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A NEW METHOD OF GENERATION OF NONLINEAR NORMAL MODES FOR NONLINEAR DYNAMICS OF DISORDERED STRUCTURES

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Abstract. *Structural response can be affected by randomness under two different points of view – either in the forcing process and/or in the structural behaviour. Various approaches can be employed to investi-gate the structural response, among which polynomial chaos (Ghanem, Spanos [1]) and perturbation approaches (Liu et al. [2], Chiostrini and Facchini [3]) can be found.*

Both kinds of approaches require the investigation of the response variation in dependence on the (random) structural parameters, thus – often dramatically – increasing the number of degrees of freedom of the examined system.

In case of linearity of the system response, a modal reduction approach can be effectively employed together with perturbation techniques; on the other hand, for some kind of nonlinear systems, such reduction approach (Bucher [4], Geschwindner [5]) might fail to give satisfactory results unless a very large number of modes is employed in the analysis (Betti et al. [6]).

The task of reducing the number of degrees of freedom of nonlinear systems has therefore to be accomplished by means of alternative procedures, such as the introduction of nonlinear normal modes.

1 INTRODUCTION

Randomness can affect structural systems by several points of view; in particular, randomness can affect the forcing process that a system undergoes, or might as well influence the structural parameters.

The study of disordered systems, as such structures are usually referred to, is a point of concern in specialized literature: a fair amount of methods to investigate the dynamics of disordered structures can in fact be found.

One of the most common methods is taken into consideration in the present work: the first application is described in Liu et al. (1986), and subsequently enhanced in Chiostrini and Facchini (1999); it can be classified as a perturbation method and makes use of sensitivity vectors to evaluate the first two moments of the response.

Unfortunately, a severe drawback of the method is that the number of degrees of freedom of the examined structure grows rapidly for increasing number of random parameters, thus leading to the solution of very large (non) linear systems.

The idea that is introduced in the present work is to investigate the possibility to compute and apply the concept of nonlinear normal modes in order to reduce the number of degrees of freedom of the resulting system, as it has recently been examined by Rizzo (2007).

It is well known that the normal modes are of fundamental importance in the theory of linear dynamic conservative and non conservative systems, as the linear normal modes can be used to decouple the equations governing the motion and analytically evaluate the dynamic response of the examined system.

Such procedure is performed making use of modal analyses and the principle of superposition to express the response of the system as a time-dependent superposition of its modal shapes.

Clearly, such an approach is generally inapplicable in the nonlinear theory. Nevertheless, it is possible to define nonlinear normal modes (NNMs) as particular synchronous periodic solutions of the non-linear motion equations, but no link of such motions to the principle of superposition can be considered.

Several techniques can be found in specialized literature for determining the response of nonlinear systems; for free vibration problems system modes can be usefully employed to construct reduced order models: such procedures have been well developed for both linear and nonlinear systems by Vakakis (1997) and by Vakakis et al. (1996) .

One such technique, introduced by Shaw and Pierre (1991, 1993, 1993), defines the normal mode of a nonlinear oscillatory system in terms of invariant manifolds in the phase space that are tangent to the linear eigenmodes at the equilibrium point. In such a formulation, a master mode is selected, and the normal mode is constructed by a formulation in which the remaining linear modes of the system, i.e., the slave modes, depend on the master mode in a manner consistent with the system dynamics. This dependence defines the invariant manifold for the nonlinear normal mode (NNM).

The construction of the NNM invariant manifold is equivalent to the determination of the constraint relationships for all of the slave coordinates with the master coordinate; once these constraint relationships are obtained, the system dynamics can be restricted to the invariant manifold, resulting in a minimalsized model that depends only on the master coordinates. By studying the dynamics of the reduced-order model, it is possible to recover the associated modal dynamics of the original nonlinear system.

Pesheck et al. (2002) used numerical solutions of the invariant manifold equations to extend the invariant manifold approach to more general systems, including strongly nonlinear ones. In this approach, the master coordinates were expressed in polar coordinate form, and a Galerkin-based solution technique was introduced to solve the invariant manifold equations.

The present method differs from the work by present authors (2009), because the modal forms are only generated regarding the lagrangian coordinates and not regarding the total coordinate, including the derived to the first one and second order.

2 FORMULATION

A system endowed with a displacement dependent nonlinear restoring force which can be expressed by means of a nonlinear stiffness matrix is taken into consideration. The stiffness is affected by randomness in one or more of the defining parameters; such random parameters will be grouped together in the vector **b**.

In specialised literature a common approach for the study of the dynamic response of nonlinear systems is to express the dependence of the system response on the random parameters at each instant of the motion, and eventually combine such dependence with the probability distribution function of the random parameters themselves.

Several examples can be found of this approach. In the following, the method proposed by Liu *et al*. (1986) and successively modified and enhanced by Chiostrini and Facchini (1999) is considered.

Let the equation of motion of a N DOF disordered nonlinear system, where the nonlinear restoring forces depend on both the system response and the uncertain parameters grouped in the vector **b**, be given in the form:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{g}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{b}) = \mathbf{f}(t) \qquad \textbf{(1)}$$

The system velocity and displacement vectors are expressed by means of a second order expansion with respect to the mean value of the random parameter vector **b**:

$$x_h(t, \mathbf{b}) \cong \bar{x}_h(t) + \frac{\partial \bar{x}_h}{\partial b_i}\left(b_i - \bar{b}_i\right) + \frac{1}{2}\frac{\partial^2 \bar{x}_h}{\partial b_i \partial b_j}\left(b_i - \bar{b}_i\right)\left(b_j - \bar{b}_j\right) \qquad \textbf{(2)}$$

where the overbar denotes that the quantity is evaluated in correspondence of the expected value of the random parameters **b**. A completely analogous equation holds for the system velocity.

The response and its derivatives up to the second order can be evaluated by means of their respective equations of motion, obtained by differentiation of the system equation of motion, evaluated in correspondence of the expected values of the random vector **b**. Thus, the first group of equations is given by

$$\mathbf{M}\ddot{\bar{\mathbf{x}}} + \mathbf{g}(\dot{\bar{\mathbf{x}}}, \bar{\mathbf{x}}, \bar{\mathbf{b}}) = \mathbf{f}(t) \qquad \textbf{(3)}$$

while the second and third groups of equations can be obtained deriving equation (1) with respect to each component of vector **b** and considering again the expected value of the random vector:

$$\begin{aligned} \mathbf{M}\ddot{\bar{\mathbf{x}}}_{,l} + \mathbf{C}_T\dot{\bar{\mathbf{x}}}_{,l} + \mathbf{K}_T\bar{\mathbf{x}}_{,l} &= \mathbf{f}_1(t) \\ \mathbf{M}\ddot{\bar{\mathbf{x}}}_{,lm} + \mathbf{C}_T\dot{\bar{\mathbf{x}}}_{,lm} + \mathbf{K}_T\bar{\mathbf{x}}_{,lm} &= \mathbf{f}_2(t) \end{aligned} \qquad \textbf{(4)}$$

where

$$\mathbf{f}_1(t) = -\frac{\partial \mathbf{g}}{\partial b_l} \tag{5}$$

$$\mathbf{f}_2(t) = -\frac{\partial \mathbf{C}_T}{\partial b_m}\dot{\bar{\mathbf{x}}}_{,l} - \frac{\partial \mathbf{K}_T}{\partial b_m}\bar{\mathbf{x}}_{,l} - \frac{\partial \mathbf{C}_T}{\partial b_l}\dot{\bar{\mathbf{x}}}_{,m} - \frac{\partial \mathbf{K}_T}{\partial b_l}\bar{\mathbf{x}}_{,m} - \frac{\partial^2 \mathbf{g}}{\partial b_l \partial b_m}$$

The symbols $\mathbf{C}_T$ and $\mathbf{K}_T$ respectively denote the tangent damping and stiffness matrices, obtained by derivation of the nonlinear restoring function:

$$\mathbf{C}_T = \frac{\partial \mathbf{g}}{\partial \dot{\mathbf{x}}}; \quad \mathbf{K}_T = \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \tag{6}$$

When the nonlinear restoring function can be expressed by only means of a response-dependent matrix times the displacement, as in:

$$\mathbf{g}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{b}) = \mathbf{g}(\mathbf{x}, \mathbf{b}) = \mathbf{K}(\mathbf{x}, \mathbf{b})\,\mathbf{x} \tag{7}$$

then the components of the tangent stiffness matrix and its derivatives can be obtained as:

$$K_{T\,ij} = \frac{\partial g_i}{\partial x_j} = \frac{\partial K_{ih}}{\partial x_j} x_h + K_{ij} \tag{8}$$

$$\frac{\partial g_i}{\partial b_l} = \frac{\partial K_{ij}}{\partial b_l} x_j; \quad \frac{\partial^2 g_i}{\partial b_l \partial b_m} = \frac{\partial^2 K_{ij}}{\partial b_l \partial b_m} x_j$$

$$\frac{\partial K_{T\,ij}}{\partial b_l} = \frac{\partial^2 K_{ih}}{\partial b_l \partial x_j} x_h + \frac{\partial^2 K_{ih}}{\partial x_m \partial x_j}\frac{\partial x_m}{\partial b_l} x_h + \\ + \frac{\partial K_{ih}}{\partial x_j}\frac{\partial x_h}{\partial b_l} + \frac{\partial K_{ij}}{\partial b_l} + \frac{\partial K_{ij}}{\partial x_m}\frac{\partial x_m}{\partial b_l}$$

and equation (4) can be simplified in the following form:

$$\mathbf{M}\ddot{\bar{\mathbf{x}}}_{,l} + \mathbf{K}_T\bar{\mathbf{x}}_{,l} = -\frac{\partial \mathbf{g}}{\partial b_l} \tag{9}$$

$$\mathbf{M}\ddot{\bar{\mathbf{x}}}_{,lm} + \mathbf{K}_T\bar{\mathbf{x}}_{,lm} = -\frac{\partial \mathbf{K}_T}{\partial b_m}\bar{\mathbf{x}}_{,l} - \frac{\partial \mathbf{K}_T}{\partial b_l}\bar{\mathbf{x}}_{,m} - \frac{\partial^2 \mathbf{g}}{\partial b_l \partial b_m}$$

Equation (3) together with (4) or (8) define a set of a total of $N\left(1 + N_b + N_b^2\right)$ scalar equations which give the evolution in time of the system response and its derivatives with respect to the random parameters. N_b is in fact the number of the random parameters of the system, which correspond to the dimension of vector **b**.

For systems with many degrees of freedom, affected by randomness in one or more parameters, the number of equations grows in a way that can be hardly manageable by current computers. In order to reduce the number of equations, the non-linear normal modes of the system can be introduced.

The approach followed in the application was introduced by a series of works by Shaw and Pierre (1991, 1993, 1993).

The new formulation, regarding the work by present authors (2009), regards the resolution of the equations (7) and (7). This equations can be resolved separately. Only the equations (7) can be resolved with the use of non-linear normal mode instead the equations (7), which are decoupled from the (7) , with a non linear dynamic analysis.

The first group of equations can be resolved by a non-linear normal mode approach by means of the explicitation of the linear part of the equations of motion. The modal shapes and associated eigenfrequencies of the linearized system can be calculated, and the LHS term of system **Errore. L'origine riferimento non è stata trovata.** can be diagonalized, obtaining

$$\ddot{\boldsymbol{\eta}} + \boldsymbol{\Omega}\boldsymbol{\eta} = \mathbf{f}(\boldsymbol{\eta}) \tag{10}$$

where $\boldsymbol{\eta} = \boldsymbol{\Phi}\mathbf{y}$ is the vector of the principal coordinates, $\boldsymbol{\Phi}$ is the matrix containing the linearized modal shapes of the system **Errore. L'origine riferimento non è stata trovata.** and $\boldsymbol{\Omega}$ is a diagonal matrix whose elements are ω_k^2.
If η_m denotes the master degree of freedom, its evolution in time is described by the expression:

$$\begin{aligned} \eta_m &= a\cos\phi \\ \dot{\eta}_m &= -a\omega_m \sin\phi \end{aligned} \tag{11}$$

and the remaining “slave” degrees of freedom are expressed in terms of the master amplitude and phase as

$$\begin{aligned} \eta_i &= P_i(a,\phi) \\ \dot{\eta}_i &= Q_i(a,\phi) \end{aligned} \tag{12}$$

The time evolution of the master amplitude and phase can be expressed by the relations

$$\begin{aligned} \dot{a} &= -\frac{f_m(\boldsymbol{\eta})}{\omega_m}\sin\phi \\ \dot{\phi} &= \omega_m - \frac{f_m(\boldsymbol{\eta})}{a\omega_m}\cos\phi \end{aligned} \tag{13}$$

and also the evolution of functions P_i and Q_i in equations (12) can be obtained by the diagonalized system (10) in the form

$$\begin{aligned} \dot{P}_i(a,\phi) &= Q_i(a,\phi) \\ \dot{Q}_i(a,\phi) &= f_i(\boldsymbol{\eta}) - \omega_i^2 P_i(a,\phi) \end{aligned} \tag{14}$$

Functions P_i and Q_i in equations (12) can be expressed in incremental terms, obtaining first order differential equations where the independent variable is time:

$$\begin{aligned} \dot{P}_i(a,\phi) &= \frac{\partial P_i}{\partial a}\dot{a} + \frac{\partial P_i}{\partial \phi}\dot{\phi} \\ \dot{Q}_i(a,\phi) &= \frac{\partial Q_i}{\partial a}\dot{a} + \frac{\partial Q_i}{\partial \phi}\dot{\phi} \end{aligned} \tag{15}$$

When the system is subject to free oscillations and therefore $\mathbf{f}(t) = \mathbf{0}$ in equations (1) and **Errore. L'origine riferimento non è stata trovata.**, by means of the substitution of equations (13) and (14) into (15), first order differential equations are obtained for the first derivatives of functions P_i and Q_i, which are independent of time. Such equations describe the geometry of the considered non normal mode (see Pesheck *et al.* (2002) for details):

$$Q_i(a,\phi) = -\frac{\partial P_i}{\partial a}\frac{f_m(\boldsymbol{\eta})}{\omega_m}\sin\phi + \frac{\partial P_i}{\partial \phi}\left[\omega_m - \frac{f_m(\boldsymbol{\eta})}{a\omega_m}\cos\phi\right] \tag{16}$$

$$f_i(\boldsymbol{\eta}) - \omega_i^2 P_i(a,\phi) = -\frac{\partial Q_i}{\partial a}\frac{f_m(\boldsymbol{\eta})}{\omega_m}\sin\phi + \frac{\partial Q_i}{\partial \phi}\left[\omega_m - \frac{f_m(\boldsymbol{\eta})}{a\omega_m}\cos\phi\right]$$

When the evaluation of the non normal mode is completely performed, the corresponding vector **y** can be evaluated by means of

$$\begin{aligned}\mathbf{y} &= \boldsymbol{\Phi}^t\boldsymbol{\eta} = \\ &= \boldsymbol{\Phi}^t\left[P_1 \quad \cdots \quad P_{m-1} \quad a\cos\phi \quad P_{m+1} \qquad P_{N_{tot}}\right]\end{aligned} \tag{17}$$

The projection in the lagrangian coordinates $\bar{\mathbf{x}}$ and the resolution of the second group of equations (13) gives the desired approximation for the dependence of the system response **x** on the random parameters vector **b**.
As outlined in the works of Liu *et al*. (1986) and Chiostrini and Facchini (1999), the response statistics can be obtained by

$$\begin{aligned}\mu_{x_h} &\cong \bar{x}_h(t) + \frac{1}{2}\frac{\partial^2 \bar{x}_h}{\partial b_i \partial b_j} E\left[\left(b_i - \bar{b}_i\right)\left(b_j - \bar{b}_j\right)\right] \\ \sigma_{x_h}^2 &\cong \frac{\partial \bar{x}_h}{\partial b_i}\frac{\partial \bar{x}_h}{\partial b_j} E\left[\left(b_i - \bar{b}_i\right)\left(b_j - \bar{b}_j\right)\right]\end{aligned} \tag{18}$$

3 APPLICATION

The method's application in the present work is make with the use of a multidimensional Duffing equation:

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} + \lambda(\mathbf{x}^{\mathbf{t}}\mathbf{C}\mathbf{x})\mathbf{K}\mathbf{x} = \mathbf{0} \tag{19}$$

In the *2*DOF system used **M**=[1 0;0 2], **K**=[10 -5;-5 5], **C**=[1 -0.5;-0.5 1] and λ_m=0.001 and 0.01.

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}(1 + \lambda_m \mathbf{x}^t\mathbf{C}\mathbf{x})\mathbf{x} = \mathbf{0} \tag{20}$$

$$\begin{aligned}&\mathbf{M}\ddot{\mathbf{x}},_{\lambda} + \mathbf{K}\left[\mathbf{x}^t\mathbf{C}\mathbf{x} + \lambda_m(\mathbf{x},_{\lambda}^t \mathbf{C}\mathbf{x} + \mathbf{x}^t\mathbf{C}\mathbf{x},_{\lambda})\right]\mathbf{x} + \\ &+ \mathbf{K}(1 + \lambda_m \mathbf{x}^t\mathbf{C}\mathbf{x})\mathbf{x},_{\lambda} = \mathbf{0}\end{aligned} \tag{21}$$

$$\begin{aligned}&\mathbf{M}\ddot{\mathbf{x}},_{\lambda\lambda} + \mathbf{K}\left[2\mathbf{x},_{\lambda}^t \mathbf{C}\mathbf{x} + 2\mathbf{x}^t\mathbf{C}\mathbf{x},_{\lambda} + \lambda_m(\mathbf{x},_{\lambda\lambda}^t \mathbf{C}\mathbf{x} + 2\mathbf{x},_{\lambda}^t \mathbf{C}\mathbf{x},_{\lambda} + \mathbf{x}^t\mathbf{C}\mathbf{x},_{\lambda\lambda})\right]\mathbf{x} \\ &+ 2\mathbf{K}\left[\mathbf{x}^t\mathbf{C}\mathbf{x} + \lambda_m(\mathbf{x},_{\lambda}^t \mathbf{C}\mathbf{x} + \mathbf{x}^t\mathbf{C}\mathbf{x},_{\lambda})\right]\mathbf{x},_{\lambda} + \\ &+ \mathbf{K}(1 + \lambda_m \mathbf{x}^{\mathbf{t}}\mathbf{C}\mathbf{x})\mathbf{x},_{\lambda\lambda} = \mathbf{0}\end{aligned} \tag{22}$$

The groups of equations (13), (13) and (13) are solved separately; in the first is solved the system (13) and the linear system associated gives ω_1=1.047 rad/s, ω_2= 3.377 rad/s and $\boldsymbol{\Phi}$=[0.369 -0.929; 0.657 0.261].

The non-linear mode shapes are generated for the first group of equation while the second and third group of equations are solved directly integrating the differential equations. this is possible because the first group of equations does not depend from the derivate ones of x. A possible future development of the method is to resolve also according to group and the third by means of use of the NNMs.

The application is made with λ_m=0.001 and are showed in figure 1, while the application with λ_m=0.01 in figure 2. The numeric simulation for the free oscillation are showed in figure 3 and 4 for λ_m=0.001 and for λ_m=0.01.

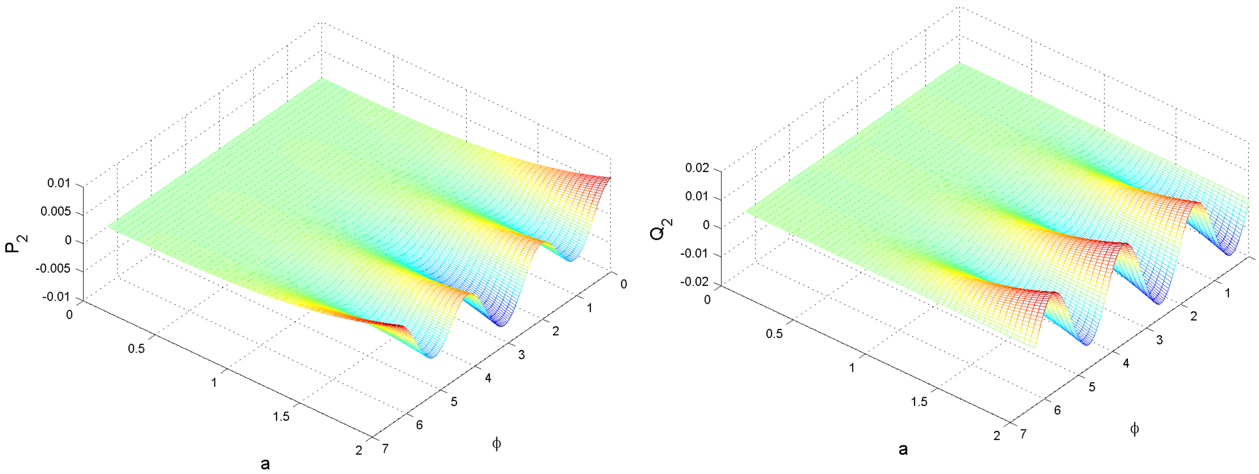

Figure 1. Non linear modal shape P_2 Q_2 for λ_m =0.001 and N_a=2, N_ϕ=4.

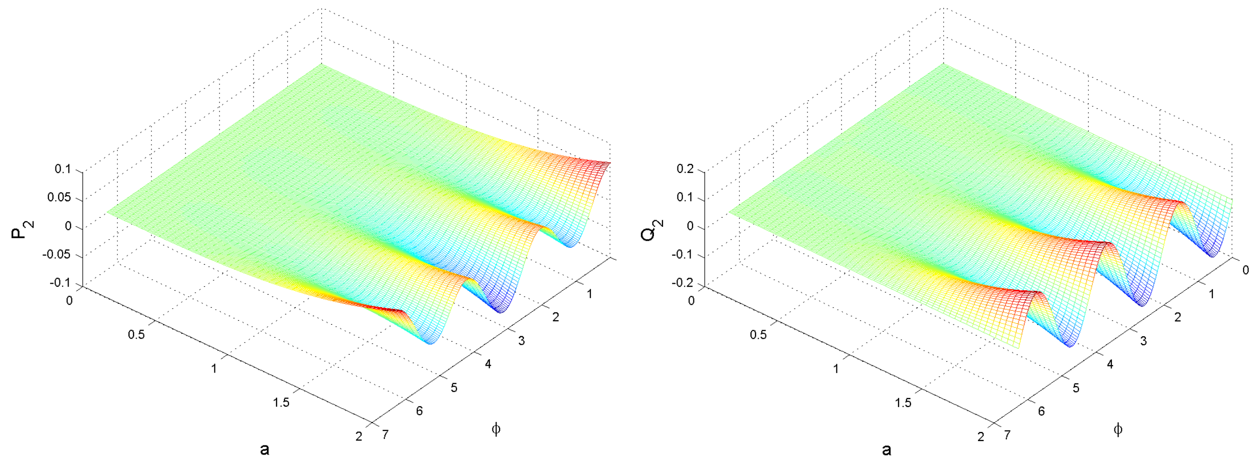

Figure 2. Non linear modal shape P_2 Q_2 for λ_m =0.01 and N_a=2, N_ϕ=4.

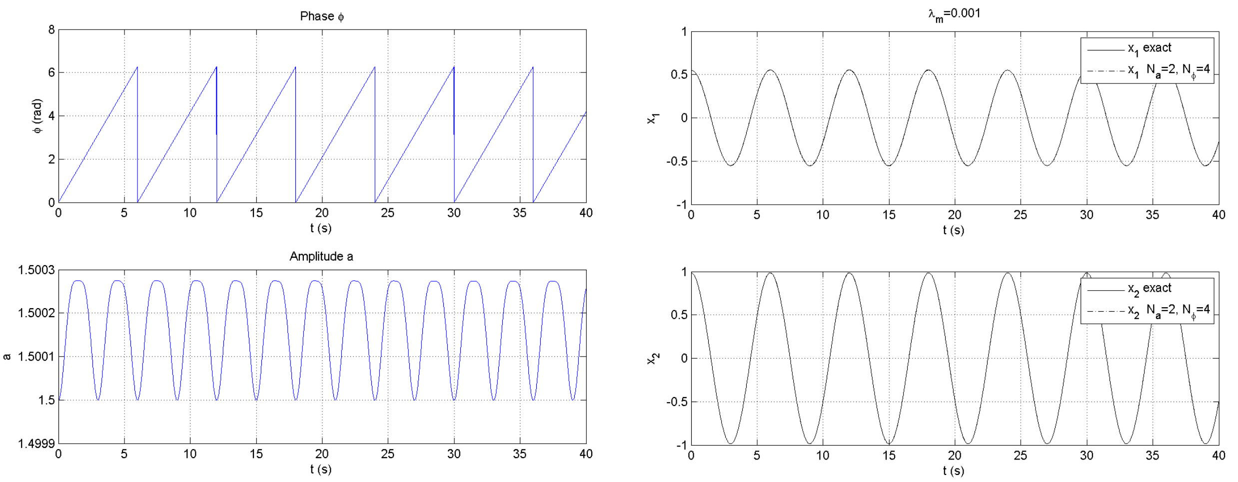

Figure 3. Numerical simulation of the amplitude a and ϕ and the system response $\mathbf{x}(t)$

obtained imposing λ_m=0.001.

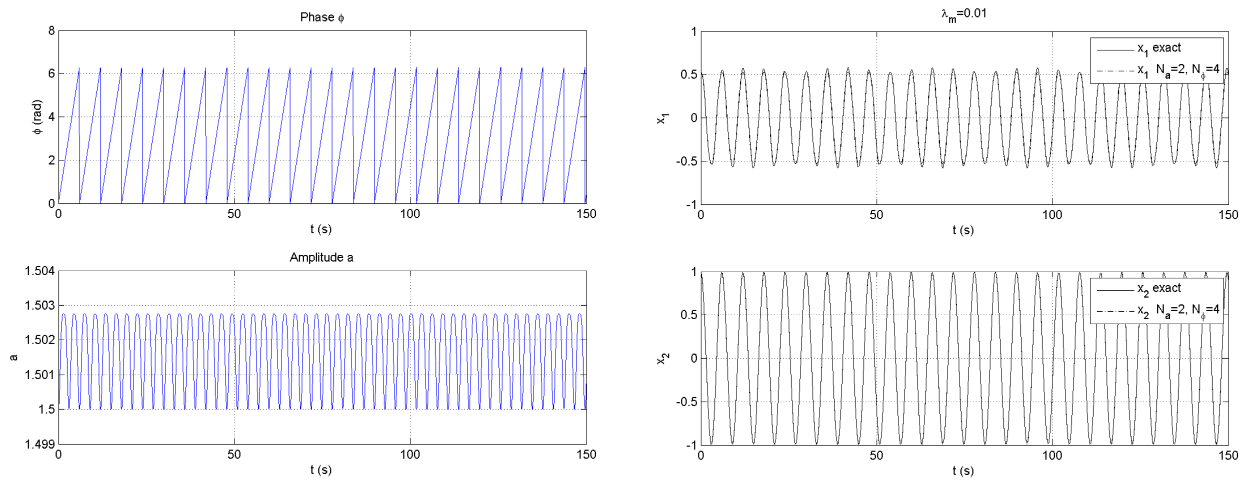

Figure 4. Numerical simulation of the amplitude a and ϕ and the system response $\mathbf{x}(t)$

obtained imposing λ_m=0.01.

The method that uses the NNM's and the exact simulations in figure 3 and 4 gives results indistinguishable. In order to gain the expected value and the standard deviation of the variable ones x_1 and x_2 the following approximations in the works of Liu *et al*. (1986) and Chiostrini and Facchini (1999) are used:

$$E[x] = x(\varepsilon_m) + \frac{1}{2} x_{,\varepsilon\varepsilon} \cdot \sigma_\varepsilon^2; \tag{23}$$

$$\sigma_x^2 \cong x_{,\varepsilon}^2 \cdot \sigma_\varepsilon^2; \tag{24}$$

$$x \cong x(\varepsilon_m) + (\varepsilon - \varepsilon_m) \cdot x_{,\varepsilon} + \frac{1}{2} x_{,\varepsilon\varepsilon} \cdot (\varepsilon - \varepsilon_m)^2. \tag{25}$$

In the following figures the variations of x_1 are brought back and x_2 regarding λ and variable t. The variation coefficient used in the numerical simulations is 50%.

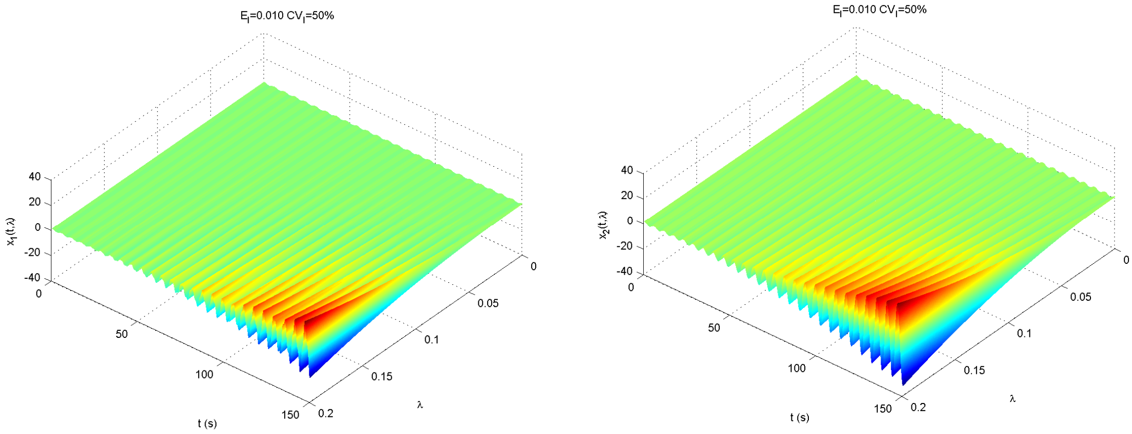

Figure 5. Numerical simulation of the system response $\mathbf{x_1}(t, \lambda)$ and $\mathbf{x_2}(t, \lambda)$ obtained with $\lambda_m = 0.01$.

In order to confront the method we have been simulated 1000 dynamic analyses with λ_m=0.01 and coefficient of variation 50%. In figure 6 the corresponding probability distribution is brought back.

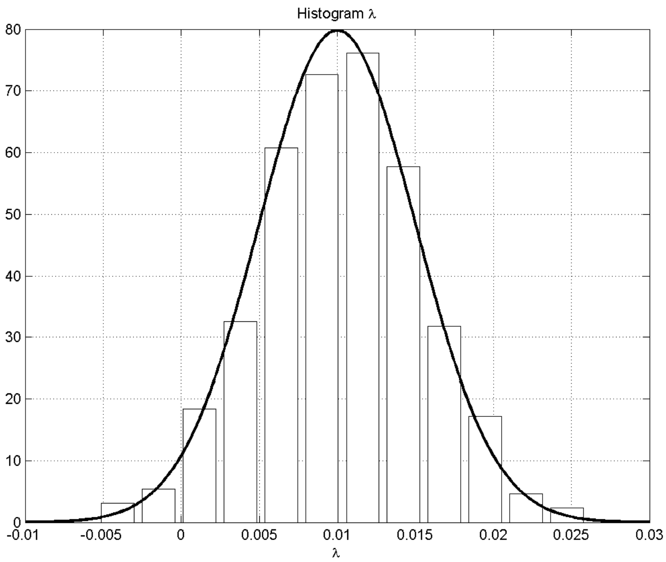

Figure 6. Normal probability distribution for the numerical simulation with $\lambda_m = 0.01$ and CV 50% .

The comparison between the adopted method and the simulations random is shown in the following figures. The method that uses the NNM's and the simulations random gives results indistinguishable.

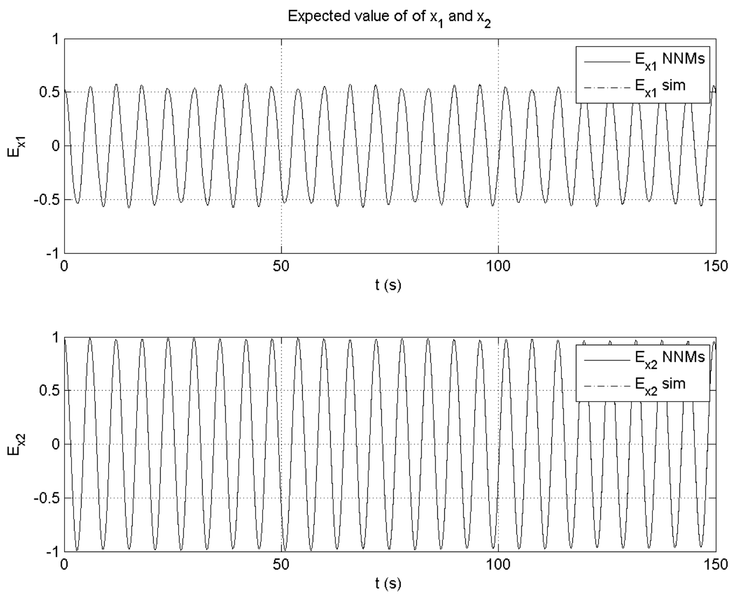

Figure 7. Numerical simulation of the expected value system response $E\mathbf{x_1}$ and $E\mathbf{x_2}$ obtained with λ_m =0.01 .

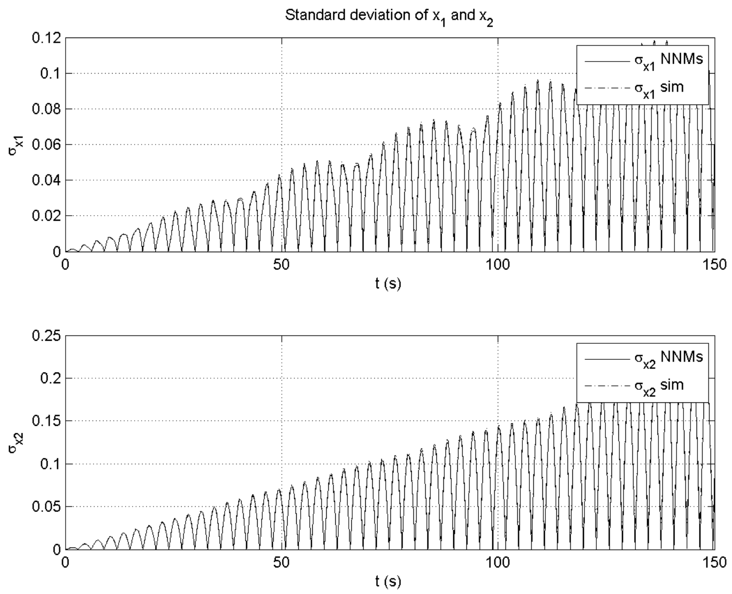

Figure 8. Numerical simulation of the standard deviation of the system response $\sigma\mathbf{x_1}$ and $\sigma\mathbf{x_2}$ obtained with λ_m =0.01 .

4 CONCLUSIONS

In the present formulation is developed a new approach that can be effectively used for the solution of the non linear dynamics of large disordered structures, when the number of degrees of freedom considerably increase owing to the presence of randomness.

The most severe problem in the present formulation is to obtain numerically the non linear modal shapes, therefore future developments of the present method will be devoted to finding a more efficient numerical method to obtain the NNM's.

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