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A PROBABILISTIC APPROACH TO FUZZY METHODS

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Abstract. *Effect algebras ([1], [2]) and D-posets ([3]) are equivalent systems important in quantum structures. In the paper an independent sequence of observables on these structures is defined by such a way that a very general version of the law of large numbers may be proved.*

1 INTRODUCTION

In multi valued logic the MV algebras play the same role as Boolean algebras in two valued logic. Therefore probability theory on MV algebras seems to be very important (see ([6]). Of course, there are interesting generalizations of MV algebras: D-posets ([3]) and equivalent effect algebras ([1], [2]). Again probability theory can be constructed on D-posets and particularly on D-posets with product ([4]).

In the paper the law of large numbers is proved for very general D-posets.

In Section 2 some basic notions are defined. The key for the law of large numbers is the new formulation of independence. It is motivated and presented in Section 3. Also the sum of independent observables is defined there. The general law of large numbers is formulated and proved in Section 4. Similarly as in [6] a local representation of a sequence of independent observables by a sequence of random variables seems to be the main idea of the proof.

2 EFFECT ALGEBRAS AND D-POSETS

The concept of an effect algebra was introduced by Foulis and Bennet [1]. We will work with an equivalent algebraic structure, called D-poset introduced by Kôpka and Chovanec ([3]).

Definition 2.1. *Effect algebra is a system* $(E, +, 0, 1)$*, where* $0, 1$ *are distinguished elements of* E *and* $+$ *is a partial binary operation on* E *such that*

1. $x + y = y + x$ *if one side is defined,*
2. $(x + y) + z = x + (y + z)$ *if one side is defined,*
3. *for every* $x \in E$ *there exists a unique* x' *with* $x' + x = 1$*,*
4. *if* $x + 1$ *is defined then* $x = 0$*.*

Every effect algebra bears a natural partial ordering given by $x \leq y$ if and only if $y = x + z$ for some $z \in E$. The poset $(E, \leq)$ is bounded, 0 is the smallest element and 1 is the largest element. In every effect algebra, a partial subtraction $-$ can be defined as follows:

$x - y$ exists and is equal to z if and only if $x = y + z$.

The system $(E, \leq, -, 0, 1)$ so obtained is a D-poset defined by Kôpka and Chovanec [3].

Definition 2.2. *The structure* $(D, \leq, -, 0, 1)$ *is called* D*-poset if the relation* $\leq$ *is a partial ordering on* D*,* 0 *is the smallest and* 1 *is the largest element on* D *and*

1. $b - a$ *is defined if and only if* $a \leq b$*,*
2. *if* $a \leq b$ *then* $b - a \leq b$ *and* $b - (b - a) = a$*,*
3. $a \leq b \leq c \Longrightarrow c - b \leq c - a, (c - a) - (c - b) = b - a$*.*

To build a probability theory we need two important mappings equivalent to probability measure and random variable. In our concept we call them state and observable.

Definition 2.3. *A state on a* D*-poset* D *is any mapping* $m : D \to [0, 1]$ *satisfying the following properties:*

1. $m(1) = 1,\ m(0) = 0$*,*

2. $a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a), \forall a_n, a \in D$,
3. $a_n \searrow a \Longrightarrow m(a_n) \searrow m(a), \forall a_n, a \in D$.

Definition 2.4. *Let $J = \{(-\infty, t); t \in R\}$. An observable on D is any mapping $x : J \to D$ satisfying the following conditions:*

1. $A_n \nearrow R \Longrightarrow x(A_n) \nearrow 1$,
2. $A_n \searrow \emptyset \Longrightarrow x(A_n) \searrow 0$,
3. $A_n \nearrow A \Longrightarrow x(A_n) \nearrow x(A)$.

Theorem 2.5. *Let $x : \mathcal{J} \to D$ be an observable, $m : D \to [0,1]$ be a state. Define a mapping $F : R \to [0,1]$ by the formula*

$$F(t) = m(x((-\infty, t))).$$

Then F is a distribution function.

Proof. If $t_n \nearrow t$, then $(-\infty, t_n) \nearrow (-\infty, t)$, hence $x((-\infty, t_n)) \nearrow x((-\infty, t))$ by 3 of Def. 4, and

$$F(t_n) = m(x((-\infty, t_n))) \nearrow m(x((-\infty, t))) = F(t)$$

by 2 of Def. 2.3, hence F is left continuous in any point $t \in R$. Similarly

$$t_n \nearrow \infty \Longrightarrow F(t_n) \nearrow 1$$

by 1 of Def. 2.4 and 1 and 2 of Def. 2.3. Moreover

$$t_n \searrow -\infty \Longrightarrow F(t_n) \searrow 0$$

by 2 of Def. 2.4 and 1 and 3 of Def. 2.3. □

Denote by $\mathcal{B}(R)$ the family of all Borel subsets of the real line R. Since F is a distribution function, there exists exactly one probability measure $\lambda_F : \mathcal{B}(R) \to [0,1]$ such that

$$\lambda_F([a, b)) = F(b) - F(a)$$

for any $a, b \in R, a < b$.

Recall that in the Kolmogorov theory the mean value $E(\xi)$ of a random variable $\xi : (\Omega, \mathcal{S}, P) \to R$ is defined as an integral

$$E(\xi) = \int_\Omega \xi dP$$

Let $g : R \to R$ be a Borel measurable function. The transformation formula states

$$E(g \circ \xi) = \int_\Omega g \circ \xi dP = \int_R g dP_\xi = \int_R g(t) dF(t),$$

where F is the distribution function of ξ. It motivates the following definition.

Definition 2.6. *An observable $x : \mathcal{J} \to D$ is integrable, if there exists*

$$E(x) = \int_R t dF(t),$$

where F is the distribution function of x.

3 INDEPENDENCE

As a motivation consider a probability space $(\Omega, \mathcal{S}, P)$, where Ω is a non-empty set, $\mathcal{S}$ is a σ-algebra of subsets of Ω and $P : \Omega \to [0,1]$ is a probability measure. Two random variables $\xi, \eta : \Omega \to R$ are independent, if

$$P(\xi^{-1}(A) \cap \eta^{-1}(B)) = P(\xi^{-1}(A)).P(\eta^{-1}(B))$$

for any Borel sets $A, B \in \mathcal{B}(R)$. Let F_1 or F_2 be distribution functions of ξ or η resp., i.e.

$$F_1(t) = P(\{\omega; \xi(\omega) < t\}),$$

$$F_2(t) = P(\{\omega; \eta(\omega) < t\}).$$

Define Borel probability measures $\lambda_{F_1}, \lambda_{F_2} : \mathcal{B}(R) \to [0,1]$ by such a way that

$$\lambda_{F_1}([a,b)) = F_1(b) - F_1(a)$$

$$\lambda_{F_2}([a,b)) = F_2(b) - F_2(a)$$

for any $a, b \in R, a \leq b$. It is very well known that there exists exactly one probability measure

$$\lambda_{F_1} \times \lambda_{F_2} : \mathcal{B}(R^2) \to [0,1]$$

such that

$$\lambda_{F_1} \times \lambda_{F_2}(A \times B) = \lambda_{F_1}(A).\lambda_{F_2}(B)$$

for any $A, B \in \mathcal{B}(R)$. We need to characterize the probability distribution of the sum $\xi + \eta$, i.e.

$$P(\{\omega; \xi(\omega) + \eta(\omega) < t\}), t \in R.$$

Theorem 3.1. *Let $\xi, \eta : \Omega \to R$ be independent random variables, $\Delta_t = \{(u,v) \in R^2; u+v < t\}, t \in R, T = (\xi, \eta) : \Omega \to R^2$. Then*

$$P(T^{-1}(\Delta_t)) = \lambda_{F_1} \times \lambda_{F_2}(\Delta_t)$$

for any $t \in R$.

Proof. We have

$$P(T^{-1}(\Delta_t)) =$$

$$= P\left(\bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} \xi^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n})) \cap \eta^{-1}((-\infty, t - \frac{i}{2^n}))\right) =$$

$$= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} P\left(\xi^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n})) \cap \eta^{-1}((-\infty, t - \frac{i}{2^n}))\right) =$$

$$= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} P\left(\xi^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n}))\right) P\left(\eta^{-1}((-\infty, t - \frac{i}{2^n}))\right) =$$

$$= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} \lambda_{F_1}\left([\frac{i-1}{2^n}, \frac{i}{2^n})\right) \lambda_{F_2}\left((-\infty, t - \frac{i}{2^n})\right) =$$

$$= \lim_{n\to\infty} \sum_{i=-\infty}^{\infty} \lambda_{F_1} \times \lambda_{F_2} \left([\frac{i-1}{2^n}, \frac{i}{2^n}) \times (-\infty, t - \frac{i}{2^n}) \right) =$$

$$= \lim_{n\to\infty} \lambda_{F_1} \times \lambda_{F_2} \left(\bigcup_{i=-\infty}^{\infty} [\frac{i-1}{2^n}), \frac{i}{2^n} \times (-\infty, t - \frac{i}{2^n}) \right) =$$

$$= \lambda_{F_1} \times \lambda_{F_2} \left(\bigcup_{n=1}^{\infty} \bigcup_{i=-\infty}^{\infty} [\frac{i-1}{2^n}, \frac{i}{2^n}) \times (-\infty, t - \frac{i}{2^n}) \right) =$$

$$= \lambda_{F_1} \times \lambda_{F_2}(\Delta_t).$$

If $T = (\xi, \eta) : \Omega \to R^2$ is a random vector, then $T^{-1} : \mathcal{B}(R^2) \to \mathcal{S}$ is a mapping such that

$$P(T^{-1}(\Delta_t)) = \lambda_{F_1} \times \lambda_{F_2}(\Delta_t), t \in R.$$

The idea may be realized also in our general case.

Definition 3.2. *Let $x_1, ..., x_n : \mathcal{J} \to D$ be observables, $\Delta_t^n = \{(u_1, ..., u_n) \in R^n; u_1 + ... + u_n < t\}, \mathcal{M}_n = \{\Delta_t^n; t \in R\}$. The observables are called to be independent, if there exists a mapping $h_n : \mathcal{M}_n \to D$ with the following properties:*
1.$t_i \nearrow t \Longrightarrow h_n(\Delta_{t_i}^n) \nearrow h_n(\Delta_t^n)$.
2.$h_n(\bigcup_{t=1}^{\infty} \Delta_t^n) = 1$.
3.$h_n(\bigcap_{t=-1}^{-\infty} \Delta_t^n) = 0$.
4.$m(h_n(\Delta_t^n)) = \lambda_{F_1} \times ... \times \lambda_{F_n}(\Delta_t^n), t \in R$.

Theorem 3.3. *Define $y_n : \mathcal{J} \to D$ by the equality $y_n((-\infty, t)) = h_n(\Delta_t^n)$. Then y_n is an observable.*

Proof. It follows by properties 1 - 3 of the previous Definition.

Definition 3.4. *Let $x_1, ..., x_n : \mathcal{J} \to D$ be independent observables. Then the observable $y_n : \mathcal{J} \to D$ defined in previous Theorem is called the sum of observables $x_1, ..., x_n, y_n = \sum_{i=1}^{n} x_i$, i.e.*

$$(\sum_{i=1}^{n} x_i)((-\infty, t)) = h_n(\Delta_t^n), t \in R.$$

Remark. There has been proved in [5] that in so-called Kôpka D-posets there exists the mapping $h_n : \mathcal{M}_n \to D$ satisfying the properties stated in previous Definition.

4 THE LAW OF LARGE NUMBERS

Recall first the classical weak law of large numbers.

Theorem 4.1. *Let (Ω, S, P) be a probability space. Let $(\zeta_n)_{n=1}^{\infty}$ be a sequence of independent random variables having the same distribution function. Let $a = E(\zeta_1) = E(\zeta_2) = \ldots$. Then the sequence of random variables*

$$\frac{\zeta_1 + \ldots + \zeta_n}{n} - a \;\; (n = 1, 2, \ldots)$$

converges in measure P to 0.

Of course, we haven't told yet, what does convergence in measure mean. In classical probability space (Ω, S, P) a sequence of random variables $(\zeta_n)_{n=1}^{\infty}$ converges to 0 in measure P, if for each real $\varepsilon > 0$

$$\lim_{n\to\infty} P\left(\zeta_n^{-1}([-\varepsilon,\varepsilon))\right) = 1.$$

In our case, the definition is similar, but at first we need to define an expression $x((a,b))$, where x is an observable and $a, b \in R$.

Definition 4.2. *Let $x : \mathcal{J} \to D$ be an observable on a D-poset D and $\alpha, \beta \in R$. Then*

$$x([a,b)) = x((-\infty,b)) - x((-\infty,a)).$$

Definition 4.3. *Let $(x_n)_{n=1}^{\infty}$ be a sequence of independent observables on a D-poset D with a state m. We say that this sequence converges in measure m to 0 if for each $0 < \varepsilon \in R$*

$$\lim_{n\to\infty} m\left(x_n([-\varepsilon,\varepsilon))\right) = 1.$$

We are able now to formulate and prove the main result of the paper. We shall use the following notation. If $y : \mathcal{J} \to D$ is an observable and α, β are real numbers, $\alpha \neq 0$, then $\alpha y + \beta : \mathcal{J} \to D$ is defined by the formula

$$(\alpha y + \beta)((-\infty,t)) = y((-\infty, \frac{1}{\alpha}(t-\beta))).$$

Theorem 4.4. *Let D be a D-poset with a state $m : D \to [0,1]$, let $(x_n)_{n=1}^{\infty}$ be an independent sequence of integrable observables having the same probability distribution, $E(x_n) = a$, $(n = 1, 2, \ldots)$. Then the sequence*

$$\frac{\sum_{i=1}^{n} x_i}{n} - a$$

converges in measure m to 0.

Proof. Denote $P_n = \lambda_{F_1} \times \ldots \times \lambda_{F_n} : \mathcal{B}(R^n) \to [0,1]$. Then $(P_n)_n$ presents a consistent system of probability measures:

$$P_n(A \times R) = P_{n-1}(A), A \in \mathcal{B}(R), n \in N$$

We will use the projection $\pi_n : R^N \to R^n$:

$$\pi_n((u_i)_{i=1}^{\infty}) = (u_1, u_2, \ldots, u_n).$$

Let's take a family of all cylinders $\mathcal{C}$, i. e. sequences with a finite number of members being fixed:

$$\mathcal{C} = \{A \subset R^N; A = \pi_n^{-1}(B), B \in \mathcal{B}(R_n), n \in N\}$$

By the Kolmogorov consistence theorem there exists a probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$P(\pi_n^{-1}(B)) = P_n(B) = \lambda_{F_1} \times \ldots \times \lambda_{F_n}(B) \tag{1}$$

for any $B \in \mathcal{B}(R^n), n \in N$. Define $\xi_n : R^N \to R$ by the formula

$$\xi_n((u_i)_{i=1}^{\infty}) = u_n.$$

We have obtained a Kolmogorov probability space $(R^N, \sigma(\mathcal{C}), P)$, where the mapping ξ_n presents a random variable. The next formula will serve as a tool for "translating" the law of large numbers to D-posets:

$$P(\xi_1 + \ldots + \xi_n < t) = P(\pi_n^{-1}(\Delta_n^t)) = P_n(\Delta_n^t) \underbrace{=}_{(1)} \lambda_{F_1} \times \ldots \times \lambda_{F_n}(\Delta_n^t) \underbrace{=}_{3.2} m(h_n(\Delta_n^t)) \underbrace{=}_{3.4}$$

$$= m\left((\sum_{i=1}^{n} x_i)((-\infty, t)) \right)$$

For simpler notation, let's introduce new two mappings: a random variable $\eta_n : R^N \to R$ and an observable $y_n : \mathcal{M}_n \to D$.

$$\eta_n = \frac{1}{n}(\sum_{i=1}^{n} \xi_i) - a.$$

$$y_n = \frac{1}{n}(\sum_{i=1}^{n} x_i) - a = (\sum_{i=1}^{n} x_i)((-\infty, n(t+a))),$$

Then

$$m\left(y_n((-\infty, t))\right) = m\left((\sum_{i=1}^{n} x_i)((-\infty, n(a+t))) \right) = P(\xi_1 + \ldots + \xi_n < n(a+t)) = P(\eta_n^{-1}((-\infty, t)))$$

The last thing we need before we can use the Theorem 4.1 is to prove, that ξ_i are independent and $E(\xi_i) = E(x_i) = a \; \forall i$.

$$P((\xi_1, \ldots, \xi_n) \in ((-\infty, t_1) \times \ldots \times (-\infty, t_n))) = P(\pi_n^{-1}((-\infty, t_1) \times \ldots \times (-\infty, t_n))) =$$

$$= \lambda_{F_1} \times \ldots \times \lambda_{F_n}((-\infty, t_1) \times \ldots \times (-\infty, t_n)) = \lambda_{F_1}((-\infty, t_1)). \ldots .\lambda_{F_n}((-\infty, t_n)) =$$

$$= P(\xi_1 < t_1). \ldots .P(\xi_n < t_n)$$

$$E(x_n) = \int_{-\infty}^{\infty} t d\lambda_{F_n}(t)$$

$$E(\xi_n) = \int_{R^N} \xi_n(u) dP(u) = \int_{R^n} u_n dP_n((u_1, \ldots, u_n)) =$$

$$= \int_{R^n} u_n d\lambda_{F_1} \times \ldots \times \lambda_{F_n}((u_1, \ldots, u_n)) = \int_R u_n d\lambda_{F_n}(u_n) = E(x_n)$$

Now all the assumptions of Theorem 4.1 are satisfied, so for all real $\varepsilon > 0$ there holds:

$$1 = \lim_{n\to\infty} P\left(\eta_n^{-1}([-\varepsilon, \varepsilon))\right) = \lim_{n\to\infty} P\left(\eta_n^{-1}((-\infty, \varepsilon))\right) - \lim_{n\to\infty} P\left(\eta_n^{-1}((-\infty, -\varepsilon))\right) =$$

$$= \lim_{n\to\infty} m\left(y_n((-\infty, \varepsilon))\right) - \lim_{n\to\infty} m\left(y_n((-\infty, -\varepsilon))\right) = \lim_{n\to\infty} m\left(y_n([-\varepsilon, \varepsilon))\right) =$$

$$= \lim_{n\to\infty} m\left((\frac{1}{n}(\sum_{i=1}^{n} x_i) - a)([-\varepsilon, \varepsilon)) \right)$$

Hence,

$$\frac{1}{n}(\sum_{i=1}^{n} x_i) - a \longrightarrow 0$$

in measure m and

$$\frac{1}{n}(\sum_{i=1}^{n} x_i) \longrightarrow a$$

in measure m. □

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