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FREE VIBRATION PROBLEM OF PRSI BRIDGES: ANALYTICAL SOLUTION AND PARAMETRIC ANALYSIS

E. Tubaldi[1], **A. Dall'Asta**[2]

[1] Dipartimento di Architettura Costruzione e Strutture, Marche Polytechnic University,
Via Brecce Bianche, 60131, Ancona (AN), Italy
etubaldi@libero.it

[2] Dipartimento di Progettazione e Costruzione dell'Ambiente, University of Camerino
Viale della Rimembranza, 63100, Ascoli Piceno (AP), Italy
a.asta@tin.it

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Abstract. *This paper analyzes the dynamic behavior of partially restrained seismically isolated (PRSI) bridges. These are a particular class of multi-span seismically isolated bridges in which isolation bearings are posed only at the top of the piers, while seismic stoppers restrain the transverse motion of the superstructure at the abutments.*

The transverse dynamic behavior of these partially-restrained bridges is described analytically by considering a two-dimensional simply supported beam model, with intermediate visco-elastic restraints whose properties are calibrated to describe the substructures' behavior. Particular simplified configurations are considered which allow to identify a minimal set of characteristic problem parameters that completely describe the dynamic response. The properties of the dynamic systems are analyzed by considering separately the undamped and the damped case.

The results of the study contribute to improve the understanding of the dynamic behavior of partially restrained seismically isolated bridges and allow to draw some conclusions useful for their preliminary assessment and design.

1 INTRODUCTION

Seismic isolation in bridges has proven to be an effective method for the mitigation of the forces induced by the seismic actions ([1],[2]). It usually consists in the introduction of isolation/dissipation devices between the bridge substructures and the superstructure, in order to decouple their motion, shift the natural period of vibration away from the dominant period of earthquake excitation and introduce additional sources of damping.

Partially restrained seismically isolated (PRSI) bridges ([3]) are a particular class of isolated bridges in which isolation bearings are posed only at the top of the piers, with seismic stoppers restraining the transverse motion of the superstructure at the abutments. This restraint is usually introduced in order to avoid the use of bi-directional joints at the abutments and to exploit the abutment contribution in resisting the inertia forces, thus reducing the forces acting on the piers.

Recently, many authors have investigated the dynamic behavior of PRSI bridges ([3], [4], [5]). Tsai [3] evaluated the effectiveness of the partially restrained seismic isolation and describes the difference of the behavior of PRSI bridges with respect to fully isolated bridges. Analytical expressions were also proposed for estimating the transverse effective period and composite damping ratio of PRSI bridges. Makris et al. [4] examined the eigenvalues of PRSI bridges under transverse and longitudinal vibration, by considering the two cases of elastomeric bearings and friction-pendulum bearings. They concluded that regardless of the value of the isolation period along the longitudinal direction, there is a certain length beyond which the transverse period of the deck will exceed the longitudinal isolation period. Tubaldi and Dall’Asta [5] analyzed the dynamic problem of PRSI bridges in a variational form, in order to obtain a simplified solution based on assumed vibration shapes which coincide with the Fourier sine-only series terms. They also defined a design procedure for dimensioning the properties of the isolation system, with the objective of controlling the internal actions on the piers. The applications to realistic PRSI bridges allowed to highlight the following results: a) the response in terms of some quantities of interest in the seismic analysis (e.g., the abutment reactions and the superstructure transverse bending moments) can be strongly influenced by higher vibration modes, b) the response according to the higher modes of vibration is less affected by the pier-bearing restraint action and depends mostly on the superstructure properties, c) neglecting the non-classically damped nature of the problem may induce some inaccuracies in the assessment of the seismic response of this particular type of system.

In summary, PRSI bridges show a complex dynamic behavior in the transverse direction and many modal properties related to the seismic response, such as modal shapes and extent of non classical damping, significantly change by varying the properties and geometry of the deck and properties and location of the intermediate supports.

This study aims at providing some contribution to the further advancement in the understanding of the dynamic behavior of PRSI bridges. Under a qualitative point of view, the dynamic properties of a linear system, such as the ones analyzed in the present study, can be measured by solving the free vibrations problem. Thus, the focus of the paper is on the free-vibration response of PRSI bridges only.

The transverse dynamic behavior of PRSI bridges is described [5] by considering a model which consists in a continuous 2-dimensional simply supported beam resting on discrete visco-elastic supports. The properties of the intermediate supports can be calibrated to represent the behavior of the pier-bearing systems. Numerous studies are devoted to the analysis of the free vibration problem of continuous beams elastically restrained by intermediate linear springs ([6],[7],[8]) or dampers ([9],[10]). However, many of these studies analyze the problem from a purely mathematical perspective and are mostly focused on the influence of the restraint on the eigenvalues or in the observation of the transition condition in which multiple coincident eigenvalues are observed. Thus, there is a need for an engineering interpretation of the complex behavior of PRSI bridges in terms of physical quantities which can be of interest in the seismic analysis, e.g., displacement and bending moment shapes, participation factors, abutment reactions, vibration periods, damping properties and extent of non classical damping.

In this paper, the free vibration problem of a set of simplified PRSI bridges configuration is considered, assuming constant stiffness, mass and support spacing for the superstructure. This allows to identify a minimal set of characteristic problem parameters that completely describe the dynamic response of partially restrained bridges and to shed light on the relationship between these parameters and the response quantities of interest in the seismic analysis. Furthermore, the consideration of the simplified cases permits deriving an analytical solution of the free-vibrations problem and developing analytical relationships between the characteristic parameters and the system response.

The influence of the characteristic problem parameters on the free vibrations is analyzed through an extensive parametric analysis by considering separately the case of support systems without damping, and the case of support system with damping, thus generating a non-classical dynamical system. The results of the parametric analysis undertaken permit to draw important information which can be useful for the preliminary assessment and design of the particular class of bridges analyzed.

2 DYNAMIC BEHAVIOR OF PRSI BRIDGES

The continuous isolated bridge with partial restraint can be modeled as a 2-dimensional simply-supported beam resting on intermediate discrete visco-elastic supports that represent the pier-bearing systems (Fig. 1).

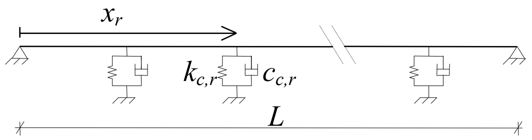

Fig. 1. Analytical 2-dimensional model for bridges with partial restraint.

Let $V=\{v(x)\in H^2[0,L]: v(0)=v(L)=0\}$ [1] be the space of displacement functions defined along the bridge length L and satisfying the kinematic (essential) boundary conditions and $u(x;t)\in U=C(V;[t_0,t_1])$ be the motion, defined in the time interval considered $[t_0, t_1]$.

The differential dynamic problem can be derived from the D'Alembert principle [11] and can be posed in the following form, $\forall\eta\in V;\forall t\in[t_0,t_1]$:

$$\begin{aligned}&\int_0^L m(x)\ddot{u}(x,t)\eta(x)dx+\int_0^L c_d(x)\dot{u}(x,t)\eta(x)dx+\sum_{r=1}^{N}c_{c,r}\dot{u}(x_r,t)\eta(x_r)+\\&+\int_0^L b(x)u''(x,t)\eta''(x)dx+\sum_{r=1}^{N}k_{c,r}u(x_r,t)\eta(x_{c,r})=-\int_0^L m(x)\ddot{u}_g(t)\eta(x)dx\end{aligned}\tag{1}$$

[1] The form $H^m(\Omega)$ denotes the space of functions defined on Ω for which the derivatives with respect to x of order less than or equal to m are on $L^m(\Omega)$.

The functions $m(x)$, $b(x)$ and $c_d(x)$ are piecewise continuous and denote the mass per unit length, the transverse stiffness per unit length and the distributed damping constant. The constants $k_{c,r}$ and $c_{c,r}$ are the stiffness and damping constant of the visco-elastic support located at the r-th position $x=x_r$. Finally, $\ddot{u}_g(t)$ denotes the ground motion input. It is noteworthy that although various models are available for describing the dissipation of energy in the deck [12],[13], in the present study the deck damping is simply described in terms of a force proportional to velocity. More cumbersome descriptions are avoided since the deck damping is usually smaller than the intermediate supports' damping and, moreover, the focus of the paper is on the influence of the intermediate supports on the response.

In the present work, particular simplified configurations of bridges are analyzed in order to identify the parameters which influence the most the bridges' dynamic properties. In these configurations, the mass per unit length, the beam transverse stiffness and the external damping coefficient are assumed constant and equal respectively to m_d, EI_d, c_d. Furthermore, the values of $k_{c,r}$ and $c_{c,r}$ at the N different intermediate supports are assumed constant and equal to $k_{c,r}=k_T/N$. Finally, the intermediate restraints are assumed equally spaced and the length of each span is set equal to $L/(N+1)$, where L is the total length of the bridge.

After these positions, the equation of motion can be expressed as:

$$\begin{aligned} & m_d \int_0^L \ddot{u}(x,t)\eta(x)\,dx + c_d \int_0^L \dot{u}(x,t)\eta(x)\,dx + \sum_{r=1}^{N} \frac{c_T}{N}\dot{u}(x_r,t)\eta(x_r) + EI_d \int_0^L u''(x,t)\eta''(x)\,dx + \\ & + \sum_{r=1}^{N} \frac{k_T}{N} u(x_r,t)\eta(x_r) = -m_d \int_0^L \ddot{u}_g(t)\eta(x)\,dx \qquad \forall \eta \in V; \forall t \in [t_0, t_1] \end{aligned} \tag{2}$$

Both the members of Eqn. (2) are divided by m_d, as usual, thus yielding:

$$\begin{aligned} & \int_0^L \ddot{u}(x,t)\eta(x)\,dx + 2\omega_d\gamma_d \int_0^L \dot{u}(x,t)\eta(x)\,dx + \sum_{r=1}^{N} \frac{c_T}{m_d N}\dot{u}(x_r,t)\eta(x_{c,r}) + \frac{\omega_d^2 L^4}{\pi^4} \int_0^L u''(x,t)\eta''(x)\,dx + \\ & + \sum_{r=1}^{N} \frac{k_T}{m_d N} u(x_r,t)\eta(x_{c,r}) = -\int_0^L \eta(x)\,dx \cdot \ddot{u}_g(t) \qquad \forall \eta \in V; \forall t \in [t_0, t_1] \end{aligned} \tag{3}$$

In deriving eqn.(3) the two following parameters have been introduced:

$$\omega_d = \sqrt{\frac{\pi^4 EI_d}{m_d L^4}} \qquad \gamma_d = \frac{c_d}{2\omega_d m_d} \tag{4}$$

The first one, ω_d, denotes the circular frequency of the first mode of vibration of the deck alone, i.e., the deck without intermediate supports [13]. The second one, γ_d, denotes the corresponding damping factor, i.e., the ratio between the energy dissipated by the external damping and the maximum strain energy attained due to deck bending, for an harmonic motion at the frequency ω_d, whose shape coincides with the first sinusoidal modal shape [13].

The properties of the system of visco-elastic intermediate supports can be conveniently described by introducing three non-dimensional parameters α, β and γ_c, defined as follows:

$$\alpha^2 = \frac{k_T L^3}{\pi^4 EI_d} = \frac{k_T}{\omega_d^2 L m_d}$$
$$\beta = \frac{1}{N} \tag{5}$$
$$\gamma_c = \frac{c_T}{2\alpha^2 \omega_d m_d L} = \frac{c_T \omega_d}{2k_T}$$

The parameter α expresses the relative importance of the total spring stiffness with respect to a generalized measure of the global deck transverse stiffness. Low values of α^2 correspond to a stiff deck relative to the springs while high values correspond to a slender deck relative to the springs. The limit case $\alpha^2=0$ corresponds to the simply supported beam with no intermediate restraints.

The parameter β measures the degree of regularity of the total support stiffness' distribution along the bridge. It is inversely proportional to the number of intermediate springs and assumes values spanning from 0 to 1. The case $\beta=1$ corresponds to the case of support stiffness concentrated at a single point while the limit case $\beta=0$ ($N \to \infty$) corresponds to a beam resting on continuously distributed springs (Winkler beam).

The parameter γ_c describes the energy dissipation in the piers or the pier/bearing systems and it has the following physical interpretation: it is the ratio between the energy dissipated by the dampers and the maximum strain energy in the springs, for a rigid transverse harmonic motion of the deck with frequency ω_d.

After substituting eqn.(5) into eqn.(3) one obtains, $\forall \eta \in V; \forall t \in [t_0, t_1]$:

$$\begin{aligned} &\int_0^L \ddot{u}(x,t)\eta(x)\,dx + 2\omega_d\gamma_d \int_0^L \dot{u}(x,t)\eta(x)\,dx + 2\alpha^2\beta\gamma_c\omega_d L \sum_{r=1}^{N} \dot{u}(x_r,t)\eta(x_r) + \\ &+ \frac{\omega_d^2 L^4}{\pi^4}\int_0^L u''(x,t)\eta''(x)\,dx + \alpha^2\beta\omega_d^2 L \sum_{r=1}^{N} u(x_r,t)\eta(x_r) = -\int_0^L \eta(x)\,dx \cdot \ddot{u}_g(t) \end{aligned} \tag{6}$$

In order to cut off also dimensional aspects related to the length, the variable $y=x/L$ can be introduced and accordingly the motion can be described by

$$\tilde{u}(y,t) = u(x,t) \tag{7}$$

Upon substitution in eqn.(6), one obtains:

$$\begin{aligned} &\int_0^1 \ddot{\tilde{u}}(y,t)\tilde{\eta}(y)\,dy + 2\omega_d\gamma_d \int_0^1 \dot{\tilde{u}}(y,t)\tilde{\eta}(y)\,dy + 2\alpha^2\beta\gamma_c\omega_d \sum_{r=1}^{N} \dot{\tilde{u}}(y_r)\tilde{\eta}(y_r) + \frac{\omega_d^2}{\pi^4}\int_0^1 \tilde{u}''(y,t)\tilde{\eta}''(y)\,dy + \\ &+ \alpha^2\beta\omega_d^2 \sum_{r=1}^{N} \tilde{u}(y_r)\tilde{\eta}(y_r) = -\int_0^1 \tilde{\eta}(y)\,dy \cdot \ddot{u}_g \quad \forall \tilde{\eta} \in \tilde{V} = \left\{ \tilde{v}(y) \in H^2[0,1] : \tilde{v}(0) = \tilde{v}(1) = 0 \right\}; \forall t \in [t_0, t_1] \end{aligned} \tag{8}$$

The corresponding local form of the problem is obtained by integrating by parts and can be written as:

$$\begin{aligned} &M\ddot{\tilde{u}}(y,t) + C\dot{\tilde{u}}(y,t) + K\tilde{u}(y,t) = -M\ddot{u}_g(t) \\ &\tilde{u}''(y,t)\tilde{\eta}'\Big|_0^1 = 0 \end{aligned} \tag{9}$$

where M, C, K are formal linear operator related to the mass, damping and stiffness:

$$M = 1$$
$$K = \left[\frac{1}{\pi^4} \frac{\partial^4}{\partial x^4} + \alpha^2 \beta \sum_{r=1}^{N} \delta\left(y - y_r\right) \right] \omega_d^2 = K_1 \omega_d^2 \qquad (10)$$
$$C = 2\left[\gamma_d + \alpha^2 \beta \gamma_c \sum_{r=1}^{N} \delta\left(y - y_r\right) \right] \omega_d = C_1 \omega_d^2$$

and $\delta(\cdot)$ denotes the Dirac delta function.

3 FREE VIBRATIONS PROBLEM

3.1 Form of the characteristic equation

Under a qualitative point of view, the dynamic properties of a linear system, such as the ones analyzed in the present study, can be measured by solving the free vibrations problem that corresponds to eqn.(9) for $\ddot{u}_g = 0$. Thus, the study will proceed with the assessment of the influence of the parameters α, β, γ_d on the response in the most significant vibration modes. For this purpose, the differential boundary problem of eqn.(9) is reduced to an eigenvalue problem through the separation of variables technique. The transverse displacement $\tilde{u}$ is decomposed into the product of a spatial function and a time-dependent function, thus yielding:

$$\tilde{u}\left(y,t\right) = \psi\left(y\right) Z\left(t\right) \qquad (11)$$

where $\psi(y)$ is a space-only dependent function while $Z(t)$ is a time function, whose expression is:

$$Z\left(t\right) = Z_0 e^{\lambda t} \qquad (12)$$

After substituting eqns.(10), (11) and (12) into eqn.(9), one obtains the characteristic equation for the free-vibration problem:

$$\left\{ \lambda^2 + 2\left[\gamma_d + \alpha^2 \beta \gamma_c \sum_{r=1}^{N} \delta\left(y - y_r\right) \right] \omega_d \lambda + \left[\frac{1}{\pi^4} \frac{\partial^4}{\partial x^4} + \alpha^2 \beta \sum_{r=1}^{N} \delta\left(y - y_r\right) \right] \omega_d^2 \right\} \psi\left(y\right) = 0 \qquad (13)$$

It is noteworthy that if λ_1 and $\psi(y)$ are solutions of eqn. (13) for a particular value of the deck circular frequency $\omega_d = \omega_{d1}$ then $\sigma\lambda_1$ and $\psi(y)$ are solution of eqn.(13) for $\omega_d = \sigma\omega_{d1}$. Thus, the particular choice of the characteristic parameters leads to a formulation such that the vibration shape $\psi(y)$ does not depend on the particular value of ω_d but it only varies by varying the parameters α, β, γ_d and γ_c. This makes it possible to obtain qualitative results that are not dependent on the deck stiffness or on the deck mass.

3.2 Analytical solution of the eigenvalue problem

In order to solve analytically the free vibrations problem it is convenient to divide the beam into a set of N_s segments, each bounded by two consecutive restraints (external or intermediate). The motion of the s-th segment is described by the function $\ddot{\tilde{u}}_s\left(z_s,t\right)$ as follows:

$$\ddot{\tilde{u}}_s\left(z_s,t\right) + 2\gamma_d \omega_d \dot{\tilde{u}}_r\left(z_s,t\right) + \frac{1}{\pi^4} \omega_d^2 \tilde{u}_s\left(z_s,t\right) = 0 \quad \text{with} \quad z_s \in \left[0, 1/N_s\right] \qquad (14)$$

The transverse displacement $\tilde{u}_s$ is decomposed into the product of the spatial function $\psi_s(y)$ and the time-dependent function $Z(t)$ of eqn.(12), thus yielding the following equation of motion:

$$\left(\lambda^2 + 2\gamma_d\omega_d\lambda\right)\psi_s\left(y,t\right) + \frac{\omega_d^2}{\pi^4}\psi_s{}^{IV}\left(y,t\right) = 0 \tag{15}$$

which is rewritten as:

$$\psi_s{}^{IV}\left(y,t\right) = \Omega^4\psi_s\left(y,t\right) \tag{16}$$

having posed $\Omega^4 = -\dfrac{\pi^4\left(\lambda^2 + 2\gamma_d\omega_d\lambda\right)}{\omega_d^2}$, or $\lambda = -\gamma_d\omega_d + i\sqrt{\left(\dfrac{\Omega^4\omega_d^2}{\pi^4}\right) - \left(\gamma_d\omega_d\right)^2}$.

The solution to eqn.(15) is:

$$\psi_s\left(y\right) = C_{4s-3}\sin\left(\Omega y\right) + C_{4s-2}\cos\left(\Omega y\right) + C_{4s-1}\sinh\left(\Omega y\right) + C_{4s}\cosh\left(\Omega y\right) \tag{17}$$

with C_{4i-3} , C_{4i-2} , C_{4i-1} , C_{4i} to be determined based on the boundary conditions at the external supports and the continuity conditions at the intermediate restraints. This involves the calculation of higher order derivatives up to the third order.

In total, a set of $4N_s$ conditions is required to determine the vibration shape along the whole beam. At the first span, i.e., for s=1, the conditions $\psi_1\left(0\right) = \psi_1{}''\left(0\right) = 0$ apply while at the last span, i.e., for $s=N_s$, the support conditions are $\psi_{N_s}\left(1/N_s\right) = \psi''_{N_s}\left(1/N_s\right) = 0$. The boundary conditions expressing the continuity of the functions ψ_{i-1} and ψ_i at each of the N_s-1 spring locations can be expressed as:

$$\begin{aligned}
&\psi_{s-1}\left(1/N_s\right) = \psi_s\left(0\right)\\
&\psi'_{s-1}\left(1/N_s\right) = \psi'_s\left(0\right)\\
&\psi''_{s-1}\left(1/N_s\right) = \psi''_s\left(0\right)\\
&\left[\psi'''_{s-1}\left(1/N_s\right) - \psi'''_s\left(0\right)\right] - \beta\left(\pi^4\alpha^2 + \frac{2\lambda\pi^4\alpha^2\gamma_c}{\omega_d}\right)\psi_s\left(0\right) = 0
\end{aligned} \tag{18}$$

By substituting eqn. (17) into the boundary (supports and continuity) conditions, a system of $4N_s$ homogeneous equations in the constants $C_1,\ldots,C_{4N_S}$ is obtained. Since the system is homogeneous, the determinant of coefficients must be equal to zero for the existence of a nontrivial solution. This procedure yields the following frequency equation in the unknown Ω:

$$G\left(\Omega, \alpha^2, \beta, \gamma_d, \gamma_c\right) = 0 \tag{19}$$

It is noteworthy that the particular choice of the set of characteristic adimensional parameters results in an function G which is independent from ω_d.

In the general case of non-zero damping, the solution of the equation must be sought in the complex domain. Since the system is continuous, an infinite set of Ω values satisfying eqn.(19) is obtained. However, only selected values of Ω are significant, because they correspond to the first vibration modes which are usually characterized by the highest participation factors. For a given value of Ω satisfying eqn. (19), the corresponding modal shape can be calculated, together with the corresponding circular frequency ω and damping factor ξ. It is remarked that the eigensolutions Ω are complex conjugate. Thus, if the couple $\Omega = \Omega_r + i\Omega_i$ and $\lambda = \lambda_r + i\lambda_i$ is a solution of eqn.(19), then the couple $\bar{\Omega} = \Omega_r - i\Omega_i$ and $\lambda = \lambda_r - i\lambda_i$ is a solution, too. The corresponding eigenvectors are complex conjugate, too. Finally, for zero-damping $\lambda = i\omega$ where $\omega = \Omega\sqrt{EI_d / m_d}$.

3.3 Generalized orthogonality conditions for vibration modes and modal properties

Eqn.(13) for mode i is multiplied by ψ_j and eqn.(13) for mode j is multiplied by ψ_i . After integrating over the entire length of the beam one obtains:

$$\lambda_i^2\int_0^1 \psi_i(y)\psi_j(y)dy + 2\omega_d\lambda_i\left[\gamma_d\int_0^1 \psi_i(y)\psi_j(y)dy + \alpha^2\beta\gamma_c\sum_{r=1}^{N}\psi_i(y_r)\psi_j(y_r)\right] + \frac{\omega_d^2}{\pi^4}\int_0^1 \psi_i^{IV}(y)\psi_j(y)dy + \alpha^2\beta\omega_d^2\sum_{r=1}^{N}\psi_i(y_r)\psi_j(y_r) = 0 \quad (20)$$

$$\lambda_j^2\int_0^1 \psi_j(y)\psi_i(y)dy + 2\omega_d\lambda_j\left[\gamma_d\int_0^1 \psi_j(y)\psi_i(y)dy + \alpha^2\beta\gamma_c\sum_{r=1}^{N}\psi_j(y_r)\psi_i(y_r)\right] + \frac{\omega_d^2}{\pi^4}\int_0^1 \psi_j^{IV}(y)\psi_i(y)dy + \alpha^2\beta\omega_d^2\sum_{r=1}^{N}\psi_j(y_r)\psi_i(y_r) = 0 \quad (21)$$

By integrating twice by parts the terms with the fourth order derivative and recalling the support boundary conditions, one finally obtains:

$$\lambda_i^2\int_0^1 \psi_i(y)\psi_j(y)dy + 2\omega_d\lambda_i\left[\gamma_d\int_0^1 \psi_i(y)\psi_j(y)dy + \alpha^2\beta\gamma_c\sum_{r=1}^{N}\psi_i(y_r)\psi_j(y_r)\right] + \frac{\omega_d^2}{\pi^4}\int_0^1 \psi_i''(y)\psi_j''(y)dy + \alpha^2\beta\omega_d^2\sum_{r=1}^{N}\psi_i(y_r)\psi_j(y_r) = 0 \quad (22)$$

$$\lambda_j^2\int_0^1 \psi_j(y)\psi_i(y)dy + 2\omega_d\lambda_j\left[\gamma_d\int_0^1 \psi_j(y)\psi_i(y)dy + \alpha^2\beta\gamma_c\sum_{r=1}^{N}\psi_j(y_r)\psi_i(y_r)\right] + \frac{\omega_d^2}{\pi^4}\int_0^1 \psi_j''(y)\psi_i''(y)dy + \alpha^2\beta\omega_d^2\sum_{r=1}^{N}\psi_j(y_r)\psi_i(y_r) = 0 \quad (23)$$

By subtracting eqn.(23) from eqn.(22), for $\lambda_i \neq \lambda_j$, one obtains the first orthogonality condition:

$$\left(\lambda_i + \lambda_j\right)\int_0^1 \psi_i(y)\psi_j(y)dy + 2\omega_d\gamma_d\int_0^1 \psi_i(y)\psi_j(y)dy + 2\alpha^2\beta\omega_d\gamma_c\sum_{r=1}^{N}\psi_i(y_r)\psi_j(y_r) = 0 \quad (24)$$

By subtracting eqn.(23) multiplied by λ_i from eqn.(22) multiplied by λ_j and dividing by $\left(\lambda_j - \lambda_i\right)$ one obtains the second orthogonality condition:

$$\frac{\omega_d^2}{\pi^4}\int_0^1 \psi_i''(y)\psi_j''(y)dy - \lambda_i\lambda_j\int_0^1 \psi_i(y)\psi_j(y)dy + 2\alpha^2\beta\omega_d^2\sum_{r=1}^{N}\psi_i(y_r)\psi_k(y_r) = 0 \tag{25}$$

which upon substitution of eqn.(24) may also read as follows:

$$\begin{aligned}&\frac{\omega_d^2}{\pi^4}(\lambda_i+\lambda_j)\int_0^1 \psi_i''(y)\psi_j''(y)dy + 2\omega_d\gamma_d\lambda_i\lambda_j\int_0^1 \psi_i(y)\psi_j(y)dy + \\ &+2\alpha^2\omega_d\beta\gamma_c\sum_{r=1}^{N}\psi_i(y_r)\psi_j(y_r) + 2\alpha^2\beta\omega_d^2(\lambda_i+\lambda_j)\sum_{r=1}^{N}\psi_i(y_r)\psi_j(y_r) = 0\end{aligned} \tag{26}$$

For the case of zero damping, the two orthogonality conditions reduce to the well expression already derived in [8]:

$$\int_0^1 \psi_i(y)\psi_j(y)dy = 0 \tag{27}$$

$$\frac{\omega_d^2}{\pi^4}\int_0^1 \psi_i''(y)\psi_j''(y)dy + 2\alpha^2\beta\omega_d^2\sum_{r=1}^{N}\psi_i(y_r)\psi_j(y_r) = 0 \tag{28}$$

Similarly to [14], the generalized orthogonality conditions are employed to derive the analytical expressions of the circular frequency of vibration ω_i and of the damping factor ξ_i for the *i*-th vibration mode. By setting $\lambda_j = \overline{\lambda}_i$ in eqn.(25) and in eqn.(24), and recalling that $\lambda_i\overline{\lambda}_i = \omega_i^2$ and $\lambda_i + \overline{\lambda}_i = -2\xi_i\omega_i$, one obtains:

$$\frac{\omega_i^2}{\omega_d^2} = \frac{\frac{1}{\pi^4}\int_0^1 \left[\psi_i''(y)\right]^2 dy + 2\alpha^2\beta\sum_{r=1}^{N}\psi_i^2(y_r)}{\int_0^1 \psi_i^2(y)dy} \tag{29}$$

$$\xi_i = \frac{\gamma_d\int_0^1 \psi_i^2(y)dy + \alpha^2\beta\gamma_c\sum_{r=1}^{N}\psi_i^2(y_r)}{\int_0^1 \psi_i^2(y)}\cdot\frac{\omega_d}{\omega_i} \tag{30}$$

The circular frequency ω_i, normalized in eqn.(29) with respect to ω_d , is simply expressed in terms of the ratio between the potential energy of the deck and the restraints, and the kinetic energy of the deck mass. It can be observed that the contribution to the potential energy of the deck is proportional to the deck curvature and becomes more and more important with respect to the other term when the number of the vibration mode increases.

The modal damping factor in eqn.(30) corresponds to the ratio between the energy dissipated by the deck and by the restraints and the kinetic energy of the deck. It should be stressed that the term of the dissipation related to the intermediate dampers qualitatively acts a mass-proportional similarly to the deck damping and, thus, the damping factor is expected to reduce significantly when the number of the vibration mode increases. Furthermore, damping factor ξ_i is not sensitive to ω_d.

4 PARAMETRIC ANALYSIS

4.1 Undamped free vibrations

This paragraph analyzes the free vibrations problem of the beam with intermediate restraints, disregarding the damping exerted by both the deck and the restraints. The values of γ_d and γ_c are assumed equal to zero at this stage, in order to describe the influence of the global stiffness of the supporting system, described by α^2, and the effects related to the distribution of its stiffness, described by β.

Different configurations of beams with intermediate restraints are considered which are representative of common short and medium span bridges (Fig. 2). They correspond to different values of the parameter β, i.e., β=1 (Fig. 2a), β=1/2 (Fig. 2b), β=1/3 (Fig. 2c). Moreover, the limit case of a beam simply supported at its extremes with no intermediate restraints (corresponding to $\alpha^2=0$) (Fig. 2d), and a beam resting on elastic continuous transverse restraints (corresponding to β=0) are also analyzed.

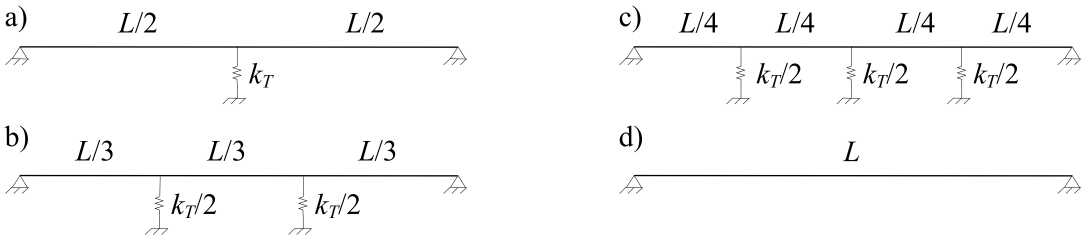

Fig. 2. Different configurations of beams with intermediate restraints analyzed.

The parameter α^2 is varied between about zero (stiff deck, flexible supports) and 10, that may be regarded as a realistic upper limit for slender superstructures on stiff supports. The results presented in the following include the modal shapes and periods of vibration. As previously shown, in the considered formulation the modal shapes depend on the parameters α and β only, and they are invariant with respect to the deck circular frequency ω_d. The vibration periods depend on the deck circular frequency, but they are divided by ω_d in order to make them independent from ω_d, too. Thus, the particular choice of the deck does not affect the following results. For the sake of completeness, the numerical results have been obtained by considering as reference the deck reported in [5], whose parameters are: L=200m, m_d=16.24t, EI_d=1100307114 kN/m^2, and ω_d=2.03Hz. Before discussing the results, it may be useful to recall that, in the limit case corresponding to β=0, the vibration modes are purely sinusoidal, irrespectively of the value of α^2 (see Appendix for the proof).

Fig. 3 reports the first three modal shapes for different values of α^2 and of β, normalized with respect to the L2 norm, i.e. $\|\psi_i\|_2 = \left(\int_0^1 |\psi_i(y)|^2 \, dy \right)^{1/2} = 1$. It is recalled that the even modes of vibration are characterized by an anti-symmetric shape and a participating factor equal to zero, and thus they do not affect the seismic response of the considered configurations.

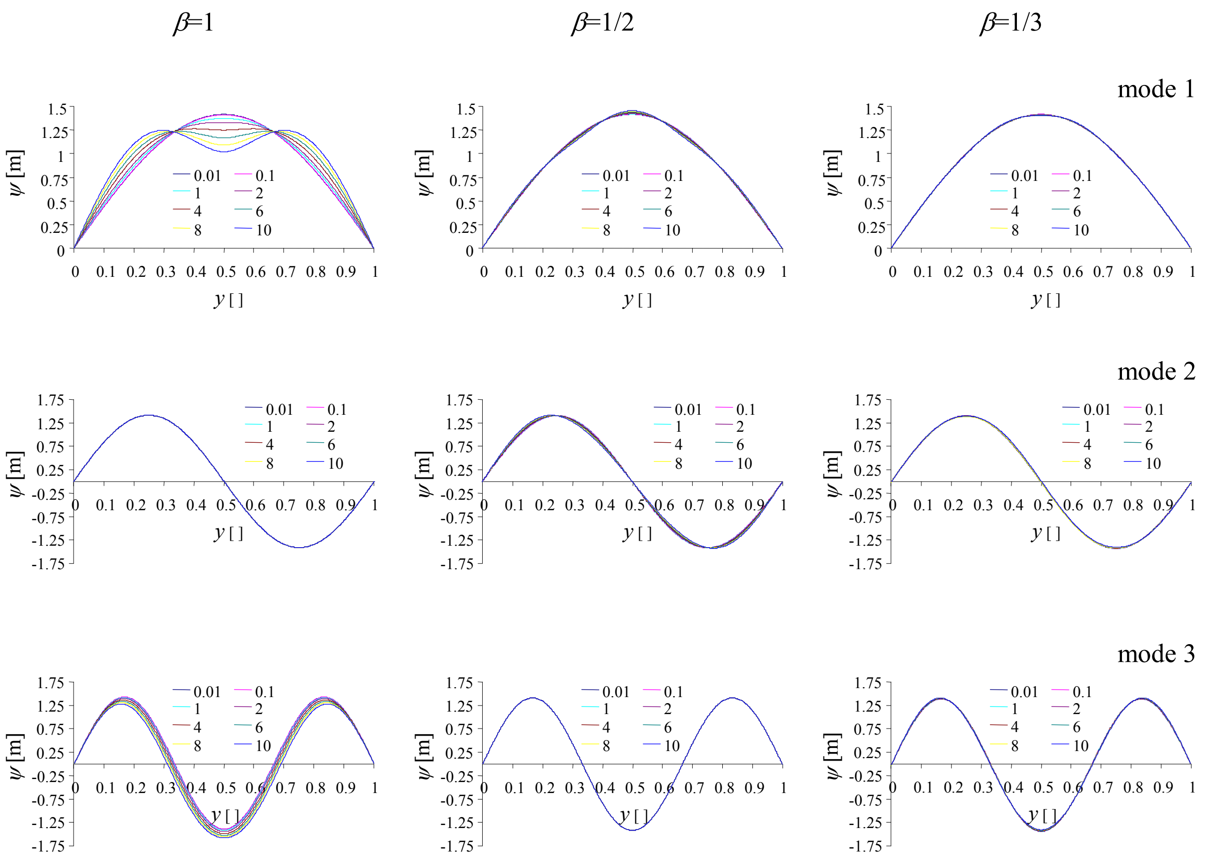

Fig. 3. Normalized first three modal shapes for different values of α^2 and for β.

The intermediate restraints have a remarkable influence on the first mode shape only for β=1. In this case, the first mode vibration shape significantly deviates from the shape of the deck vibrating alone (α^2=0), for high values of α^2. On the other hand, for β=1/2 and β=1/3, the vibration shape is very close to the shape of the deck vibrating alone, even for high values of α^2. This can be explained recalling that the vibration shape tends, for decreasing β values, to the sinusoidal shape of a simply-supported beam on continuous elastic restraints.

Globally, the influence of the restraints on the higher modes shapes is almost negligible, for any case of β, although a uniform trend is not observed, since the influence of the intermediate supports depend on their distance from the shape mode nodes.

In order to have a full insight into the influence of the restraints on the response along the deck, it is of interest to analyze the effects of the variation of the support properties on the second order derivative $\psi_i''(y)$ of the displacement shape, since this quantity is related to the deck's bending moments and strain energy distribution. Fig. 4 reports the normalized shape of $\psi_i''(y)$, for the first three vibration modes and for different values of the parameters α^2 and β.

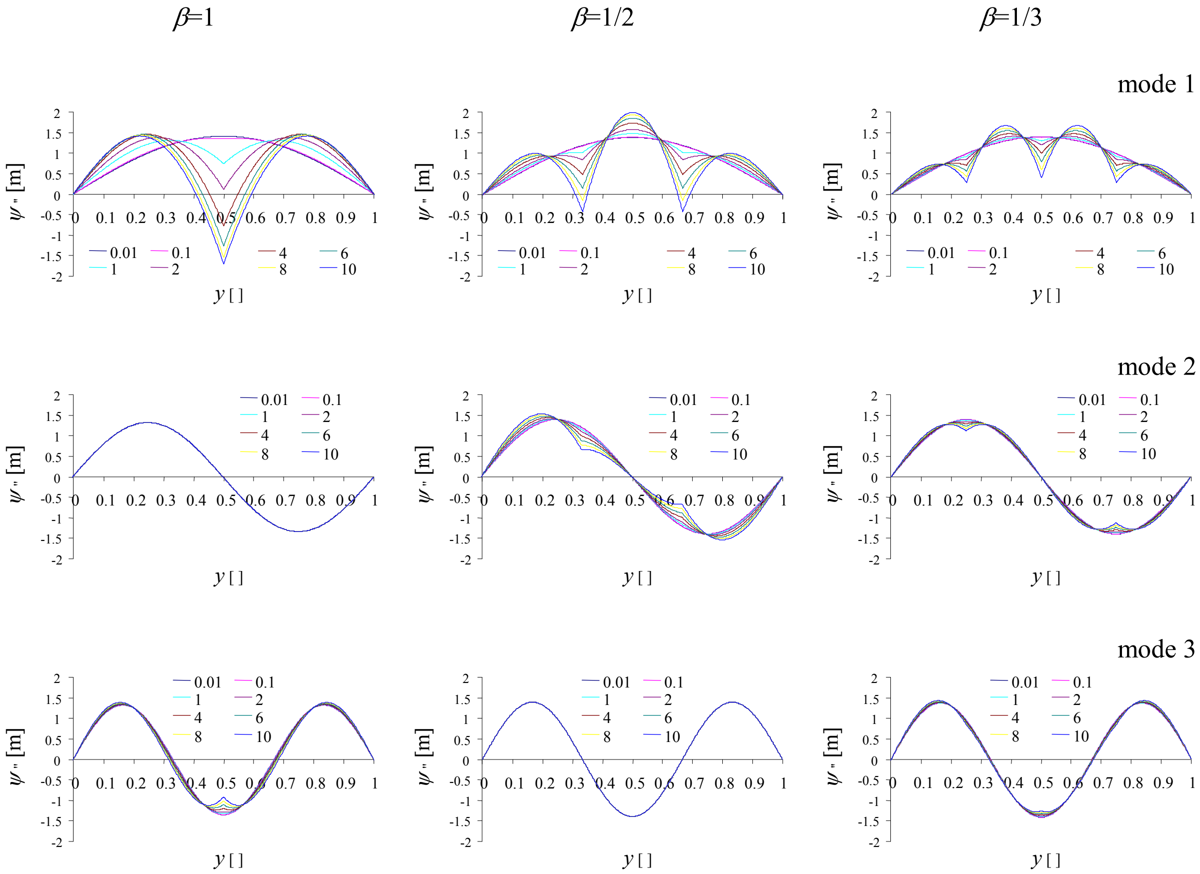

Fig. 4. Normalized second order derivatives of the first three modal shapes for different values of α^2 and β.

The intermediate supports influence more significantly the shape of the second order derivative than the displacement shape itself. Notable variations (with respect to the case corresponding to α^2=0) are observed in the first mode shape, for all the three configurations considered. The variations in the higher modes are less significant and different from configuration to configuration.

In conclusion, the variation of the intermediate supports' properties induces minor variations in the distribution of the modal shapes, and thus in the distribution of the piers shear and inertia forces, directly related to them. Significant variations are limited to the first mode and to the configuration with a single support. On the other hand, the distribution of internal actions in the deck is strongly affected by the intermediate restraints, for the modes and configuration reported.

In order to describe synthetically the variation of the functions ψ_i and ψ_i'' defined along the bridge, the following scalar variation index is introduced:

$$\delta_{f_i} = \frac{\left\| f_i - f_{d,i} \right\|_2}{\left\| f_{d,i} \right\|_2} \quad \text{for} \quad f_i = \psi_i, \psi_i'' \tag{31}$$

where the subscript "i" denotes the i-th vibration mode while the subscript "d" refers to the limit case corresponding to $\alpha^2=0$.

Other scalar quantities are of interest in the assessment of the system's seismic response and their variation due to the intermediate supports can be described synthetically in a similar way. In particular, the following results report the variation of the vibration periods T_i, of the participation factors ρ_i, and of the values of the third order derivative at the beam ends $R_i = |\psi_i'''(0)| = |\psi_i'''(L)|$, that is proportional to the transverse reaction at the abutments. The variation indexes for these quantities are defined as:

$$\delta_{f_i} = \frac{f_i - f_{d,i}}{f_{d,i}} \qquad \text{for } f_i = R_i, \rho_i, T_i \tag{32}$$

Fig. 5 shows the influence of parameters α^2 and β on the first-mode variation indexes. Fig. 5a highlights the different trend of the mode shape (displacement) and of its second order derivatives (bending moments). The variation index of the abutment reactions (Fig. 5b) assume positive and negative values spanning from -10% to around 40%-50%. Similar trends are observed for all the configuration considered. Finally, minor variations are observed for the participation factor while the vibration period strongly decreases with increasing α^2. Its sensitivity to α^2 is very high in the initial part of the range explored.

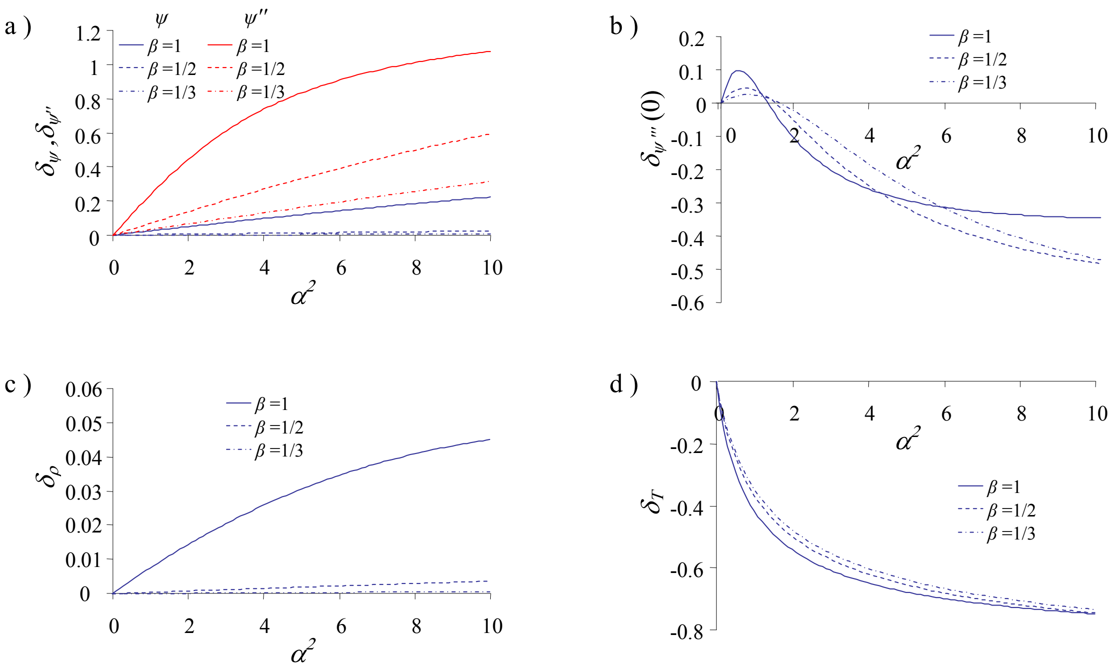

Fig. 5. Variation of δ with α^2 and β for mode 1 and relative to: a) displacements $\psi(y)$ and bending moments $\psi''(y)$, b) abutment shear $\psi'''(0)$, c) participation factor ρ, and d) vibration period T.

Fig. 6 and Fig. 7 show the influence of parameters α^2 and β on the variation indexes of the third and fifth mode. As previously noted, even modes are not of interest. Thus, the results concern odd terms only.

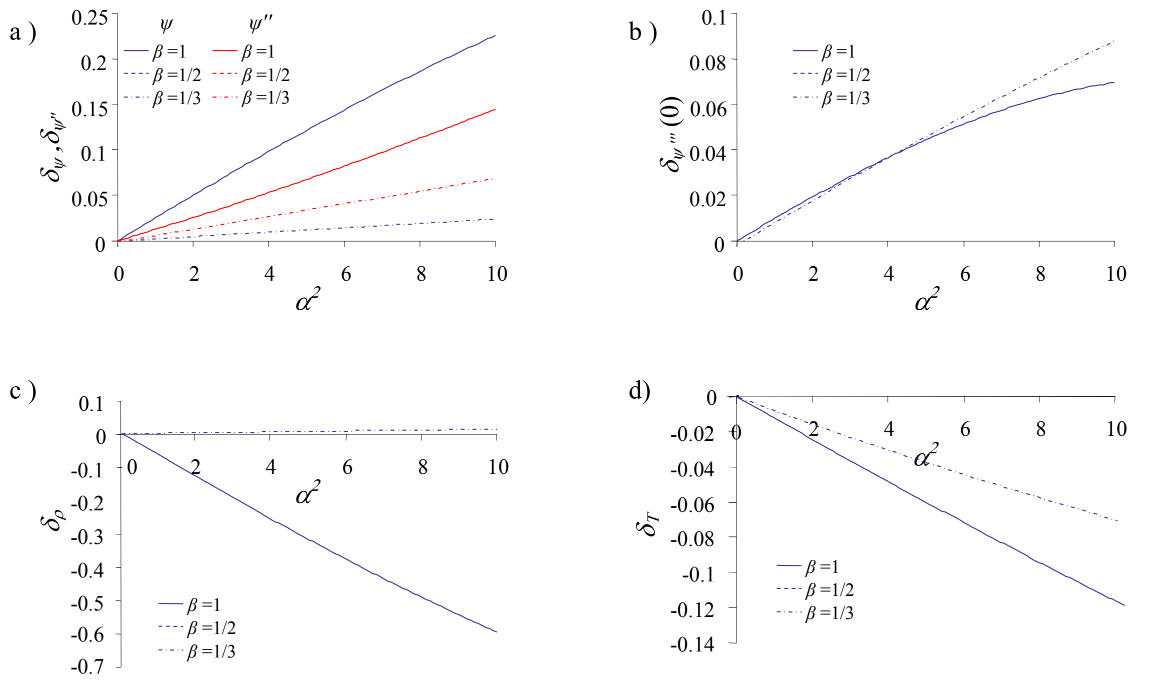

Fig. 6. Variation of δ with α^2 and β for mode 3 and relative to: a) displacements $\psi(y)$ and bending moments $\psi'''(y)$, b) abutment shear $\psi'''(0)$, c) participation factor ρ, and d) vibration period T.

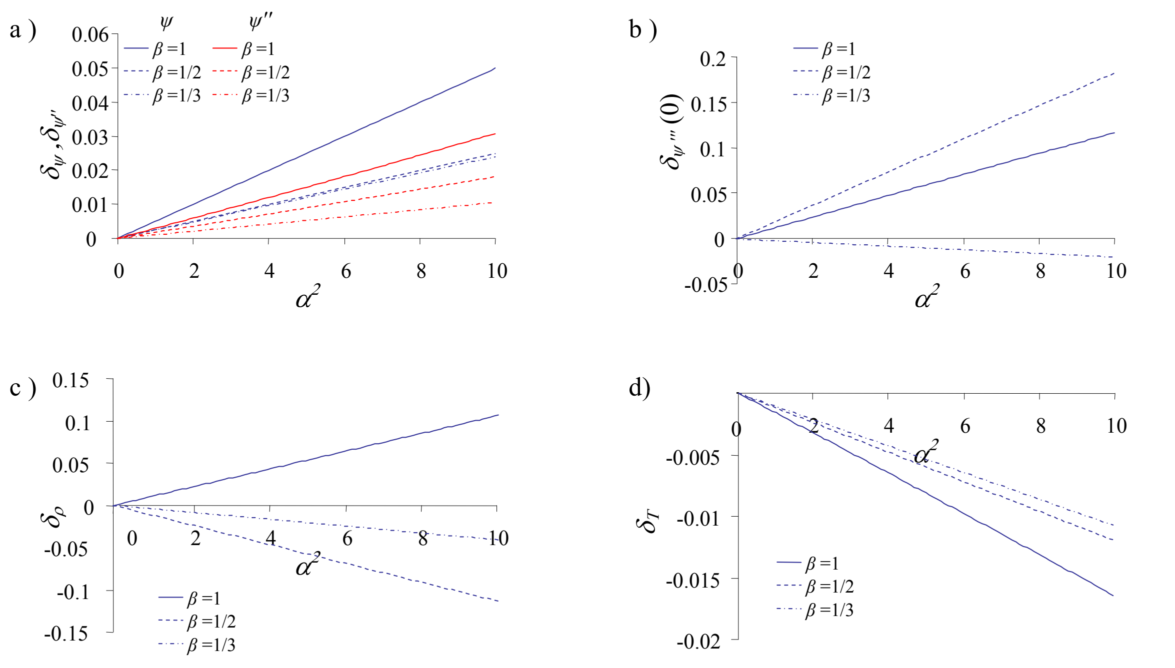

Fig. 7. Variation of δ with α^2 and β for mode 5 and relative to: a) displacements $\psi(y)$ and bending moments $\psi''(y)$, b) abutment shear $\psi'''(0)$, c) participation factor ρ, and d) vibration period T.

Generally, different trends can be observed in the higher-modes variation indexes. Variations in displacement and bending decrease for higher modes and higher number of supports. The reaction at the abutments, related to the third order derivative, is strongly influenced by the presence of intermediate supports, without distinction for the considered configurations, and it shows large variations with different sign. The intermediate supports in general reduce the vibration period but their influence is lower and lower for increasing order of modes. Finally, the participation factors are not significantly influenced by the intermediate restraints, with the exception of the second mode's participation factor, for β=1.

4.2 Non-classically damped free vibrations

This paragraph analyzes the damped free vibrations problem which corresponds to considering the deck damping and intermediate restraints with visco-elastic behavior. The solution to this problem is reported in Appendix A. A damping factor γ_d =0.02 is employed to describe the dissipation in the deck [5] while different values of γ_c from 0 to 0.25 are considered to describe the dissipation of the intermediate restraints. It is noteworthy that only the intermediate dampers are the cause of the non-classical damping, since the deck damping is assumed proportional to the mass. The most relevant effects of the variation of the intermediate dampers properties on the system free-vibration response are discussed below.

Fig. 8 shows the influence of the intermediate supports damping γ_c on the global dissipative properties, described by the damping factor ξ, for vibration modes 1, 3 and 5. The definition and the analytical expression of ξ are reported in Appendix A. Different values of the parameters describing the intermediate supports stiffness (α^2) and the bridge configuration (β) are considered.

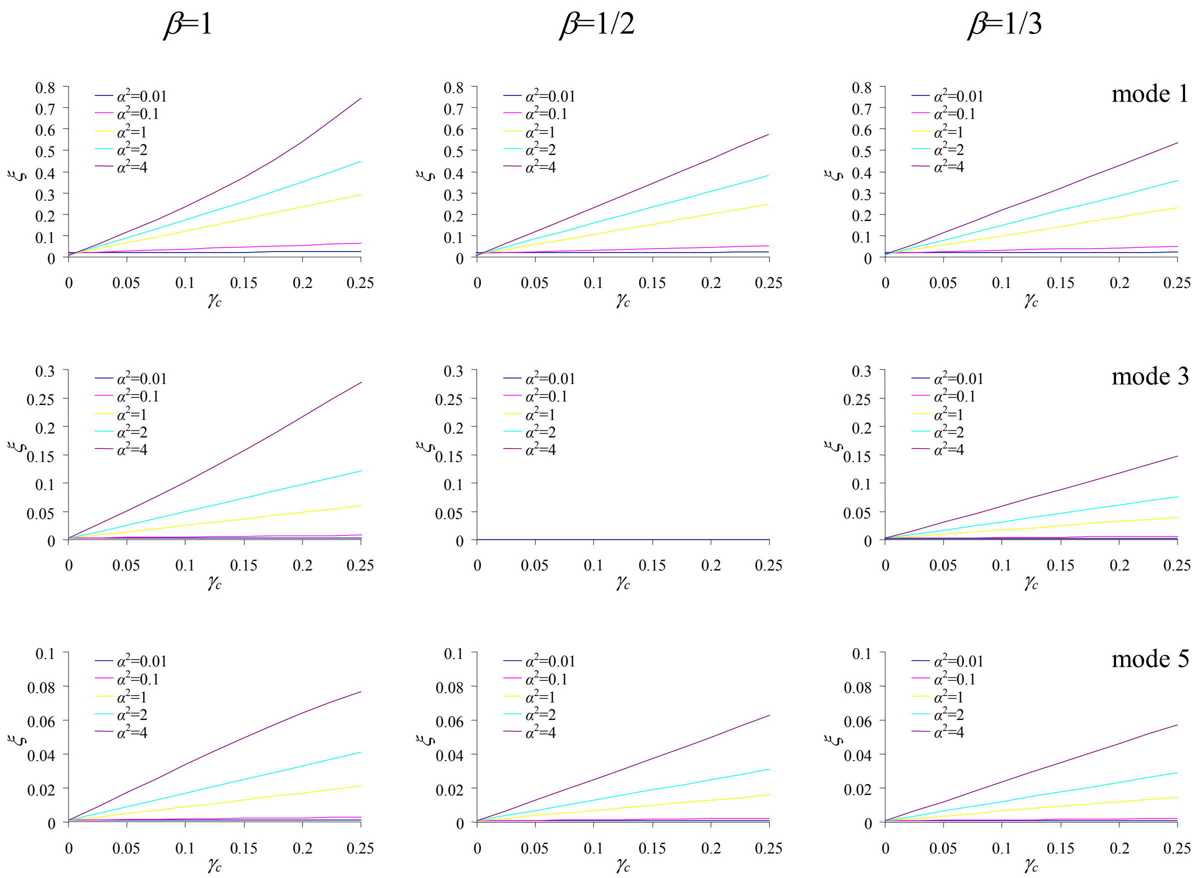

Fig. 8. Damping factor ξ vs. γ_c for different values of α^2 and β.

In general, the values of ξ corresponding to the first vibration mode (maximum value of 0.7 for γ_c =0.25) are significantly higher than those corresponding to the third (maximum value of 0.3 for γ_c =0.25) and the fifth vibration mode (maximum value of 0.08 for γ_c =0.25). It is also observed that the intermediate supports damping, which is proportional to the displacement, have a reduced efficiency in damping the higher modes and its decay rate is approximately proportional to $1/i^2$ (i denotes the mode index). Furthermore, the above figures demonstrate that the value of ξ increases almost linearly for increasing γ_c and for increasing α^2 while decreases slightly for decreasing β.

In a non-classically damped system such as those analyzed, the dampers may significantly affect the modal shape and the vibration motion may considerably vary with respect to the undamped case. In order to illustrate the modifications introduced by the intermediate dampers on the vibration shape with respect to the undamped case, a particular configuration is firstly analyzed which corresponds to the values of α^2=4, β=1 and γ_c=0.13, leading to ξ=0.3. Fig. 9 shows the real and imaginary part of the normalized first modal shape ($\psi^R_{1,ncd}$ and $\psi^I_{1,ncd}$) and of its second order derivative ($\psi''^R_{1,ncd}$ and $\psi''^I_{1,ncd}$) for the non-classically damped case. In the same figure, the normalized shape ($\psi_{1,cd}$) and its second order derivative ($\psi''_{1,cd}$) corresponding to the classically-damped case (γ_c=0) are also shown for comparison.

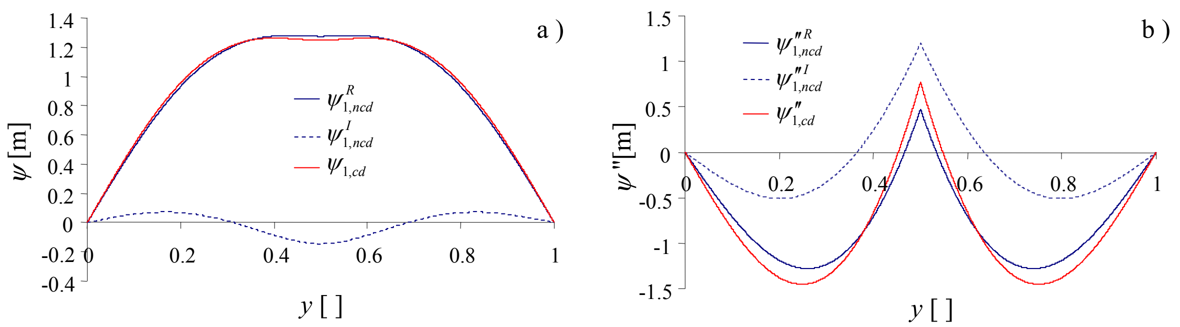

Fig. 9. (a) Real and imaginary part of the normalized first modal shape and (b) of the second order derivative for the non-classically damped case and for the classically-damped case (γ_c=0).

The intermediate dampers influence not only the imaginary part of the eigenvector and of its derivatives, but also, to a less extent, the real part. Furthermore, the imaginary part is quite small with respect to the corresponding real part in the displacement shape, but it assumes an high relevance in the moment shape.

For design purposes, it can be interesting to analyze the envelope of the free vibration response in the first classically ($\psi_{1,cd}$) and non-classically damped ($\psi_{1,ncd}$) modes obtained neglecting the amplitude decay term.

$$\psi_{1,ncd} = \max_t \left\{ \psi_1^R \cos(\omega_1 t) - \psi_1^I \sin(\omega_1 t) \right\} \tag{33}$$

Fig. 10a plots the envelope (acronym "env") of the modal shape. The envelope of the second-order derivative, derived in a similar manner, is also reported in Fig. 10b.

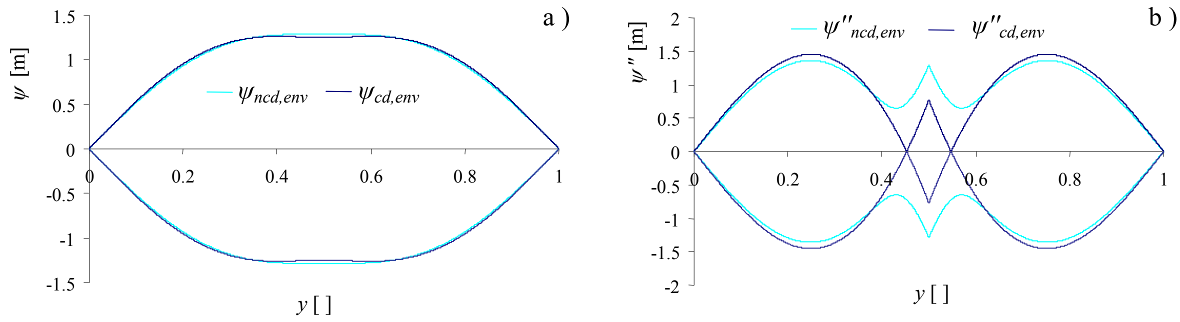

Fig. 10. Envelope of a) the first modal shape and b) of its second-order derivative for the non-classically damped case and the classically-damped case.

It is interesting to observe that the displacements envelopes accounting for and disregarding damping are very similar. On the other hand, the envelope of the curvatures for the non-proportionally damped case significantly differs from the corresponding envelope for the proportionally damped case. In conclusion, with reference to this case it is observed that neglecting the non proportionality of the damping yields sufficiently accurate estimates of the displacement shape but inadequate estimates of the maximum values of the internal actions attained during the vibration motion.

In order to provide information about the extent of non proportionality in the response, the following non-proportionality indexes are introduced for displacements, second order derivatives and end reactions:

$$\mu_{f_i} = \frac{\left\| \mathrm{Im}\left(f_i\right) \right\|_2}{\left\| \mathrm{Re}\left(f_i\right) \right\|_2} \quad \text{with } f_i = \psi_i, \psi_i'' \qquad \mu_{R_i} = \frac{\mathrm{Im}\left(R_i\right)}{\mathrm{Re}\left(R_i\right)} \tag{34}$$

The non-proportionality indexes generally increase for increasing α^2 and increase almost linearly for increasing γ_c ,except for the variation index of ψ in case β=1,mode 1. They also decrease for increasing β as expected, since when β tends to zero the system tends to a simply supported beam resting on continuously distributed visco-elastic restraints, whose damping is proportional and whose vibration modes are real.

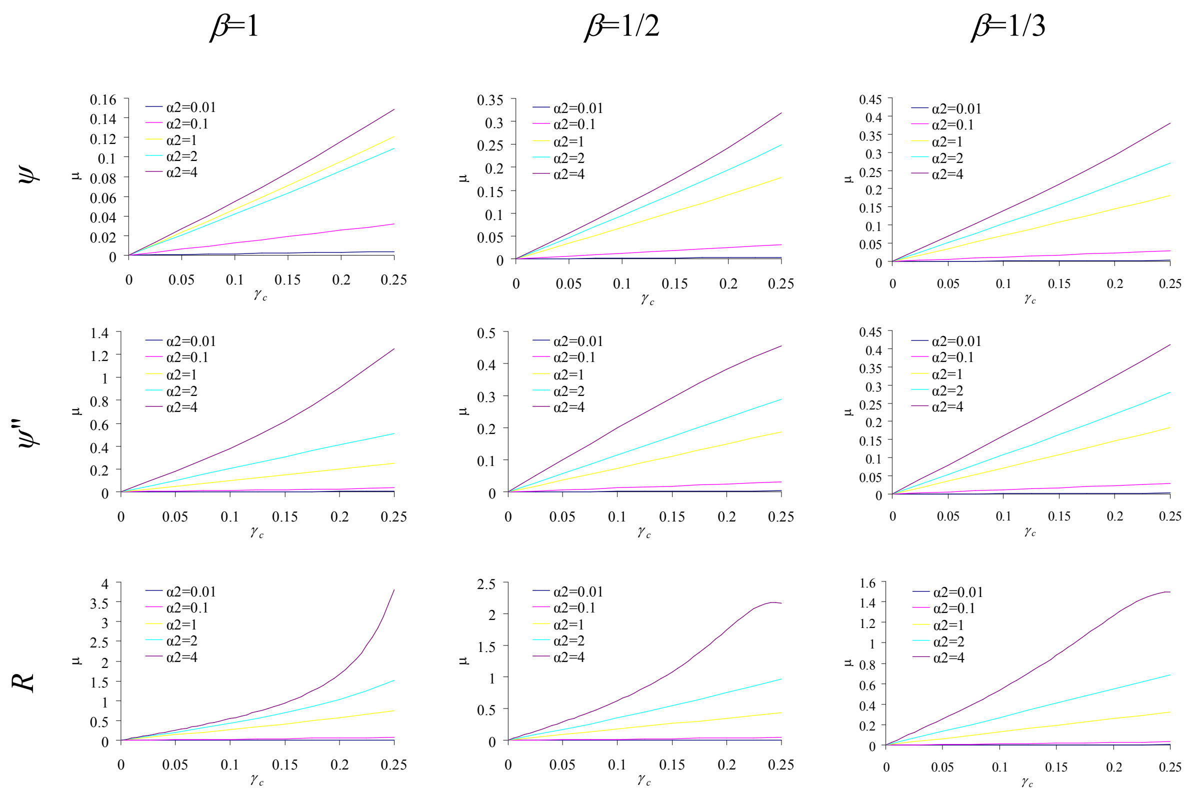

Fig. 11. Variation with γ_c of non-proportionality indexes relative to mode 1, for different values of α^2.

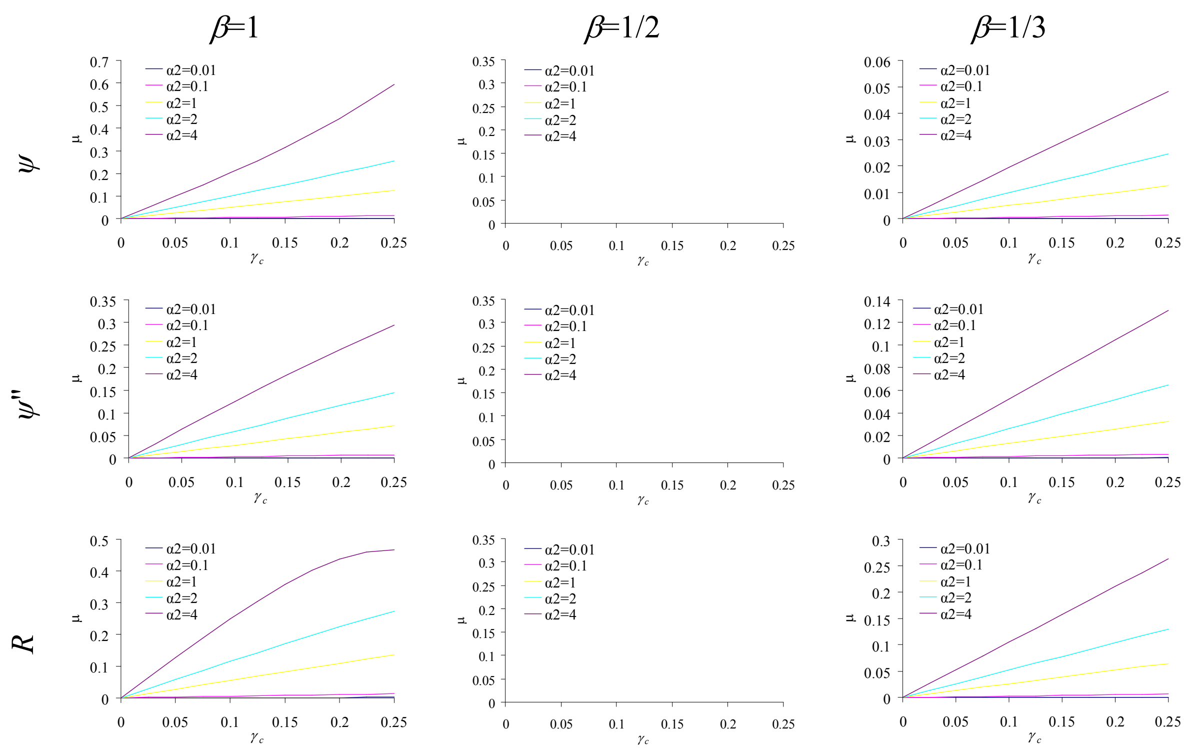

Fig. 12. Variation with γ_c of non-proportionality indexes relative to mode 3, for different values of α^2.

CONCLUSIONS

This paper examines the dynamic behavior of partially restrained seismically isolated bridges by studying the transverse free vibrations of a two-dimensional simply-supported beam model resting on intermediate visco-elastic supports.

Particular simplified configurations with constant deck properties and uniform equally spaced supports are considered, with two main aims: to seek a reduced set of characteristic parameters describing completely the dynamic system and to obtain analytical solutions useful to make explicit the relationship between the bridge properties and the dynamic response. Three parameters are identified that describe a) the global ratio between deck and supports stiffness (α), b) the regularity of the support stiffness distribution (β), and c) the global dissipative properties of the supports (γ_c).

A parametric analysis is carried out by varying the values assumed by these parameters, in order to highlight their influence on the dynamic properties of interest for the seismic response assessment. The reported results show that (a) variations in the distribution of displacements, related to the piers and inertia forces, are remarkable only for the first mode of the configuration with only one intermediate support, while minor variations are observed in the other cases; (b) variations in the curvatures' shape, related to the deck bending moments, are very significant and cannot be neglected for all the modes and configuration reported; (c) the abutment reactions are also strongly affected to the presence of intermediate supports, without distinction for the considered configurations; (d) the damping promoted by the intermediate supports is proportional to the displacements, and its effectiveness decreases rapidly with increasing mode order; (e) for certain values of the characteristic parameters the non classical damping induced by the intermediate supports can play an important role in the dynamic response and, thus, simplified approaches neglecting it can lead to not accurate estimates of the maximum values of the response parameters attained during the motion; (f) non-classical damping induces minor variations on the displacement shape and on related quantities rather than on higher order derivatives.

REFERENCES

[1] G.C. Lee, Y. Kitane, I.G. Buckle, Literature review of the observed performance of seismically isolated Bridges. *Report on Progress and Accomplishments: 2000-2001, MCEER Publications*, State University of New York, Buffalo, USA, 2001.

[2] M.C. Kunde, R.S. Jangid, Seismic behavior of isolated bridges: A state-of-the-art review. *Electronic Journal of Structural Engineering*, **3**, 140-170, 2003.

[3] M.H. Tsai, Transverse earthquake response analysis of a seismically isolated regular bridge with partial restraint. *Engineering structures*, **30**, 393-403, 2008.

[4] N. Makris, G. Kampas, D. Angelopoulou, The eigenvalues of isolated bridges with transverse restraints at the end-abutments. *Earthquake Engineering and Structural Dynamics*, **39**, 869-886, 2009.

[5] E. Tubaldi, A. Dall'Asta, A design method for seismically isolated bridges with abutment restraint. *Engineering Structures*, **33**, 786-795, 2011.

[6] C.M. Albarracín, L. Zannier, R.O. Grossi, Some observations in the dynamics of beams with intermediate supports, *Journal of Sound and Vibration*, **271**, 475-480, 2004.

[7] M.J. Maurizi, D.V. Bambilla, P.M. Bellès, M.A. De Rosa. Free vibration of Bernoulli Euler beams with intermediate elastic support: a concise thematic recension. *Journal of Sound and Vibration*, **281**, 1238-1239, 2005.

[8] P.A. Hassanpour, E. Esmailzadeh, W.L. Cleghorn, J.K. Mills, Generalized Orthogonality Condition for Beams with Intermediate Lumped Masses Subjected to Axial Force. *Journal of Vibration and Control*, **16**, 665-683, 2010.

[9] M. Gurgoze, On the eingenvalues of a viscously damped cantilever carrying heavy massed and restrained by linear and torsional spring. *Journal of Sound and Vibration*, **208**, 153-158, 1997.

[10] M.A. De Rosa, M. Lippiello, M.J. Maurizi, H.D. Martin, Free vibration of elastically restrained cantilever tapered beams with concentrated viscous damping and mass *Mechanics Research Communications*, **37**, 261-264, 2010.

[11] C.A. Truesdell, R.A. Toupin, *The Classical Field Theories*, Handbuch der Physik, vol. III-1, Springer-Verlag, Berlin, Germany, 1960.

[12] D.L. Russell, On mathematical models for the elastic beam with frequency-proportional damping. *Control and Estimation in Distributed Parameter Systems*, H.T. Banks, Editor, SIAM, Philadelphia, PA, 1992 .

[13] A.K. Chopra, *Dynamics of structures: Theory and Applications to earthquake engineering, 2nd Edition*, Prentice Hall, Englewood Cliffs, N.J, 2001.

[14] G. Oliveto, A. Santini. Complex modal analysis of a flexural vibrating beam with viscous end conditions. Journal of Sound and Vibration, **200**, 327-45, 1997.