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A COMPARATIVE STUDY OF UNCERTAINTY PROPAGATION METHODS IN STRUCTURAL PROBLEMS

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Abstract. Several uncertainty propagation algorithms are available in literature: (i) Monte-Carlo simulations based on response surfaces, (ii) approximate uncertainty propagation algorithms and (iii) non probabilistic algorithms. All of these approaches are based on some a priori assumptions about the nature of design variables uncertainty and on the models and systems behavior. Some of these assumptions could misrepresent the original problem and, consequently, could yield to erroneous design solutions, in particular where the prior information is poor or inexistent (complete ignorance). Therefore, when selecting a method to solve an uncertainty based design problem, several aspects should be considered: prior assumptions, non-linearity of the performance function, number of input random variables and required accuracy. It could be useful to develop some guidelines to choose an appropriate method for a specific situation.

In the present work some classical structural problems will be studied in order to investigate which probabilistic approach, in terms of accuracy and computational cost, better propagates the uncertainty from input to output data. The methods under analysis will be: Univariate Dimension Reduction methods, Polynomial Chaos Expansion, First-Order Second Moment method, and algorithms based on the Evidence theory for epistemic uncertainty. The performances of these methods will be compared in terms of moment estimations and probability density function construction corresponding to several scenarios of reliability based design and robust design. The structural problems presented will be: (1) the static, dynamic and buckling behavior of a composite plate, (2) the reconstruction of the deformed shape of a structure from measured surface strains.

1 INTRODUCTION

The design for reliability, as well as robust design, is phased in over last decades in the structural design. Although these concepts are well-known in many engineering fields, the high computational cost of the mathematical approaches needed to perform these kinds of analysis, have set back their application in the aerospace structural design. Although in this last field, the problems that deal with the input variable uncertainties are known since the beginning of the aviation history, they are coped with deterministic methods based on the safety factor approach. The diffusion of components based on composite materials, in secondary and primary aerospace structures, and the dropping of aerospace and aviation companies' profit have reawaken the interest in design philosophies that deal with the uncertainty in a more effective way. For this reason, mathematicians and researchers have been urged on the study of new numerical approaches for an accurate Uncertainty Propagation (UP) from input to output data. Traditionally, both the reliability and the robustness of a design configuration have been studied using the Montecarlo simulation; although it is the most accurate method, its computational cost could be prohibitive. For this reason several alternative approaches have been developed to face UP.

Most of the available UP algorithms have particular characteristics that make them appropriate for some specific problems but their capabilities are not fully exploited in all kinds of applications. First of all it is possible to distinguish between algorithms for the study of aleatory uncertainty and approaches that deal with the epistemic uncertainty. This classification can be based on the prior hypotheses needed to simulate the prior uncertainty. In order to model the epistemic uncertainty by means of probabilistic (aleatory) algorithms, some prior hypotheses should be adopted to transform the epistemic information into a probability distribution function (epistemic algorithms do not need such assumptions). On the other hand, a probabilistic problem may be studied by means of an epistemic algorithm if the prior probability density functions are transformed into set-based information.

The UP algorithms based on the probability theory are usually classified into five categories [1]: 1) Simulation based methods: these techniques are based on the simulation of the problem in proper trial points, selected according to the stochastic characteristics of the input variables. MonteCarlo Simulation (MCS) is certainly the most known and used of these methods. 2) Local expansion based methods: these algorithms, also known as perturbation methods, are based on the local series expansion of output functions in terms of input random parameters. The methods based on Taylor expansion, such as the FOSM (First Order Second Moment) or the SOSM (Second Order Second Moment) methods, belong to this class. 3) Most Probable Points (MPP) based methods: this class includes the First and Second Order Reliability Methods (FORM and SORM, respectively). 4) Functional expansion based methods: they rely on a stochastic expansion of the performance function. The most known method of this class is the Polynomial Chaos Expansion (PCE). 5) Numerical integration based methods: these techniques are based on the numerical solving of integral equations for the statistical moments. These methods don't yield directly the performance joint probability function, but the corresponding statistical moments; by using the Pearson System and knowing the first four statistical moments, the probability distribution function can be obtained.

Several factors affect the choice of a suitable UP approach: (i) the identification and the classification of the input uncertainty, (ii) the definition of the required outputs (the first two statistical moments in robust design and the probability density function or the most probable points in a reliability based analysis),(iii) and the mathematical characteristics of the studied model (if the first order interactions cannot be neglected the Univariate Dimension Reduction method does not yield accurate prediction while the performance function is non linear the methods based in Taylor local expansion are not accurate). This last information can be obtained using some numerical tools, such as the sensitivity analysis.

The main objective of this work is a comparative study of some of the most common and newest UP algorithms for both aleatory and epistemic uncertainties. As far as the first ones, the limits and merits of the Univariate Dimension Reduction method (UDR), of the Polynomial Chaos Expansion (PCE), and of the First Order Second Moments algorithm (FOSM) will be analyzed and discussed. These methods will be tested and compared on some numerical test functions and a classical structural problem: the probabilistic study of static, dynamic and buckling behavior of a composite plate. The sensitivity analysis has been performed in order to study the mathematical characteristics of the model. In the second part of the present work a probabilistic approach based on the UDR is compared with an epistemic approach based on the evidence theory. The structural application used as a test case for the comparison is an inverse problem: reconstruction of the deformed shape of a beam from measured surface strains using the inverse Finite elements Method (iFEM) [2][3].

2 UNCERTAINTY PROPAGATION ALGORITHMS

In this section, a review of some uncertainty propagation algorithms will be presented in order to set the framework for the assessment and comparison, through some structural applications, discussed in Sec. 3.

2.1 The Univariate Dimension Reduction method (UDR)

This method involves an additive decomposition of a multidimensional integral function to multiple one-dimensional integral functions. The technique is suitable for calculating the stochastic moments of a system response function, as Rahman and Xu have shown [4]-[6].
The stochastic moments of a probability distribution may be calculated as follows

$$m_l = \zeta[Y^l(X)] = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} y^l(X) f_X(X) dx \qquad l = 0,1,\dots\dots,m \tag{1}$$

where m_l is the lth-order statistical moment (i.e., m=1 is the mean value, m=2 is the variance, etc.), $f_X(X)$ is the system response joint probability density function, $y(X)$ is the deterministic response when the input variables assume the values collected in the vector $X = \{x_1,\dots\dots\dots,x_n\}^T$, and $Y(X) = y^l(X) f_X(X)$ is the performance function. The latter can be approximated as the sum of univariate functions, each one depending on only one random variable at a time and the other variables being fixed to nominal values

$$Y(x_1,\dots\dots,x_n) \cong \tilde{Y}(X) \equiv \sum_{j=1}^{N} Y(\mu_1,\dots,\mu_{j-1},x_j,\mu_{j+1},\dots,\mu_N) - (N-1)\cdot y(\mu_1,\dots\dots,\mu_N) \tag{2}$$

where μ_j is the first moment of the stochastic variable x_j, $Y(\mu_1,\dots,\mu_{j-1},x_j,\mu_{j+1},\dots,\mu_N)$ is the stochastic response of the system only depending on the x_j random variable, and $y(\mu_1,\dots,\mu_N)$ is the deterministic response of the system depending on the nominal value of the N input variables. Adopting the dimension-reduction procedure, the expression of statistical moments (1) can be rewritten as:

$$m_l = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \sum_{j=1}^{N} \left(Y\left(\mu_1, \dots, \mu_{j-1}, x_j, \mu_{j+1}, \dots, \mu_N\right) - (N-1) \cdot y\left(\mu_1, \dots, \mu_N\right)\right)^l dx \tag{3}$$

To solve the univariate integration in the context of the UDR method, Xu and Rahman [4] suggest the use of the moment based quadrature rule. The evaluation of integration points x_j involves the solution of the following equation

$$x_j^n - r_{j,1} x_j^{n-1} + r_{j,2} x_j^{n-2} - \dots\dots + (-1)^n r_{j,n} = 0 \tag{5}$$

where the coefficients r_j are solution of the following linear system of equations

$$\begin{bmatrix} \mu_{j,n-1} & -\mu_{j,n-2} & \mu_{j,n-3} & \dots & (-1)^{n-1}\mu_{j,0} \\ \mu_{j,n} & -\mu_{j,n-1} & \mu_{j,n-2} & \dots & (-1)^{n-1}\mu_{j,1} \\ \mu_{j,n+1} & -\mu_{j,n} & \mu_{j,n-1} & \dots & (-1)^{n-1}\mu_{j,2} \\ \dots & \dots & \dots & \dots & \dots \\ \mu_{j,2n-2} & -\mu_{j,2n-3} & \mu_{j,2n-4} & \dots & (-1)^{n-1}\mu_{j,n-1} \end{bmatrix} \begin{bmatrix} r_{j,1} \\ r_{j,2} \\ r_{j,3} \\ \dots \\ r_{j,n} \end{bmatrix} = \begin{bmatrix} \mu_{j,n} \\ \mu_{j,n+1} \\ \mu_{j,n+2} \\ \dots \\ \mu_{j,2n-1} \end{bmatrix} \tag{6}$$

$\mu_{j,i}$ $(i = 1, \dots, n)$ represents the i[th] stochastic moment of the j[th] input variable. Thus, the univariate integral can be numerically solved as

$$\int_{-\infty}^{\infty} y^l\left(\mu_1, \dots, x_j, \dots, \mu_N\right) f_{X_j}\left(x_j\right) \cdot dx_j \cong \sum_{i=1}^{n} w_{j,i} y^l\left(\mu_1, \dots, x_j, \dots, \mu_N\right) \tag{7}$$

where f_{x_j} is the probability density function of input variable x_j. The weight $w_{j,i}$ appearing in Eq. (7) are evaluated using the following expression:

$$w_{i,j} = \frac{\sum_{k=0}^{n-1} (-1)^k \mu_{j,(n-h-1)} \cdot q_{j,(ik)}}{\prod_{k=1, k\neq 1}^{n} \left(x_{j,i} - x_{j,k}\right)} \qquad q_{j,i_0} = 1\,;\; q_{j,ik} = r_{j,k} - x_{j,i} \cdot q_{j,i(k-1)} \tag{8}$$

2.2 The Polynomial Chaos Expansion (PCE)

The Polynomial Chaos Expansion was introduced by Wiener [7] and is based on the approximation of each random variable by means of a suitable polynomial expansion about centered normalized Gaussian variables.

Any set $X = \{x_1, \dots, x_n\}^T$ of independent Gaussian variables can be expressed as function of a set $\xi = \{\xi_1, \dots, \xi_n\}$ of independent normal variables

$$X = f(\xi) \tag{9}$$

Hence, a performance function $y = Y(X)$ could be transformed into a function expressed in terms of ξ and, afterwards, approximated by means of the Polynomial Chaos Expansion (PCE) on the vector space

$$Y(X) = a_0\Gamma_0 + \sum_{i_1=1}^{\infty} a_{i_1}\Gamma_1(\xi_1) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} a_{i_1 i_2}\Gamma_2(\xi_{i_1}, \xi_{i_2}) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} a_{i_1 i_2 i_3}\Gamma_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3}) + \ldots \tag{10}$$

where $a = [a_0, \ldots\ldots, a_n]$ is the vector of the expansion unknown terms and $\Gamma_p(\xi_1, \ldots\ldots, \xi_n)$ are the multidimensional Hermite polynomials (only if the input random variables are defined by a normal probability distribution) of order p.

Cameron and Martin have shown that this kind of series is convergent in the L_2-sense [8]. In order to simplify the notation a univocal relation between the functional Γ and a new functional Ψ is defined. Hence, the PCE expansion, expressed by Eq. (10), can be rewritten as follows

$$Y(X) = \sum_{k=0}^{\infty} \beta_k \Psi_k(\xi(X)) \tag{11}$$

In the present work the classical convention is adopted:

- $\Psi_0 = 1$: is the 0th-order polynomial
- β_k are the constant coefficients of the expansion
- Ψ_k are multivariate Hermite polynomials, orthogonal in the L_2-space. These polynomials are the product of the proper set of univariate Hermite polynomials [9].

The expansion is normally truncated at a selected order P

$$Y(X) \approx \hat{Y}(X) = \sum_{k=0}^{P} \beta_k \Psi(\xi)_k \tag{12}$$

The number of unknown coefficients β_k (*13*) can be evaluated using the following expression

$$P + 1 = \frac{(p+n)!}{p!n!} \tag{13}$$

The procedure described above is general, but the Hermite polynomials can be used only in the cases of input variables with Gaussian probability distribution function. Xiu and Karniadakis [9] have extended the PCE applicability to all kinds of input distribution function, adopting the Wiener-Askey scheme for non Gaussian input distribution. They have proposed to use the Askey scheme to combine the non Gaussian input distribution with orthogonal polynomial family; in this way the expansion convergence for all kind of input PDF. As well as the Hermitte polynomials are orthogonal in the Hilbert space, in the same way all polynomials, adopted in the Wiener-Askey [9] scheme are orthogonal in the Hilbert space and form an Hilbert basis of the corresponding space.

The set $\beta = \{\beta_0, \ldots\ldots\beta_n\}^T$ of the PCE unknown coefficients, can be approximated by a new vector $\hat{\beta}$, obtained solving the following least squares problem

$$\hat{\beta} = \arg\min_{\beta} \sum_{i=1}^{N} \left(Y(X_i) - \sum_{k=0}^{P} \beta_k \psi_k(\xi_i) \right)^2 \qquad (12)$$

where N is the training points set size; generally, it is convenient that $N > p+1$.

2.3 The First Order Second Moment algorithm (FOSM)

In this approach a performance function $Y(X)$ is approximated by means of a first order Taylor-series expansion around the design point [10]

$$Y(X) \cong Y(\overline{X}) + \sum_{i=1}^{n} \left. \frac{\partial Y}{\partial x_i} \right|_{\overline{X}} (x_i - \overline{x}_i) \qquad (13)$$

Substituting Eq. (13) in the expectation definition (mean)

$$E[Y(X)] = E[Y(\overline{X})] + E\left[\sum_{i=1}^{n} \left. \frac{\partial Y}{\partial x_i} \right|_{\overline{X}} (x_i - \overline{x}_i) \right] \qquad (14)$$

and considering that:

$$E\left[\sum_{i=1}^{n} \left. \frac{\partial Y}{\partial x_i} \right|_{\overline{X}} (x_i - \overline{x}_i) \right] = \sum_{i=1}^{n} \left. \frac{\partial Y}{\partial x_i} \right|_{\overline{X}} E[(x_i - \overline{x}_i)] = 0 \qquad (15)$$

$$E[(x_i - \overline{x}_i)] = E(x_i) - \overline{x}_i = \overline{x}_i - \overline{x}_i = 0 \qquad (16)$$

the performance function mean value, estimated by means of FOSM, assumes the following expression

$$E[Y(X)] = Y(\overline{X}) \qquad (17)$$

Now, given the variance definition

$$Var[Y(X)] = E[(Y(X) - E(Y(X))^2)] \equiv \sigma^2[Y(X)] \qquad (18)$$

and substituting in it Eq. (13), the variance assumes the following expression

$$\sigma[Y(X)] = \sum_{i=1}^{n} \sum_{j=1}^{m} \left. \frac{\partial Y}{\partial x_i} \right|_{\overline{x}_i} \left. \frac{\partial Y}{\partial x_j} \right|_{x_j} \cdot COV(x_i, x_j) \qquad (19)$$

where $COV(x_i, x_j)$ is the covariance matrix.

2.4 The Evidence Theory

The Evidence Theory is a non probabilistic approach, used to characterize the effect of epistemic uncertainty on a system.

Given a design variable x_1, the prior information, or evidence, consists of *n* intervals, obtained from s sources $\left[x_{1,i}^{l}, x_{1,i}^{u}\right]$ (with $i = 1,......s$) that enclose the supposed true value. Clearly, the traditional probability theory cannot handle this type of evidence, without making some assumptions that can pervert the nature of the information. Several combination rules have been formulated to handle this kind of prior information [11],[12]; in this work, the Dempster-Shafer combination rule is adopted.

When a source provides a set information, this means that the variable can assume any value inside the interval. The probability that a variable x_1 assumes the value $\bar{x}_1$ is not defined by a probability distribution function but is included between a maximum probability (plausibility), and a minimum probability (belief**)**. In order to define the plausibility and the belief, the basic probability assignment (m) must be introduced; *m* defines a mapping of the variable prior information. Formally the basic probability assignment function is defined by means of the following expressions

$$m : \mathrm{P}(x_1) \rightarrow [0,1] \tag{20}$$

$$m(\varnothing) = 0 \tag{21}$$

$$m = 1 \; if \;\; \bar{x}_1 \in S_i \;\; with \; i = 1,......., s \tag{22}$$

where $\mathrm{P}(x_1)$ represents the power set of x_1 (defined, according to the axiomatic set theory [12] as the set of all subset of S), while $\varnothing$ is the null set and S_i is the i-th evidence set. According to the previous equations, the basic probability assignment assumes any value included between 0 and 1; if $\bar{x}_1$ does not belong to any subset, the basic probability assignment assumes value 0, while if $\bar{x}_1$ belongs to every subset, it assumes the value 1. Once defined the basic probability assignment *m*, the plausibility and belief probability measures can be introduced. Given a set $C = [D(x_1)^{-}, \bar{x}_1]$, where $\bar{x}_1$ is a generic value of the variable x_1 on its domain $D(x_1)$, while $D(x_1)^{-}$ represents the lower domain boundary, the plausibility can be expressed by

$$Pl(C_1) = \sum_{C_1 \cap s_{x_1}^{i} \neq 0} m_{x_{1,i}}\left(S_{x_1}^{i}\right) \tag{24}$$

while the belief is defined as:

$$Bel(C_1) = \sum_{S_{x_1}^{i} \subseteq C_1} m_{x_1}\left(S_{x_1}^{i}\right) \tag{25}$$

In other words, the plausibility is the sum of all basic probability assignments of the sets $S_{x_1}^{i}$ which intersect the set of interest C_1, hence it represents the maximum probability that a variable x_1 assumes a given value $\bar{x}_1$. On the other hand, the belief is defined as the sum of all

basic probability assignments of the sets $S_{x_1}^i$ that $S_{x_1}^i \subseteq C_1$ hence it is a measure of the minimum probability that a variable x_1 assumes a given value $\bar{x}_1$.
The probability lies between the plausibility and the belief

$$Bel(C_1) \leq P(C_1) \leq Pl(C_1) \tag{26}$$

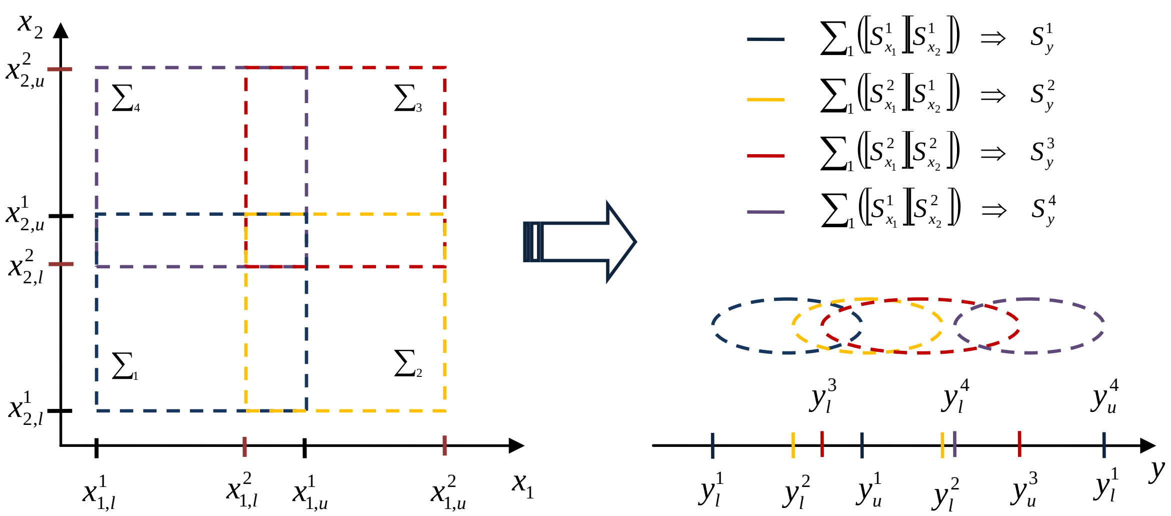
and, only when plausibility and belief are overlapped, it can be univocally defined.
In a problem with n input variables there is the need to transfer the basic probability assignment values m_{x_j}, evaluated for each variable, into an equivalent information in the n-dimensional design variables space. Assuming that all the variables are uncorrelated, the probability of each elementary set in the design variables space is defined by

$$m_{x_1,x_2,\ldots\ldots,x_n}\left([S_{x_1}^j],\ldots\ldots\ldots\ldots\ldots\ldots,[S_{x_{n1}}^j]\right) = m_{x_1}\left([S_{x_1}^j]\right)\cdot\ldots\ldots\ldots\ldots\cdot m_{x_1}\left([S_{x_n}^j]\right) \tag{27}$$

Once defined the uncertainty acting on the design variables, its effects on the performance function can be evaluated. Given a generic function $y = Y(X)$, linking the output with the input variables $X = \{x_1,\ldots\ldots\ldots x_n\}$, the evidence about y must be estimated from the joint body of evidence previously described in Eq. (27). By means of two optimization problems, for each evidence-set of the input variables space, the lower and upper boundary of the corresponding set into the output space are evaluated

$$\begin{array}{ll} find & \bar{X} = \{\bar{x}_1,\ldots\ldots,\bar{x}_n\} \in \Sigma_j\left([S_{x_1}^j],[S_{x_2}^j],\ldots\ldots\ldots,[S_{x_n}^j]\right) \\ t.c & \max/\min\ y = f(X)\ \square \\ & \Downarrow \\ & S_y^j = [y_l^j, y_u^j] \end{array} \tag{28}$$

The above optimization problems yield an evidence set on output-space S_y^j for each set Σ_j of the joint body of evidence (Fig 4.5). Hence, in order to propagate the uncertainty from in put to output two optimizations for each set Σ_j have to be performed.



3 NUMERICAL EXAMPLES

In this section some numerical examples will be presented in order to verify the accuracy of the UP methods described above. Some test functions and a structural problem (static, dynamic and buckling behavior of a composite plate) will be the test cases considered for assessing the methods for stochastic uncertainty. A comparison between stochastic and epistemic approaches when applied to a structural shape sensing problem will be then discussed.

3.1 Test functions

Two test functions are used to compare the performance of the UP methods introduced in the previous paragraphs (Table 1).

Function	PDF	PDF Parameters
$y = x_1^k x_2^k + 2x_3^4 \qquad k = 2,3,5$	Gaussian	$\mu = [1,1,1]$ $\sigma = 0.1, 0.2, 0.4, 0.8$
$y = \sin x_1 + a\sin^2 x_2 + bx_3^4 \sin x_1$	Gaussian	$\mu = [\pi/4, \pi/4, \pi/4]$ $\sigma = 0.05, 0.1, 0.2, 0.5$

Table 1: Test functions used to test the UP methods

The first example is a three-variate function, chosen to compare the performance of the UP algorithms against the first order interaction of the input variables. The input variables follow a Gaussian distribution centered in $X = [1,1,1]$ and four values of standard deviation are tested ($\sigma = 0.1, 0.2, 0.4, 0.8$). In addition, the effect of the interactions among the variables is studied changing the value of k. The analysis of the accuracy of each method is performed comparing the predicted values of the statistical moments with those evaluated using a MonteCarlo Simulation, based on 10^6 observations. In this example the effect of an increasing input variability is combined with that of an increasing interaction effect.

In Table 2 the main effects and the interactions are listed for each value of k. These indices are evaluated by means of the Polynomial Chaos Expansion [13]. We can observe that changing the value of k the interaction $x_1 x_2$ increases its effect on the output, becoming gradually the most important factor.

	Main Effects and Interactions		
	k=2	k=3	k=5
X_1	0,011	0,0838	0,0575
X_2	0,011	0,0838	0,0572
X_3	0,9587	0,4734	0,0006
X_1X_2	0,01927	0,359	0,8808
X_1X_3	9,60E-25	8,78E-25	5,82E-04
X_2X_3	4,84E-26	8,34E-25	1,22E-03

Table 2: Main effects and interaction

In Fig (1) the errors in the estimation of the mean value are plotted in function of the input variables standard deviation and against different values of k. As a general rule, when the input variability increases, all the UP methods here discussed become less accurate. This phenomenon is negligible if the first order interactions are marginal; on the contrary, in problems

where the interaction effects are more important ($k = 3$ or $k = 5$) the results become more sensitive to the input variability.
As shown in Fig (1A) the UDR yields a good estimation of the mean values when the interaction between the variables is low $(k = 2)$, also in the case of high input variability $(\sigma = 0.8)$. Increasing the effect of interaction, the accuracy of this method greatly decays, in particular for higher values of input variability.

Similar behaviors are shown in Fig (1B-D); the output function is approximated with the Polynomial Chaos Expansion, truncated at different orders. Also in this case, for higher values of k and for a higher input uncertainty, the mean value is poorly approximated. When using the PCE, however, the reduced accuracy is not due to the interaction effects, but it is caused by the non-linearity of the output function: for example, if $k = 2$ we have a 4th order function, while if $k = 3$ we have a 6th order function. Hence, it is clear that a Polynomial Chaos Expansion truncated at the 5th order better describes the problem than an expansion trounced at the 2nd order, but, for $k = 5$, it does not guarantee adequate accuracy. Increasing the order of the expansion, the error in the prediction gradually vanishes.

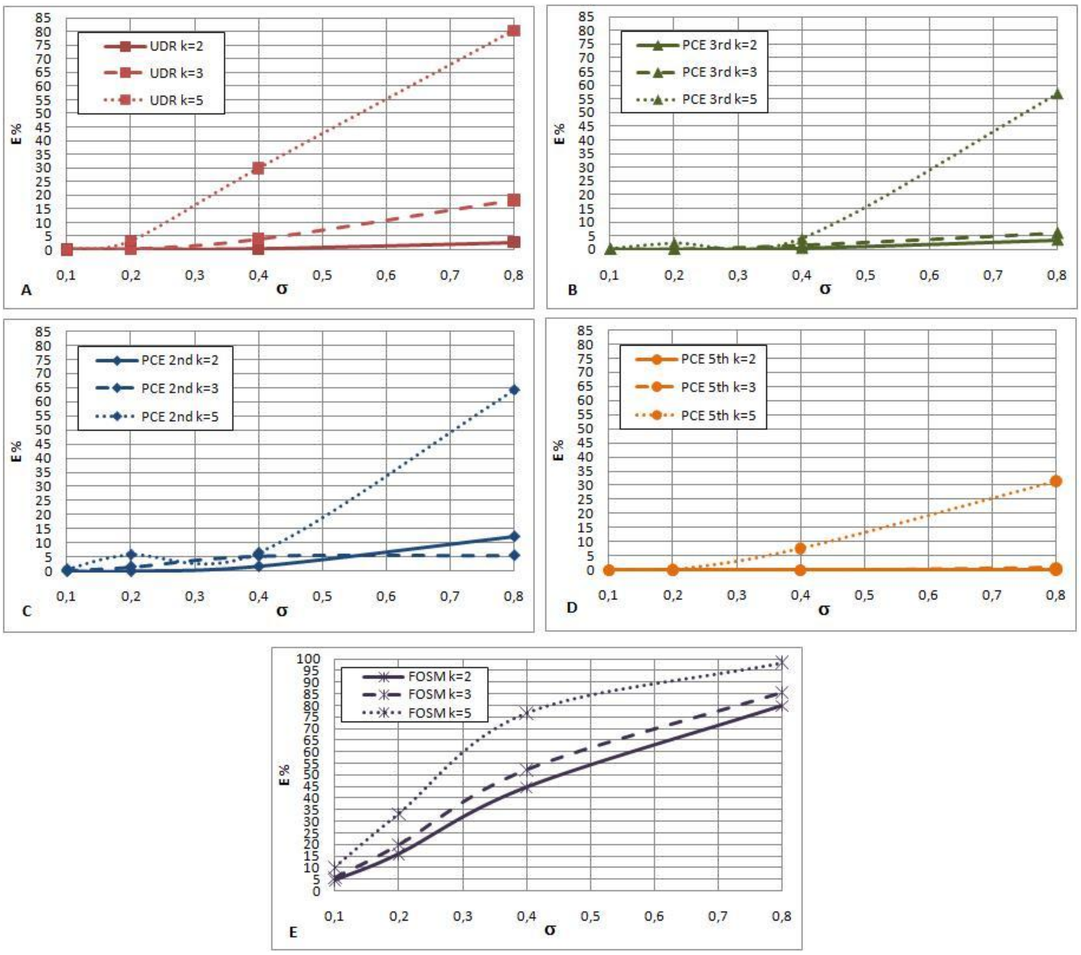


Fig 1 Error in the extimation of the output mean value: A) Error due to the Univariate dimension reduction method (UDR) B-D) Error due to the Polinomial Chaos Expansion (PCE) respectivelly of 2nd, 3rd and 5th order. E) Error due to the FOSM alghorithm

Therefore, there is a substantial difference between the UDR and the PCE. In the UDR the lack of accuracy, is inherent to its mathematical formulation and cannot be reduced. On the other hand, the accuracy of the PCE results can be improved increasing the order of the expansion.

In Fig (1E) the errors are shown on the output function mean value when computed using the FOSM algorithm. In this case the approximation is based on the hypothesis that the output function has a linear behavior in the studied domain; the errors are quite high also for small input variability levels.

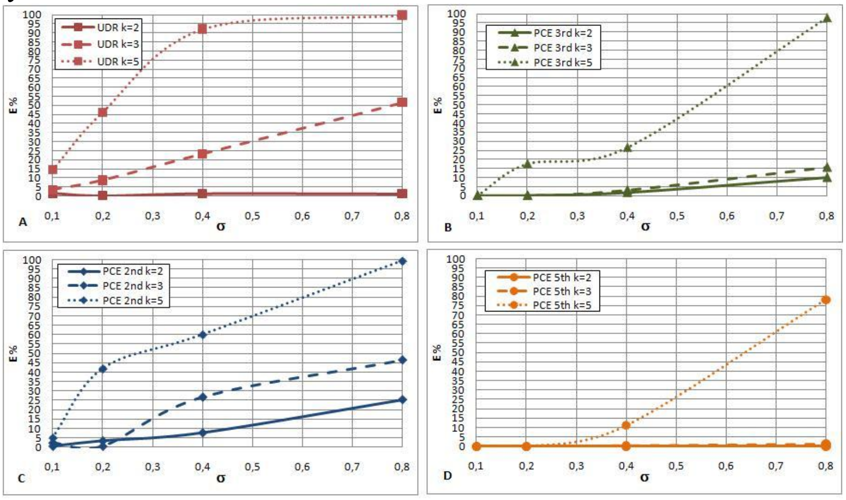


Fig 2 Error in the extimation of the output variance: A) Error due to the Univariate dimension reduction method (UDR) B-D) Error due to the Polinomial Chaos Expansion (PCE) respectivelly of 2^{nd}, 3^{rd} and 5^{th} order.

The decay of UDR accuracy in the prediction of the statistical moments (due mainly to the first order effects) is more evident in the evaluation of the variance Fig (2-A) and of the higher order moments: Skewness (Fig 3-A) and Kurtosis (Fig 4-A). As already observed in the evaluation of mean value, for quite small interaction effects ($k = 2$), the UDR approximation does not affect the accuracy of the results. This is not true for the higher order moments.

Results regarding higher order statistical moments (Figs. 6.4B-D, 6.5B-D, 6.6B-D) confirm that the interaction between the variables does not affect the accuracy of the Polynomial Chaos Expansion; anyhow, a higher order expansion is required in order to have a good estimation of the variance, skewness and kurtosis.

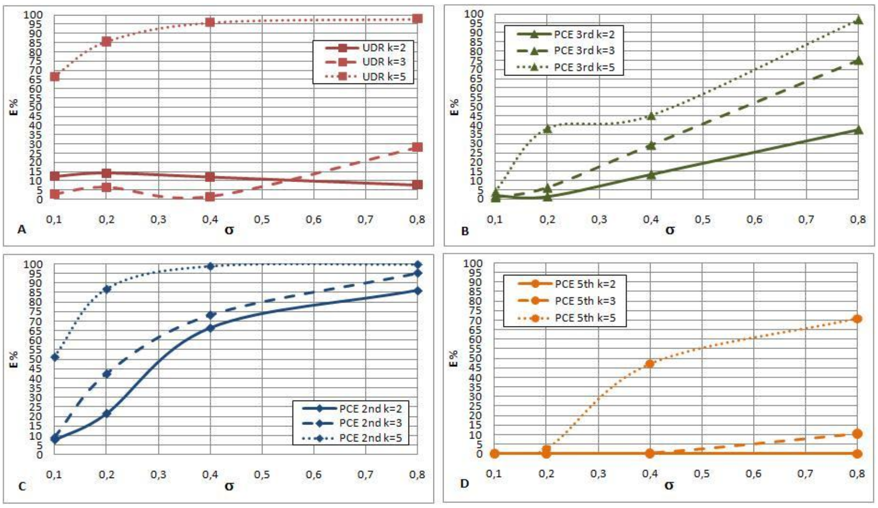


Fig 3: Error in the extimation of the output skewness: A) Error due to the Univariate dimension reduction method (UDR) B-D) Error due to the Polinomial Chaos Expansion (PCE) respectivelly of 2^{nd}, 3^{rd} and 5^{th} order.

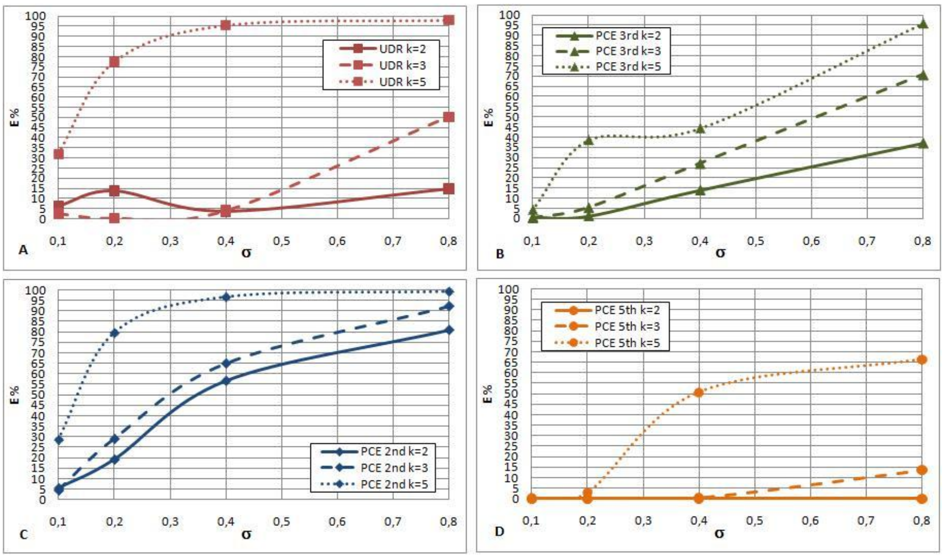


Fig 4: Error in the estimation of the output kurtosis: A) Error due to the Univariate dimension reduction method (UDR) B-D) Error due to the Polinomial Chaos Expansion (PCE) respectivelly of 2nd, 3rd and 5th order.

In Table 3 the number of observation points needed to perform each analysis are listed. The UDR methods needs only 16 observed data (it requires only $5n+1$, where n are the stocastic input variables).

FOSM	UDR	PCE 2nd	PCE 3rd	PCE 5th
49	16	31	61	168

Tab 3: Number of observations for each methods

The UDR method is the cheapest one and, as seen in the present example, if there is a negligible interaction between the input variables it yields a good estimation of the statistical moments. The computational cost of the PCE grows considerably increasing the order of expansion and the problem dimension.

The second function here considered (see Tab (1)) is the Ishigami function, commonly used to test the uncertainty propagation algorithms and the sensitivity in order to understand their behavior with non-linear and non-monotonic functions. The three variables follow a Gaussian distribution, centered in $X=[\pi/4, \pi/4, \pi/4]$; the standard deviation ranges from 0.05 to 0.5. The accuracy of each method is assessed comparing the predicted values of the statistical moments with those evaluated using a MonteCarlo Simulation, based on 10^6 observations.

In Table 6.5 the percentage errors on the estimation of the mean value and variance are listed for different values of the input standard deviation. All methods yield a good estimation of the mean value (the error is always less than 1%). The differences between the methods are more evident when considering the variance evaluation. The FOSM method yields a very poor estimation in particular for high values of input variability: for example, the error with an input standard deviation of $\sigma = 0.2$ is around 12.3%, while with $\sigma = 0.5$ is around 56%. The UDR method leads to a good estimation of the variance (error around 2%); there is no evident correlation between the input variability and the estimation error. The PCE is very accurate, if the order of the expansion is sufficient to describe the problem; we can observe that a 5th order expansion is very accurate in the prediction of variance, and that the 7th order expansion yields the exact solution.

Standard Deviation Input	***FOSM***		***UDR***		***PCE 2nd***	
	Mean	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**
0,05	3,336E-02	1,204E+00	2,515E-02	1,934E+00	2,363E-03	2,398E-01
0,1	7,351E-02	1,968E+00	4,734E-02	1,952E+00	2,837E-02	1,997E+00
0,2	1,609E-01	1,235E+01	7,092E-02	2,162E+00	1,754E-01	2,334E+00
0,5	5,364E-01	5,575E+01	3,560E-01	1,818E+00	5,596E-01	6,521E+00
Standard Deviation Input	**PCE 3rd**		**PCE 5th**		**PCE 7th**	
	Mean	**Variance**	**Mean**	**Variance**	**Mean**	**Variance**
0,05	0,000E+00	0,000E+00	0,000E+00	0,000E+00	0,000E+00	0,000E+00
0,1	0,000E+00	2,058E-02	0,000E+00	0,000E+00	0,000E+00	0,000E+00
0,2	4,731E-03	3,456E-02	0,000E+00	3,456E-02	0,000E+00	0,000E+00
0,5	1,189E-01	2,878E+00	4,749E-03	1,641E-02	0,000E+00	2,553E-02

Tab 4: Mean values and variance estimation errors

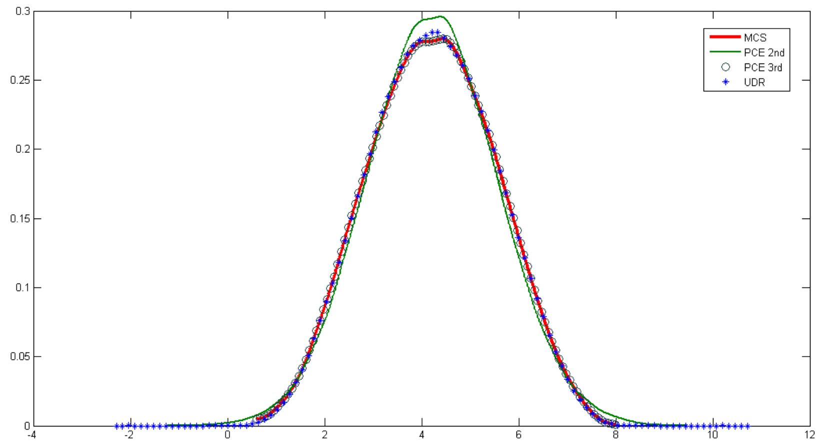


Fig 5: Probability density function in the case that all input variables have a standard deviation of 0.2

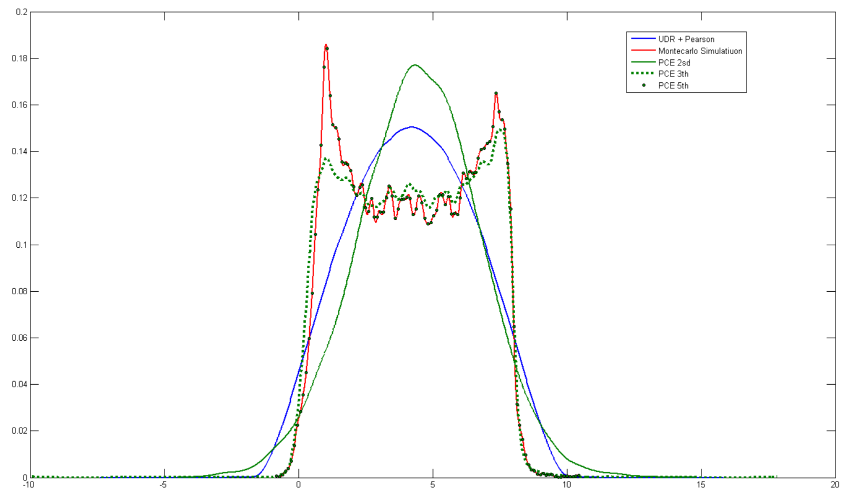


Fig 6: Probability density function in the case that all input variables have a standard deviation of 0.5.

In a robust design framework it is important the accurate evaluation of the first two statistical moments (mean and variance), but for the evaluation of the reliability degree this information

is not enough. Hence, the knowledge of the probability distribution function is needed. One of the main problems of the UDR approach is that it does not yield directly the probability distribution, but only the statistical moments. Anyway, it is possible to obtain the PDF, knowing the first four statistical moments, by means of the Pearson System. In fig (5) and fig (6) the probability distribution function, evaluated with the UDR and the Pearson System, is compared with the PDF obtained using the PCE with different expansion orders and the one obtained by means of MCS (10^6 training points). The curves plotted in Fig (5) and in Fig (6) are referred, respectively, to the case of an input standard deviation of $\sigma = 0.2$ and of $\sigma = 0.5$.
In the first case (fig 5) a good agreement among all plotted curves can be observed. The UDR method coupled with the Pearson System yields a very good approximation of the probability density function. There is only a small discrepancy in the description of the tails: the tails of PDFs, obtained with the UDR method and the 2nd order PCE, end with an asymptotic behavior, while in the one obtained with MCS the tails are limited. On the contrary there is a perfect correspondence between the probability function obtained with a 3rd order PCE and the one obtained with the MonteCarlo Simulation.
In the second case $(\sigma = 0.5)$ it is possible to appreciate a bigger discrepancy among the methods (fig 5-A). Although the UDR method is able to predict with a good accuracy the mean and the variance of the output, it fails in the estimation of the PDF. This is mainly due to the fact that the higher order moments are predicted with low-accuracy and, as well known, the Pearson System is based on the relation between skewness and kurtosis. Also the 3rd order PCE gives not an accurate probability distribution. Only by means of a 5th order PCE a good PDF approximation can be obtained.
In this example we have seen that, although the UDR approach is adequately accurate to be used in a robust design problem, it cannot be used in a reliability based problem.

3.2 Composite plate mechanical behavior

In this example the performances of UDR and PCE are tested on the static and dynamic response analysis of a symmetric composite plate $\left(0^\circ/90^\circ/90^\circ/0^\circ\right)$ with all edges clamped. The material properties, the fiber angles, and the plies thickness are considered affected by uncertainty and are described by means of Gaussian distributions. In tab (5) all plate properties are reported in terms of mean value and standard deviation. The stochastic moments of the maximum deflection (***w***), the first natural frequency (***f***), the maximum Von Misses stress ($\boldsymbol{\sigma_{VM}}$), the maximum τ_{xz} and the maximum τ_{yz} are evaluated by means of the UDR and of the PCE. The static and dynamic responses of the plate have been obtained using the Refined Zigzag plate Theory (RZT) [14]-[18]; a Rayleigh-Ritz solution procedure has been adopted to find maximum deflection, first natural frequency and stresses distribution. The results are compared with those obtained using a MCS based on 10^5 observations.
Finally, the first four statistical moments, evaluated using the UDR and PCE, are compared with those obtained by a MonteCarlo Simulation (Tab 6). In Tab (6) one can observe that approximately all approaches give the same results, the main difference between the UDR and the 2nd order PCE is in the computational cost needed to perform the analysis, indeed are needed 71 observations and 360 training points, respectively.

		Mean	*PDF*	*SD*
Mechanical properties	**E11 [Mpa]**	1,58E+05	Gaussian	7,895
	E22 [Mpa]	9,58E+03	Gaussian	0,4792
	E33 [Mpa]	9,58E+03	Gaussian	0,4792
	G12 [Mpa]	5,93E+03	Gaussian	0,2965
	G13 [Mpa]	5,93E+03	Gaussian	0,2965
	G23 [Mpa]	3,23E+03	Gaussian	0,1613
	v12	0,32	Deterministic	-
	v13	0,32	Deterministic	-
	v23	0,49	Deterministic	-
	ρ [T/mm^3]	1,90E-09	Deterministic	-
Orientation angles	θ_1	0	Gaussian	3
	θ_2	90	Gaussian	3
	θ_3	90	Gaussian	3
	θ_4	0	Gaussian	3
Thicknesses	**t1 [mm]**	1	Gaussian	0,05
	t2 [mm]	1	Gaussian	0,05
	t3 [mm]	1	Gaussian	0,05
	t4 [mm]	1	Gaussian	0,05

Tab 5: Plate properties: θ_1 is the fiber orientation of the first ply, t_1 is the ply thickness

		w	f	σ_{VM}	τ_{yz}	τ_{xz}
MCS	Mean	21,37	434,24	2194,65	28,76	14,01
	Variance	3,36	200,36	13570,68	0,70	0,45
	SD	1,84	14,15	116,49	0,84	0,67
	SKW	0,01	0,04	0,23	0,15	0,16
	KURT	3,05	3,03	3,06	3,01	3,03
UDR	Mean	21,36	434,24	2194,50	28,76	14,01
	Variance	3,35	201,67	13570,68	0,78	0,45
	SD	1,83	14,20	116,49	0,88	0,67
	SKW	0,00	0,00	0,23	0,15	0,16
	KURT	3,03	3,03	3,00	3,02	3,03
2nd PCE	Mean	21,37	434,24	2194,64	28,76	14,01
	Variance	3,34	205,67	13570,68	0,72	0,45
	SDY	1,83	14,34	116,49	0,85	0,67
	SKW	0,01	0,00	0,23	0,15	0,16
	KURT	3,03	3,03	3,00	3,02	3,03

Tab 6: Stochastic moments: SD is the standard deviation, SKW is the skewness, KURT is the curtosis, w is the maximum deflection, f is the first modal frequency , VM is the maximum Von Mises stress.

3.3 Structural shape sensing

The inverse Finite Element Method (iFEM), developed by Tessler for plate and shell structures [18] and specialized by Gherlone for beams and frames [19], is aimed at the reconstruction of the displacement field of a structure starting from in situ measurements of surface strains [19]; this represents an inverse problem [20].

In the present work a cantilevered aluminum beam with a circular thin-walled cross-section, subjected to different load conditions, has been studied. In lieu of the experimental measures of surface strains, high-fidelity forward FE analyses (MSC/NASTRAN) have been carried out for the example problem (Table 7). These results have also been used to verify the accuracy of the nodal displacements and rotations obtained by iFEM.

Element type (name)	N° of elements along the external circumference	N° of elements along the beam length (*L* = 20) [dm]	N° of elements	N° of nodes
Shell element (QUAD4)	114	360	41,040	41,156

Tab 7: Sensor configuration: 1

The position of a strain gauge, used to measure surface strains, is defined by three coordinates: the first one, *x*, indicates the position along the longitudinal beam axis, the second one, θ, is an angle representing the circumferential position on the beam and the coordinate β indicates the strain gauge orientation (i.e., it represents the rotation of the strain gauge with respect to the beam axis (Fig 7). For the current application, six strain gauges are used; their nominal positions are reported in Table 8 and their location is also represented in Figure8.

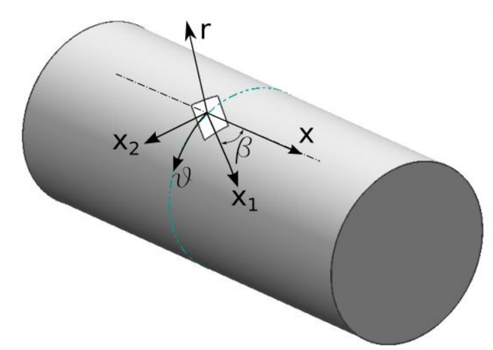


Fig 7. Location of a strain gauge on the beam external surface. [20]

Strain gauges	x	θ	β
1	10	-120	0
2	10	-120	45
3	10	0	0
4	10	0	45
5	10	120	0
6	10	120	45

Tab 8 Strain gauge nominal positions

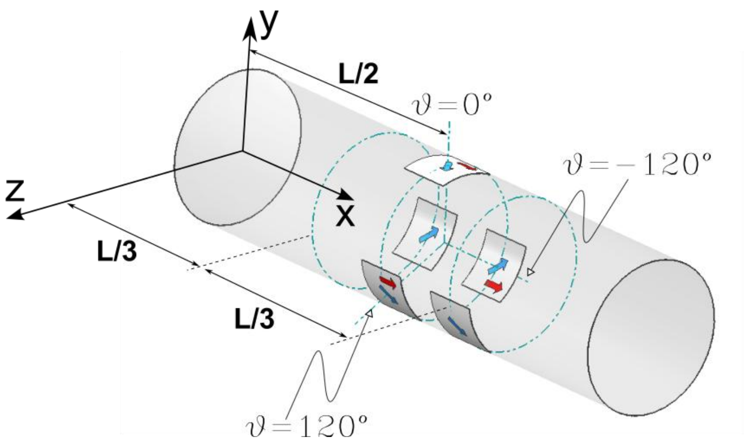


Fig 8 Sensors position [20]

In this example three different load conditions are considered (Fig (9-A)): 1) a shear force applied along y-axis, 2) the torque moment and 3) the bending moment around the z-axis. The free end displacements and rotations (Fig (9-B)) are computed by means of the iFEM and are compared with the ones obtained using the forward FEM solution. Hence, the iFEM accuracy is evaluated by means of the following error:

$$E \equiv \frac{Value(FEM) - Value(iFEM)}{Value(FEM)} \tag{29}$$

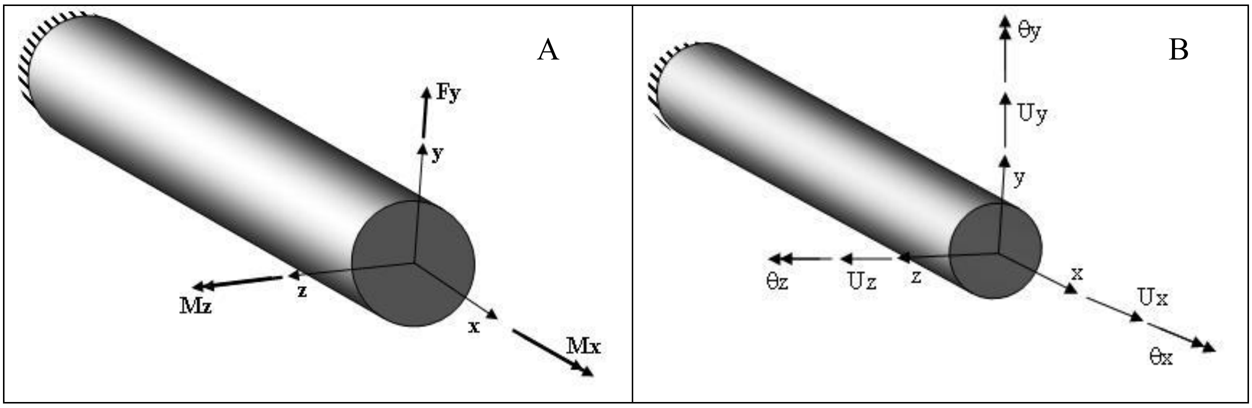


Fig 9 (A)Applied load, (B) Studied degrees of freedom.

The aim of the present application is to verify the robustness of the iFEM in evaluating the displacement field when the sensor positions are considered affected by uncertainty. For this purpose a probabilistic approach is compared with a non-probabilistic method based on the evidence theory. The main issue is the definition of the uncertainty that affects the coordinate values describing the sensors position. In order to obtain this kind of information, three technicians have been interviewed. They have given three different estimations of the error in the strain gauge location; all these experts are equally trusted. The second expert (see Tab 9) defined the errors using disjoint sets.

Expert	x [mm]		θ [°]		β [°]	
	LOWER	UPPER	LOWER	UPPER	LOWER	UPPER
1	-5	5	-5	5	-4	4
2	-5	-1	-5	5	-4	-1,5
	1	5	-5	5	1,5	4
3	-1	1	-5	5	-1,5	1,5

Tab 9: Sensor coordinates defined by means of interval sets

In order to use a probabilistic approach to propagate the uncertainty from input to output, there is the need to transform the input epistemic uncertainty into probabilistic information. Several hypotheses are then needed about the shape of the probability distribution and its standard deviation. In the present example we have assumed that the uncertainty in the sensor position is described by means of a Gaussian distribution, having the standard deviations listed in Tab (10).

Input variables	PDF	Standard Deviation
x	Gaussian	0,0233
θ	Gaussian	1,1666
β	Gaussian	1,3333

Tab 10: Probabilistic assumptions of sensor position

The information obtained by the sensitivity analysis [21] are used to select which input variables should be considered and which could be neglected during the uncertainty propagation process, performed both using the evidence theory and a probabilistic approach (the UDR method, having verified that there are not significant interactions between variables).. Then, once the first four statistical moments are known, the corresponding probability distribution function is evaluated by means of the Pearson System.
The probability that the iFEM error on the FEM reference displacements and rotations is greater than a given threshold value, is finally evaluated.

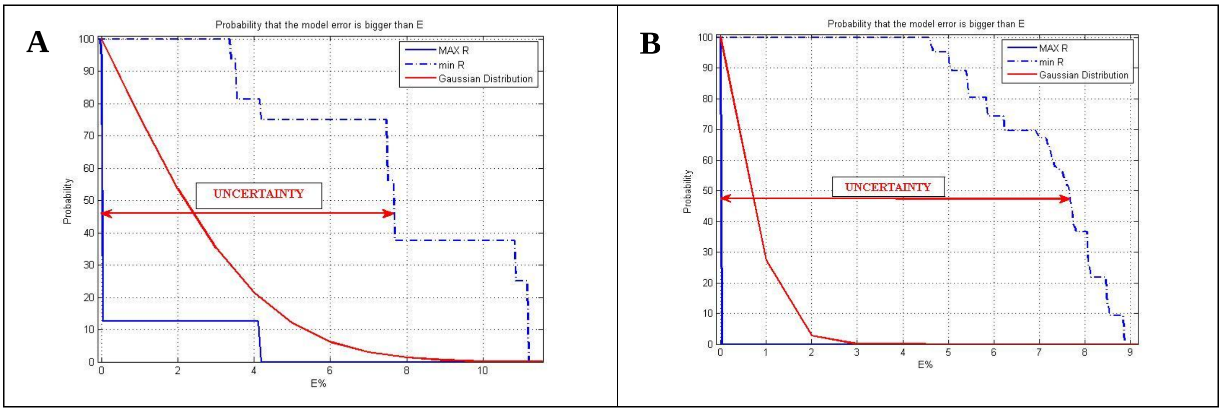


Fig 10 Shear load (Fy): Probability that the error E about Uy (A) and θz (B) is bigger than a given threshold value. Three curves are plotted: the first one represents the maximum model reliability (labeled with MAX R), the second one represents the minimum reliability of the model (labeled with min R), the red one represents the curve obtained using the assumption of the Gaussian distribution.

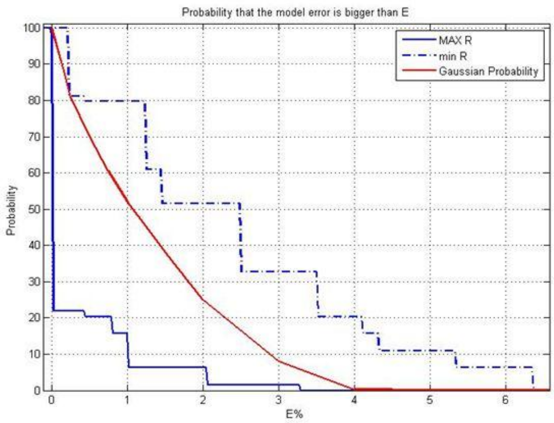


Fig 11 Torque Moment Probability that the error E about θx (B) is bigger than a given threshold value. Three curves are plotted: the first one represents the maximum model reliability (labeled with MAX R), the second one represents the minimum reliability of the model (labeled with min R), the red one represents the curve obtained using the assumption of the Gaussian distribution.

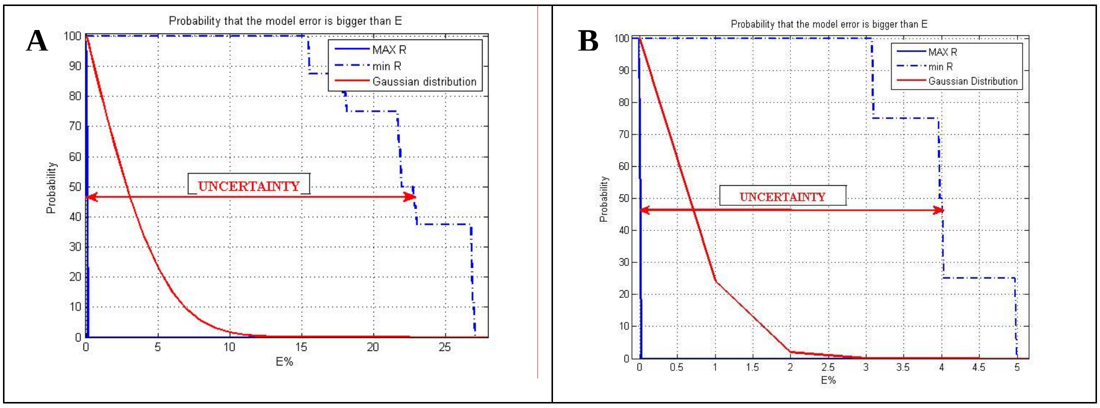


Fig 12 Bending Moment (z):): Probability that the error E about Uy (A) and θz (B) is bigger than a given threshold value. Three curves are plotted: the first one represents the maximum model reliability (labeled with MAX R), the second one represents the minimum reliability of the model (labeled with min R), the red one represents the curve obtained using the assumption of the Gaussian distribution.

The evidence theory does not furnish a unique measure of the probability, but it gives two different probability curves: the plausibility, that describes the curve of the maximum reliability of the system and the belief, that describes the minimum reliability of the system (fig 10-12). According to what it was said in Section 2.4, the true reliability curve is included between the plausibility and the belief. Actually, the area included between the maximum reliability curve and the minimum one represents a region of uncertainty; this means that, without further information, no prediction about the actual behavior of the model can be made (we only know that the true error is included between the two probability curves). For this reason, the belief curve, that represents a conservative estimation of the model behavior, is used during the design phase. In this study we have compared the results obtained assuming that the position error is described by means of Gaussian distributions with those obtained assuming that each sensor is located inside an interval. In this last case no hypothesis has been made about the probability that a sensor is in a given point (inside the region). As shown in Fig (10-12) the maximum reliability curves give almost null prediction errors, otherwise the minimum reliability curves indicate bigger probability to have large errors; in particular the evaluation of the y-displacement and z-rotation is quite sensible to the sensor position uncertainty Fig (10) and Fig (12). In most cases the reliability curves, based on the Gaussian distribution hypothesis, underestimate considerably the prediction errors. In particular the Gaussian hypothesis furnishes probability values close to those given by the maximum reliability curves.

4. Conclusion

In the present work a comparative study of some uncertainty propagation algorithms is performed and discussed.

Methods for both aleatory and epistemic uncertainty are considered; in particular, a brief review of Univariate Dimension Reduction method (UDR), Polynomial Chaos Expansion (PCE), and First Order Second Moments algorithm (FOSM) - for aleatory uncertainty - and of Evicence Theory – for epistemic uncertainty - is presented.

Then, selected example problems are considered to assess and compare the available methods; some test functions are used as preliminary test cases, then structural applications are studied, ranging from the mechanical behavior of a composite plate to the shape sensing of a beam starting from measured surface strains. As for the latter application, an epistemic uncertainty propagation approach (Evidence Theory) has been compared with a probabilistic uncertainty propagation algorithm (UDR); the considered problem is a classical example of epistemic uncertainty, therefore probabilistic approaches may be applied after introducing some prior assumptions whose correctness may not be guaranteed.

Although this study is limited to some particular examples, interesting general conclusions can be drawn.

If there is no significant interaction between variables, the UDR is the most efficient method for statistical moments estimation. Its accuracy decreases when the interactions cannot be neglected; in particular, the evaluation of the 3rd and 4th statistical moments is more sensitive to the interaction effects and, therefore, also the evaluation of the corresponding Probability Distribution Function (PDF), by means of the Pearson System, can be compromised. The accuracy and the computational cost of the PCE depend on the truncation order of the expansion. However, the PCE is a useful approach when the knowledge of the PDF is desired. Moreover, the UDR method leads to the best compromise between accuracy and computational cost when performing a probabilistic study of the mechanical behavior of a composite plate.

The transformation of the epistemic knowledge into a probabilistic knowledge could often cause a loss of information and consequently the underestimation of the uncertainty effects. The evidence theory, in the particular case of the shape sensing problem, seems to be the more robust and conservative approach. The use of a probabilistic approach is not wrong but it requires too strong prior assumptions. In other words, the correct use of the probabilistic approach would require the experimental probabilistic characterization of the sensors position.

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