity, it is reasonable to consider that $(\boldsymbol{\phi}_k^T \mathbf{f}_0)$ does not vary significantly for different values of k. Using the ordering of the eigenvalues, one can select a small number ϵ such that $\lambda_1/\lambda_q < \epsilon$ for some value of q, where λ_j is the eigenvalue of the generalized eigenvalue problem defined in Eqn. (15). Based on this, we can approximate the solution using a truncated series as

$$\hat{\mathbf{u}}(\theta) \approx \sum_{k=1}^{q} c_k \widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\boldsymbol{\phi}_k \tag{60}$$

where c_k, $\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))$ and $\boldsymbol{\phi}_k$ are obtained following the procedure described in the previous section by letting the indices j, k only upto q in Eqs. (54) and (55). The accuracy of the series (60) can be improved in two ways, namely, (a) by increasing the number of terms q, or (b) by increasing the order of the spectral functions $\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))$.

Model reduction techniques have been widely used within the scope of proper orthogonal decomposition (POD) method [35–37]. Here the eigenvalues of a symmetric positive definite matrix (the covariance matrix of a snapshot the system response) are used for model reduction. In spite of this similarity, the reduction method proposed here is different from a POD since it only considers the operator and not the solution itself. Reduction based on eigen-solution is of classical nature in various areas of applied mathematics, engineering and physics and extensive studies exist on this topic. It should be noted that the truncation in series (60) introduces errors. A rigorous mathematical quantification of error arising due to this truncation is beyond the scope of this article. The ratio of the eigenvalues λ_1/λ_q gives a good indication, but the projection of the force on the eigenvector $(\boldsymbol{\phi}_k^T \mathbf{f}_0)$ is also of importance. Since this quantity is problem dependent, care should be taken while applying this reduction method.

6 Post processing: Moments of the solution

For the practical application of the method developed here, the efficient computation of the response moments and pdf is of crucial importance. A simulation based algorithm is proposed

in this section. The coefficients c_k in Eqn. (51) can be calculated from a reduced set of equations given by (53). The reduced equations can be obtained by letting the indices j, k upto $q < n$ in Eqns. (54) and (55). After obtaining the coefficient vector $\mathbf{c} \in \mathbb{C}^q$, the statistical moments of the solution can be obtained from Eqns. (61) and (62) using the Monte Carlo simulation. The spectral functions used to obtain the vector $\mathbf{c}$ itself, can be reused to obtain the statistics and pdf of the solution. The mean vector can be obtained as

$$\bar{\mathbf{u}} = \mathrm{E}\left[|\hat{\mathbf{u}}(\theta)|\right] = \sum_{k=1}^{q} |c_k| \mathrm{E}\left[\left|\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right|\right] \boldsymbol{\phi}_k \tag{61}$$

where $|\bullet|$ is the absolute value of the complex quantities. The covariance of the solution vector can be expressed as

$$\boldsymbol{\Sigma}_u = \mathrm{E}\left[\left(|\hat{\mathbf{u}}(\theta)| - \bar{\mathbf{u}}\right)\left(|\hat{\mathbf{u}}(\theta)| - \bar{\mathbf{u}}\right)\right] = \sum_{k=1}^{q}\sum_{j=1}^{q} |c_k c_j| \, \Sigma_{\Gamma_{kj}} \boldsymbol{\phi}_k \boldsymbol{\phi}_j \tag{62}$$

where the elements of the covariance matrix of the spectral functions are given by

$$\Sigma_{\Gamma_{kj}} = \mathrm{E}\left[\left(\left|\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right| - \mathrm{E}\left[\left|\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right|\right]\right)\left(\left|\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right| - \mathrm{E}\left[\left|\widehat{\Gamma}_k(\boldsymbol{\xi}(\theta))\right|\right]\right)\right] \tag{63}$$

Based on the results derived in the paper, a hybrid reduced simulation-analytical approach can thus be realized in practice. The method is applicable to general structural dynamics problems with general non-Gaussian random fields. In the following section this approach has been applied to a physical problem.

7 Illustrative application: The stochastic dynamics of an Euler-Bernoulli beam

In this section we apply the computational method to a cantilever beam with stochastic bending modulus. Fig. (1) shows the configuration of the cantilever beam with a harmonic point load at its free end. We assume that the bending modulus is a homogeneous stationary Gaussian ran-

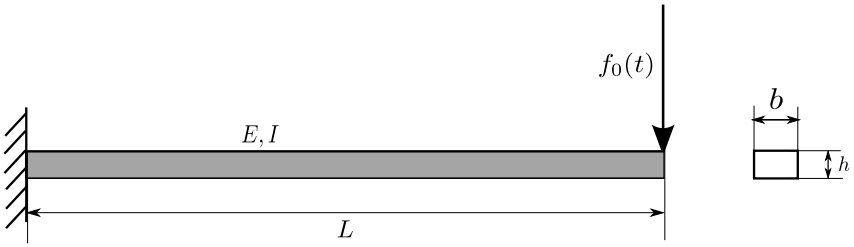


Figure 1: Schematic diagram of the Euler-Bernoulli beam with a point load at the free end.

dom field of the form

$$EI(x,\theta) = EI_0(1 + a(x,\theta)) \tag{64}$$

where x is the coordinate along the length of the beam, EI_0 is the estimate of the mean bending modulus, $a(x,\theta)$ is a zero mean stationary Gaussian random field. The autocorrelation function of this random field is assumed to be

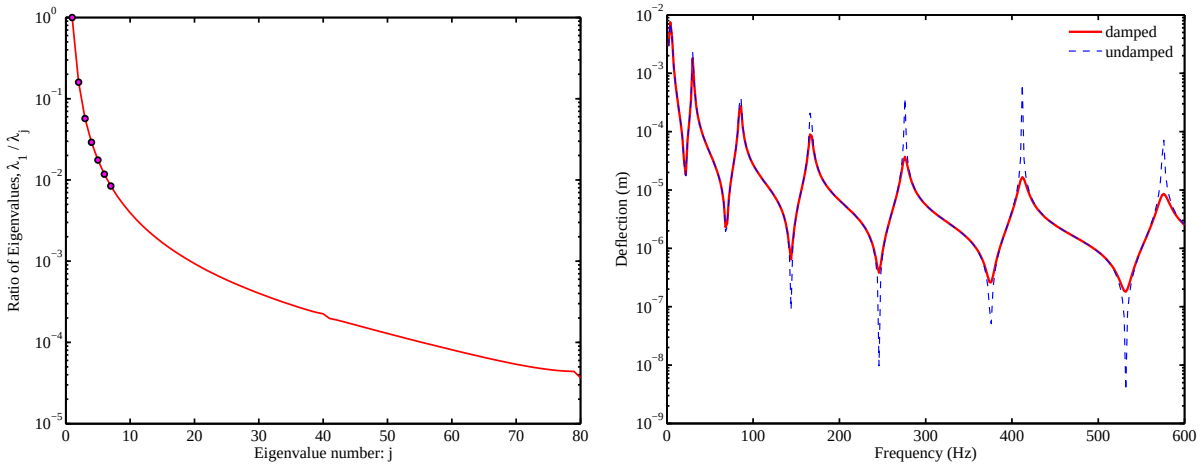
$$C_a(x_1, x_2) = \sigma_a^2 e^{-(|x_1 - x_2|)/\mu_a} \tag{65}$$

where μ_a is the correlation length and σ_a is the standard deviation. We use the base-line parameters as the length $L = 1m$, cross-section $(b \times h)$ $39 \times 5.93mm^2$ and Young's modulus $E = 2 \times 10^{11}$ Pa. In study we consider deflection of the tip of the beam under harmonic loads of amplitude $\tilde{f}_0 = 1.0N$. The correlation length considered in this numerical study is $\mu_a = L/2$. The number of terms retained (M) in the Karhunen-Loève expansion (8) is selected such that $\nu_M/\nu_1 = 0.01$ in order to retain 90% of the variability. For this correlation length the number of terms M comes to 18. For the finite element discretization, the beam is divided into 40 elements. Standard four degrees of freedom Euler-Bernoulli beam model is used [38]. After applying the fixed boundary condition at one edge, we obtain the number of degrees of freedom of the model to be $n = 80$.

7.1 Results

The proposed method has been compared with a direct Monte Carlo Simulation (MCS), where both have been performed with $10,000$ samples. For the direct MCS, Eqn. (23) is solved for each sample and the mean and standard deviation is derived by assembling the responses. The calculations have been performed for all the four values of σ_a to simulate increasing uncertainty. This is done to check the accuracy of the proposed method against the direct MCS results for varying degrees of uncertainty.

Fig. 2(a) presents the ratio of the eigenvalues of the generalized eigenvalue problem (15) for which the ratio of the eigenvalues is taken with the first eigenvalue. We choose the reduced basis of the problem based on $\lambda_1/\lambda_q < \epsilon$, where $\epsilon = 0.01$, and they are highlighted in Fig. 2(b) shows the frequency domain response of the deterministic system for both damped and undamped conditions. We have applied a constant modal damping matrix with 1% damping



(a) Ratio of eigenvalues of the generalized eigenvalue problem.

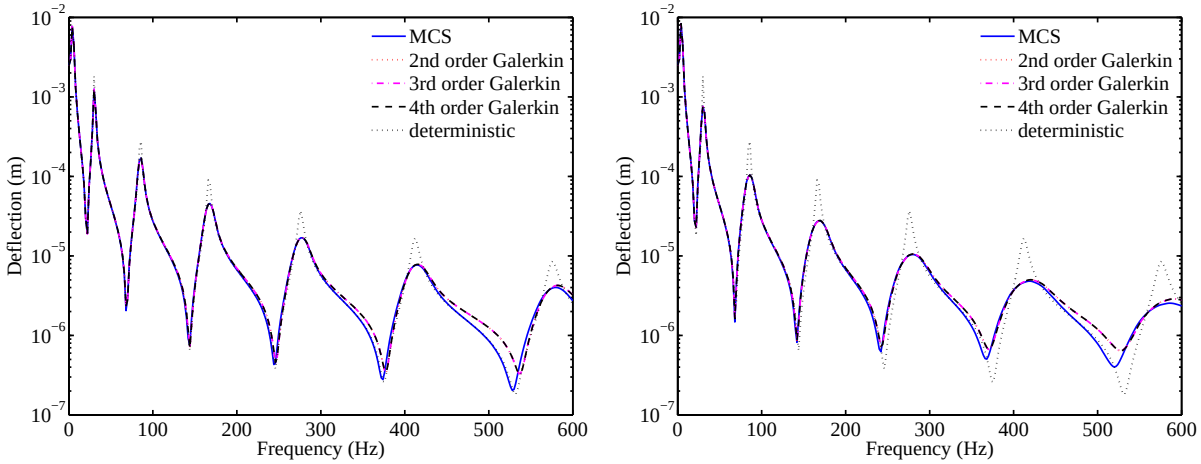
(b) Frequency domain reponse of the tip of the beam under point load for the undamped and damped conditions (constant modal damping)

Figure 2: The eigenvalues of the generalized eigenvalue problem involving the mass and stiffness matrices given in Eqn. 15. For $\epsilon = 0.01$, the number of reduced eigenvectors $q = 7$ such that $\lambda_1/\lambda_j < \epsilon$.

factor for all the modes. Here the mass and damping matrices are assumed to be deterministic in nature. However, the proposed theoretical approach is general and equally applicable for random mass, stiffness and damping matrices. The frequency range of interest for the present study is $0 - 600$ Hz with an interval of 2 Hz. In Fig. 2(b), the tip deflection is shown on a log

scale for a unit amplitude harmonic force input. The resonance peak amplitudes of the response of the undamped system definitely depends on the frequency resolution of the plot.

The frequency response of the mean deflection of the tip of the beam is shown in Fig. 3 for the cases for cases of $\sigma_a = \{0.05, 0.10, 0.15, 0.20\}$. The figures show a comparison of the direct MCS simulation results with different orders of the solution following Eqn. (31), where the orders $s = 2, 3, 4$. A very good agreement between the MCS simulation and the proposed



(a) Beam deflection for $\sigma_a = 0.05$.

(b) Beam deflection for $\sigma_a = 0.1$.

(c) Beam deflection for $\sigma_a = 0.15$.

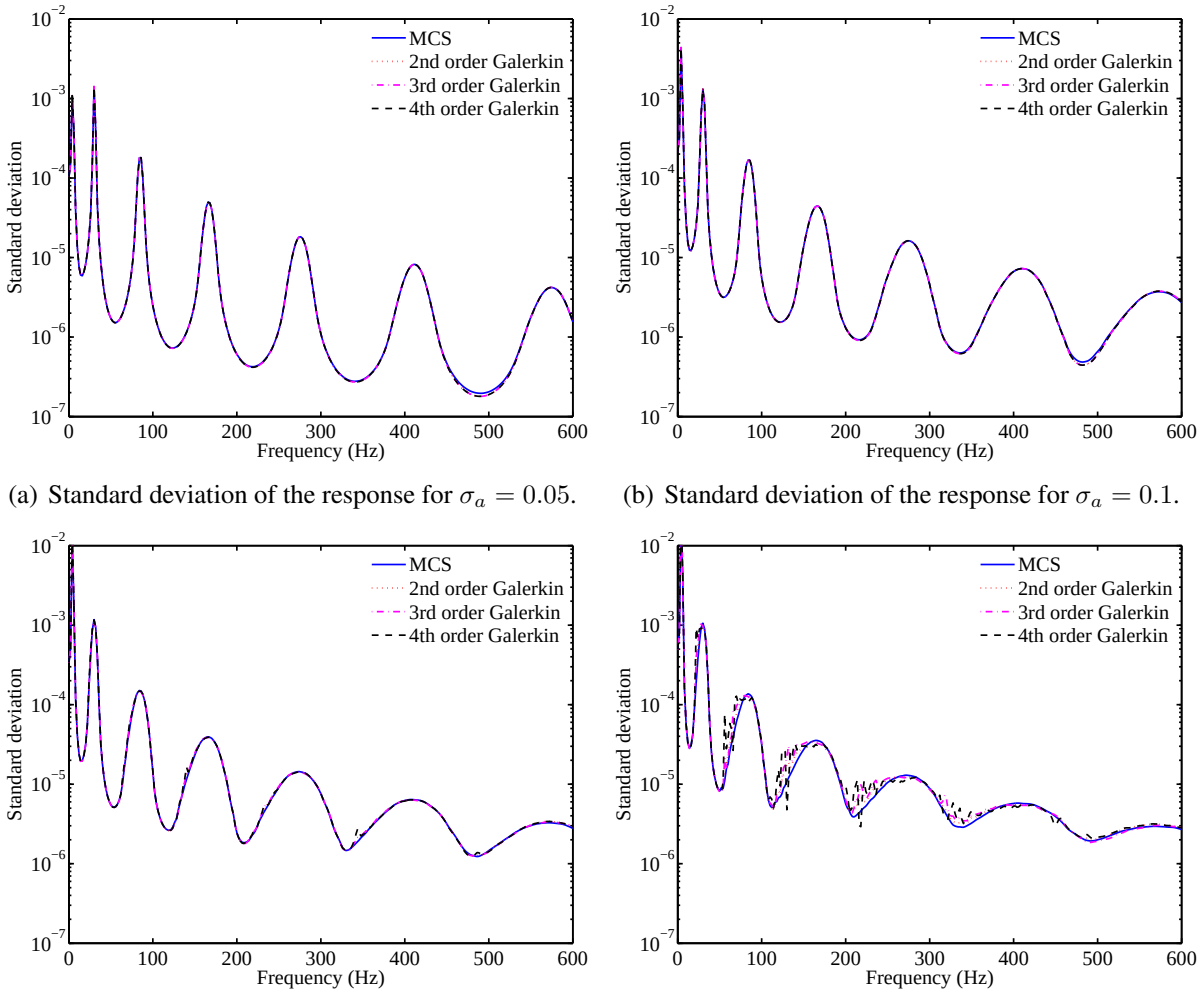
(d) Beam deflection for $\sigma_a = 0.2$.

Figure 3: The frequency domain response of the deflection of the tip of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.05, 0.10, 0.15, 0.20\}$. The proposed Galerkin approach needs solution of a 7×7 linear system of equations only.

spectral approach can be observed in the figures. All the results have been compared with the response of the deterministic system which shows that the uncertainty has an added damping effect at the resonance peaks. This can be explained by the fact that the parametric variation of the beam, results in its peak response for the different samples to get distributed around the resonance frequency zones instead of being concentrated at a particular frequency, and when the subsequent averaging is applied, it smooths out the response peaks to a fair degree. The same explanation holds for the anti-resonance frequencies. It can also be observed that increased variability of the parametric uncertainties (as is represented by the increasing value of σ_a) results

in an increase of this added damping effect which is consistent with the previous explanation.

The standard deviation of the frequency domain response of the tip deflection for different spectral order of solution of the reduced basis approach is compared with the direct MCS and is shown in Fig. 4, for different values of σ_a. We find that the standard deviation is maximum



(a) Standard deviation of the response for $\sigma_a = 0.05$.

(b) Standard deviation of the response for $\sigma_a = 0.1$.

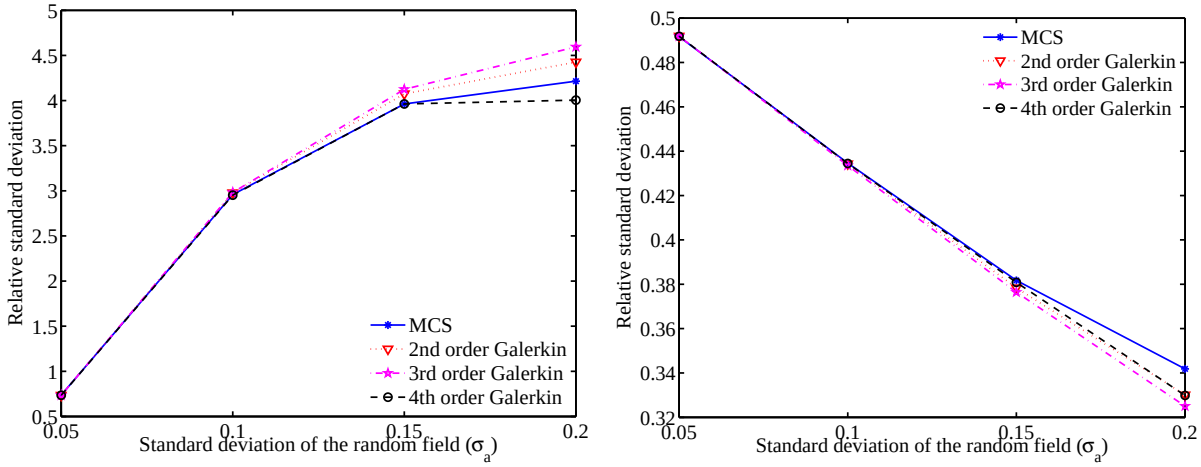
(c) Standard deviation of the response for $\sigma_a = 0.15$.

(d) Standard deviation of the response for $\sigma_a = 0.2$.

Figure 4: The standard deviation of the tip deflection of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ sample MCS and for $\sigma_a = \{0.05, 0.10, 0.15, 0.20\}$.

at the resonance frequencies which is expected due to the differences in the resonance peak of each sample. It is again observed that the direct MCS solution and the reduced order approach give almost identical results, which demonstrate the effectiveness of the proposed approach.

Fig. 5 shows the relative standard deviation of the response of the beam at two frequencies 154 Hz and 412 Hz, which correspond to the anti-resonance and resonance frequencies of the cantilever beam respectively. The relative standard deviation values have been obtained for a set of 4 values of σ_a, which represents the different degrees of variability of the system uncertainty. The results obtained with the Galerkin approach for the different order of spectral functions have been compared to the direct MCS, and a good agreement between the two results have been obtained. It is interesting to point out here that the standard deviation decreases with the values of σ_a for the anti-resonance frequency while it increases for the resonance frequencies.



(a) Relative standard deviation at 154 Hz. (b) Relative standard deviation at 412 Hz.

Figure 5: The relative standard deviation of the deflection of the tip (which has been obtained by normalizing the value of the standard deviation with the deterministic system response at the same frequency) versus the standard deviation (σ_a) of the random field of the Euler-Bernoulli beam under unit amplitude harmonic point load at the free end. The two frequencies, namely 154Hz and 412Hz are considered. They correspond to off-resonance and resonance frequencies respectively. The plots are shown for 4 different values of $\sigma_a = \{0.05, 0.10, 0.15, 0.20\}$ and calculated with $10,000$ random samples.

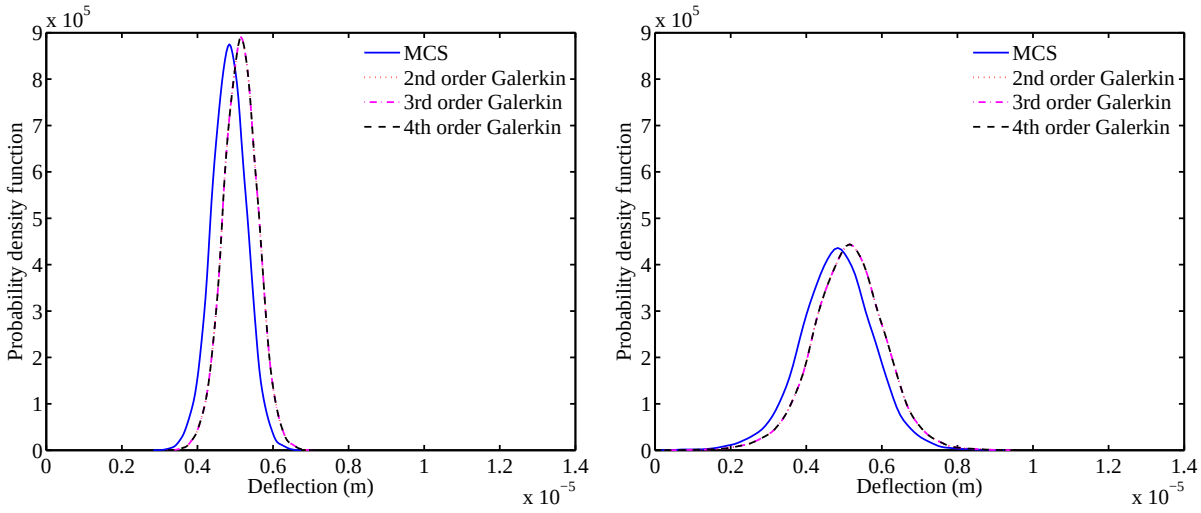
This is consistent with the results shown in Fig. 4 which shows that an increased value of the variance of the random field has the effect of an increasing added damping on the system, when an averaging is done over the sample space. Thus the resonance response is expected to reduce with the increased variability of the random field while the anti-resonance response will increase.

The probability density function of the deflection of the tip of the cantilever beam for different degrees of variability of the random field is shown in Fig. 6. The probability density functions have been calculated at the frequency of 412 Hz, which is a resonance frequency of the beam. The results indicate that with the increase in the degree of uncertainty (variance) of the system, the lower values of deflection has a higher probability which is absolutely consistent with the standard deviation curve shown in Fig. 5(a) and the comparison of the mean deflection of the stochastic system with the deterministic response in Fig. 3. This shows that the increase in the variability of the stochastic system has a damping effect on the response.

The results establish the applicability of this spectral reduced basis method with Galerkin error minimization technique as a satisfactory working model for providing solution of the stochastic structural systems. The method is found to be consistent with the direct MCS approach, while being much more computationally efficient than the latter.

8 Conclusions

We have considered the discretized stochastic partial differential equation for structural dynamic systems with generally non-Gaussian random fields. In the classical spectral stochastic finite element approach, the solution is projected into an infinite dimensional orthonormal basis functions and the associated constant vectors are obtained using the Galerkin type of error minimization approach. Here an alternative approach in the frequency domain is proposed. The solution is projected into a finite dimensional reduced vector basis and the associated complex coefficient functions are obtained. The coefficient functions, called as the *spectral functions*, are



(a) PDF of the response at 210Hz for $\sigma_a = 0.05$.

(b) PDF of the response at 210Hz for $\sigma_a = 0.1$.

(c) PDF of the response at 210Hz for $\sigma_a = 0.15$.

(d) PDF of the response at 210Hz for $\sigma_a = 0.2$.

Figure 6: The probability density function (PDF) of the tip deflection of the Euler-Bernoulli beam at 210 Hz under unit amplitude harmonic point load at the free end. The response is obtained with $10,000$ samples and for $\sigma_a = \{0.05, 0.10, 0.15, 0.20\}$.

expressed in terms of the spectral properties of the mass and stiffness matrices appearing in the discretized governing equation. It is shown that then the resulting series converges to the exact solution in probability 1. This is a stronger convergence compared to the classical polynomial chaos which converges in the mean-square sense in the Hilbert space.

Using the spectral functions, a Galerkin error minimization approach in the complex domain has been developed in a reduced vector basis. It is shown that the number of unknown constants can be obtained by solving a system of linear equations which have a dimension much smaller than the dimension of the original discretized equation. A simple numerical approach to obtain the reduced dimension has been suggested based on the ratio of the eigenvalues of the stiffness matrix corresponding to the baseline model. A numerical approach using a general-order spectral function has been developed. Based on these, a hybrid analytical-simulation approach is proposed to obtain the statistical properties of the solution.

The computational efficiency of the proposed reduced spectral approach has been demon-

strated for large linear systems with non-Gaussian random variables. It may be possible to extend the underlying idea to the class of non-linear problems. For example, the proposed spectral approach can be used for every linearisation step or every time step. Further research is necessary in this direction.

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