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# Bi-hierarchical finite element for the analysis of the non-axisymmetric free vibration of shells of revolution

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**Abstract.** *A new hierarchical finite element is developed to analyze the free vibration of isotropic shells of revolution with linear varying thickness. It is a bi-hierarchical four nodes quadrilateral element with three degrees of freedom (three displacements) by node. The specificity of this element is a double increase of the hierarchical mode number independently according to both axial and radial directions. The advantage is that the solution is accurate for high ratio dimensions as well flattened axisymmetric (plates) shells that slim ones (high cylindrical shells) with different shapes and boundary conditions. The second advantage is the possibility of using only one element to idealize regular shape shells. Through the application of this element to some numerical examples, the comparisons with other studies show clearly that this element gives good results accuracy with simple idealization for axisymmetric and non-axisymmetric shells vibration (thick and thin).*

## 1 INTRODUCTION

The propose of this work is to present a new bi-hierarchical finite element to idealize isotropic shells of revolution with linear varying thickness. Shells of revolution can be of different shape and dimensions. Thin and thick plates and cylindrical shells cannot be idealized by the same finite elements in standard or hierarchical method. The dimensions ratio and the shape of the shell influence on the choice of the finite element and the mesh. For modeling for example a storage tank, one must have two elements, one for the cylinder and the other for the plate. The new proposed element can idealize the two shells.

The literature on finite element approximation of shells of revolution is huge. Let us just mention a few different approaches. Bodies of revolution compounds of thin shell segments, thick shell segments, and rings have been analyzed by using quadrilateral isoparametric elements of revolution of eight nodes [1] and Mindlin Reissner axisymetric finite elements which take in account the shearing strains and the rotational inertia effects [2]. Other methods were implemented of which a three-dimensional method of analysis to determine the frequencies and eigen modes of paraboloïd solids and paraboloïd shells of revolution with variable thickness [3] and a spline finite element method for the analysis of the free vibration of shells of revolution which is based on the theory of the thin shells [4].

The proposed element in this work is a four nodes volumetric and axisymetric bi-hierarchical finite element with only displacements as degrees of freedom. The word "bi-hierarchical" comes from the fact that this element has two hierarchical mode numbers, one in the axis direction and the second in the radius direction. These two hierarchical mode numbers increase independently. This element can than idealize flattened shells like plates and high shells like cylinders and conic shells (thins and thicks) or structures composed of them. For the first kind of shells, the radius direction hierarchical mode number is increased to have the accuracy of the results, and for the second kind, the axis hierarchical mode number is increased.

## 2 POTENTIAL ENERGY

The potential energy stored in the shell is in the form of strain energy due to the effect of both stretching and bending. The strain energy expression can be written as

$$U(t)=\frac{1}{2}\int_{R_i}^{R_e}\int_{0}^{H}\int_{0}^{2\pi}\left(\sigma_r\varepsilon_r+\sigma_z\varepsilon_z+\sigma_\theta\varepsilon_\theta+\tau_{rz}\gamma_{rz}+\tau_{r\theta}\gamma_{r\theta}+\tau_{z\theta}\gamma_{z\theta}\right).r.dr.dz.d\theta z \tag{1}$$

And in matrix form

$$U(t)=\frac{1}{2}\int_{R_i}^{R_e}\int_{0}^{H}\int_{0}^{2\pi}\left(\{\varepsilon\}^T.\{\sigma\}\right)r.dr.dz.d\theta \tag{2}$$

Where $\{\varepsilon\}=\begin{Bmatrix}\varepsilon_r\\ \varepsilon_z\\ \varepsilon_\theta\\ \gamma_{rz}\\ \gamma_{r\theta}\\ \gamma_{z\theta}\end{Bmatrix}$ (3-a) and $\{\sigma\}=\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{rz}\\ \tau_{r\theta}\\ \tau_{z\theta}\end{Bmatrix}$ (3-b) are the stain and stress vectors

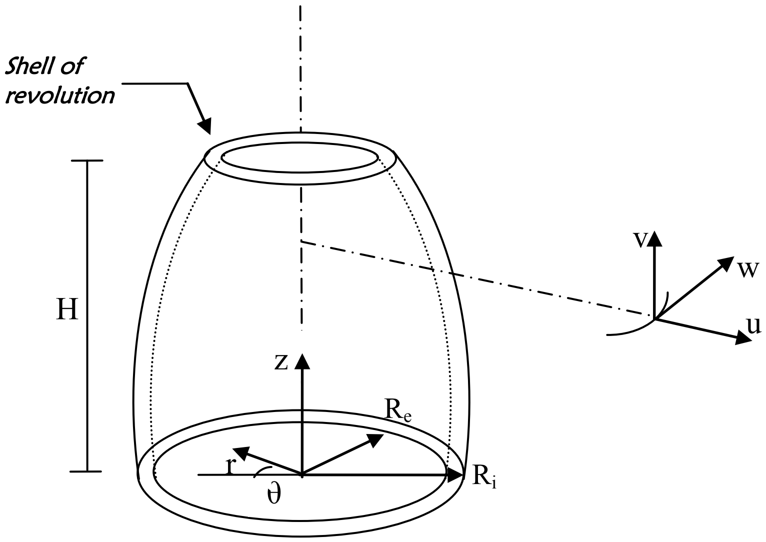



Figure 1: Shell of revolution

The constitutive relationship between stress and strain is the generalized Hooke's law. For an isotropic homogeneous structure, it is given as

$$\{\sigma\} = [D]\{\varepsilon\} \tag{4}$$

Where $[D]$ is the elasticity matrix, $E$ is Young's modulus, and $\upsilon$ is Poisson ratio of the structure.

Eq. (4) into Eq. (2), the potential energy is given in terms of strain as

$$U(t) = \frac{1}{2}\int_{R_i}^{R_e}\int_0^H\int_0^{2\pi}\left(\{\varepsilon\}^T.[D]\{\varepsilon\}\right)r.dr.dz.d\theta \tag{5}$$

The strain-displacement relations are given by:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \\ \gamma_{r\theta} \\ \gamma_{z\theta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial z} \\ \frac{u}{r} + \frac{1}{r}\cdot\frac{\partial w}{\partial \theta} \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \\ \frac{1}{r}\cdot\frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial r} - \frac{w}{r} \\ \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} \end{Bmatrix} \tag{6}$$

The strain vector can be expressed in terms of the displacement vector as follows:

$$\{\varepsilon\} = [d]\{\delta\} \tag{7}$$

Where $$\{\delta\} = \{u, v, w\} \tag{8}$$

and $[d]$ is a differential operator matrix

The stress vector can be expressed in terms of the displacement vector, as:

$$\{\sigma\}=\frac{E}{(1+\nu).(1-2\nu)}.\begin{Bmatrix} (1-\nu).\frac{\partial u}{\partial r}+\nu.\left(\frac{\partial v}{\partial z}+\frac{u}{r}+\frac{1}{r}.\frac{\partial w}{\partial \theta}\right) \\ (1-\nu).\frac{\partial v}{\partial z}+\nu.\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}.\frac{\partial w}{\partial \theta}\right) \\ (1-\nu).\left(\frac{u}{r}+\frac{1}{r}.\frac{\partial w}{\partial \theta}\right)+\nu.\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial z}\right) \\ \frac{1-2\nu}{2}.\left(\frac{\partial u}{\partial z}+\frac{\partial v}{\partial r}\right) \\ \frac{1-2\nu}{2}.\left(\frac{1}{r}.\frac{\partial u}{\partial \theta}+\frac{\partial w}{\partial r}-\frac{w}{r}\right) \\ \frac{1-2\nu}{2}.\left(\frac{1}{r}.\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}\right) \end{Bmatrix} \tag{9}$$

## 3 KINETIC ENERGY

The kinetic energy of the shell can be written as

$$T(t)=\frac{1}{2}\int_{R_i}^{R_e}\int_{0}^{H}\int_{O}^{2\pi}\rho.\left[\left(\frac{\partial u(r,z,\theta,t)}{\partial t}\right)^2+\left(\frac{\partial v(r,z,\theta,t)}{\partial t}\right)^2+\left(\frac{\partial w(r,z,\theta,t)}{\partial t}\right)^2\right].r.dr.dz.d\theta \tag{10}$$

where $\rho$ is the density. Equation can be written as follows

$$T(t)=\frac{1}{2}\int_{R_i}^{R_e}\int_{0}^{H}\int_{O}^{2\pi}\rho.\{\dot{\delta}\}^T.\{\dot{\delta}\}.r.dr.dz.d\theta \tag{11}$$

where $\{\delta\}$is the displacement vector defined by Eq. (8) , and differentiation is with respect to the time, t.

## 4 HIERARCHICAL FINITE ELEMENT FORMULATION

The hierarchical finite element method known under the name of the p-version of the finite element method is more precise and its convergence is faster than that of the h-method. Indeed, when the exact solution is analytical everywhere the rate of convergence is exponential, whereas that of the h-method is only algebraic. The quality of the solutions is not very sensitive to the distortions of the elements, which allows the use of flattened elements or great ratio on sides without penalizing the precision too much. In addition, as a hierarchical formulation is adopted for the representation of displacements, the matrix of stiffness relative to a given degree imbricates those of lower degrees. This makes it possible to obtain in an economic way a sequence of solutions instead of only one solution as it is the case of the h-method [5], [6] et [7].

The hp version of the finite element method has as a characteristic to increase the precision by increasing both the degree of the polynomial of interpolation and the number of finite elements as for the standard finite element method [7].

### 4.1 Idealization of the shell

The shell is divided into four nodes hierarchical axisymmetric quadrilateral isoparametric finite elements (Fig.1). The element size is arbitrary. They may all be of the same size or may all be different. The shell can also be idealized by only one element if the thickness of the shell is varying linearly or if it is constant.

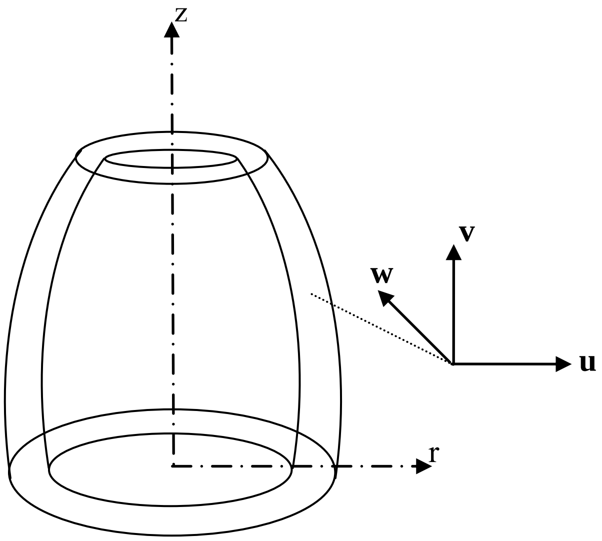



Figure 2: Displacements of a shell point

The equation of motion of the shell admit the representation of the Radial, circumferential and axial displacement components u, v, and w following respectively R, Z and θ in the following form

$$\begin{cases} u(r,z,\theta) = \bar{u}(r,z).\cos n\theta \\ v(r,z,\theta) = \bar{v}(r,z).\cos n\theta \\ w(r,z,\theta) = \bar{w}(r,z).\sin n\theta \end{cases} \tag{12}$$

The displacement functions $\bar{u}(r,z)$, $\bar{v}(r,z)$ and $\bar{w}(r,z)$ can be expressed in terms of the nodal displacements of the finite elements by means of an appropriate set of interpolation functions. title should be written centered, in 14pt, boldface Roman, all capital letters. It should be single spaced if the title is more than one line long.

### 4.2 Shape functions selection

The hierarchical shape functions are generally selected in the Serendipity space. In this paper, the shape functions are built from the shifted Legendre orthogonal polynomials introduced by [8]. They are so different from those introduced by [9]. These polynomials are defined in the interval [0,1].

They can be classified in three categories, nodal shape functions, side shape functions, internal shape functions. Their recurring form is:

$$\begin{cases} f_i(x) = 1 - x \\ f_i(x) = x \\ f_{i+2}(x) = \int_0^x P_i(\alpha).d\alpha \end{cases} \tag{13}$$

where $P_i(\alpha)$ are the shifted Legendre polynomials defined by :

$$\begin{cases} P_0(\alpha)=1 \\ P_1(\alpha)=2.\alpha-1 \\ P_{i+1}(\alpha)=\frac{1}{i+1}\cdot\left[\left(-2i-1+(4i+2)\alpha\right)\cdot P_i(\alpha)-i\cdot P_{i-1}(\alpha)\right] \end{cases} \qquad i=1,2,3,\ldots \tag{14}$$

The shape functions are given on the basis of one-dimensional hierarchical finite element. The origin of the non-dimensional coordinates is at the left end of the element. For the C° continuous problems, the first two linear shape functions of the standard finite element method are retained. The higher order C° shape functions vanish at each end of the elements, they are used to describe the displacement function inside the element. These functions are generated by using the recursive formula (16).

Having the shape functions, the displacement functions $\bar{u}(r,z)$, $\bar{v}(r,z)$ and $\bar{w}(r,z)$ can be expressed:

$$\begin{cases} \bar{u}(r,z)=\sum_{1}^{n} N_i(r,z).u_i \\ \bar{v}(r,z)=\sum_{1}^{n} N_i(r,z).v_i \\ \bar{w}(r,z)=\sum_{1}^{n} N_i(r,z).w_i \end{cases} \tag{15}$$

where

$$N_i(r,z)=f_k(r).g_l(z) \tag{16}$$

with: k = 1,..,p+1 et l = 1,..,q+1

p: Hierarchical modes number according to ξ,

q: Hierarchical modes number according to η, f, g: Shape functions

The matrix form of the expression (15) is:

$$\{\bar{\delta}\}=[N].\{q\} \tag{17}$$

Where:

$$\{\bar{\delta}\}=\begin{Bmatrix} \bar{u}(r,z) \\ \bar{v}(r,z) \\ \bar{w}(r,z) \end{Bmatrix} \tag{18}$$

is the generalized displacement vector,

$$\{q\}=\{u_1,v_1,w_1,\ldots,u_i,v_i,w_i,\ldots\}^T \tag{19}$$

With i=1,…,(p+1)(q+1)

is the nodal displacement vector

$$[N]=\left[[N_1],[N_2],\ldots,[N_i],\ldots,[N_{(p+1)(q+1)}]\right] \tag{20}$$

is the shape functions matrix, where $\left[N_i\right]$ is a sub-matrix given by:

$$\left[N_i\right]=\begin{bmatrix} f_k(r).g_l(z) & 0 & 0 \\ 0 & f_k(r).g_l(z) & 0 \\ 0 & 0 & f_k(r).g_l(z) \end{bmatrix} \quad (21)$$

$f_k(r)$ and $g_l(z)$ are the nodal, edge and internal shape functions of the element.

### 4.3 Shell stiffness matrix

Defining the stiffness properties of the shell is reduced to evaluating the stiffness of a typical element.

Expression (12) can be written

$$\{\delta\}=[\theta_n]\{\bar{\delta}\} \quad (22)$$

Where

$$[\theta_n]=\begin{bmatrix} \cos n\theta & 0 & 0 \\ 0 & \cos n\theta & 0 \\ 0 & 0 & \sin n\theta \end{bmatrix} \quad (23)$$

Substitute Eq. (22) into Eq. (7), then one can write

$$\{\varepsilon\}=[d]\{\delta\}=[d][\theta_n]\{\bar{\delta}\}=\left[\bar{d}\right]\{\bar{\delta}\} \quad (24)$$

Where $\left[\bar{d}\right]$ is an operator vector .Eq. (24) into Eq. (5), the potential energy can be written, as

$$U(t)=\frac{1}{2}\int_{R_i}^{R_e}\int_0^H\int_0^{2\pi}\left(\left(\left[\bar{d}\right]\{\bar{\delta}\}\right)^T.[D].\left[\bar{d}\right]\{\bar{\delta}\}\right).r.dr.dz.d\theta \quad (25)$$

To evaluate the stiffness matrix of an element, the global cylindrical coordinates (r , z) may be expressed in terms of the local non-dimensional coordinates (ξ , η) which are varying from 0 to 1 whose origin is the node lower right of the element.

After transformations, the potential energy can be written, as

$$U(t)=\frac{k\pi}{2}\int_0^1\int_0^1\left(\{q\}^T.[N]^T.\left[\bar{d}\right]^T.[D].\left[\bar{d}\right].[N]\{q\}\right).|J|.r.d\xi.d\eta=\frac{1}{2}\{q\}^T.[K].\{q\} \quad (26)$$

Where

$$[K_e]=k\pi\int_0^1\int_0^1\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}\left([B_i]^T.[D].[B_j]\right)r.|J|d\xi.d\eta \quad (27)$$

is the element stiffness matrix.

$[B]=\left[\bar{d}\right][N]$ and k = 2 for n = 0 and k = 1 for n = 1,2,… (n: circumferential wave number),

### 4.4 Mass matrix

The kinetic energy can be written

$$T(t)=\frac{1}{2}\int_{R_i}^{R_e}\int_{0}^{H}\int_{O}^{2\pi}\rho.\left\{\dot{\bar{\delta}}\right\}^{T}.[\theta_n]^{T}.[\theta_n]\left\{\dot{\bar{\delta}}\right\}r.dr.dz.d\theta=\frac{1}{2}\{\dot{q}\}^{T}.[M_e]\{\dot{q}\} \tag{28}$$

Where

$$[M_e]=k\pi.\int_{0}^{1}\int_{0}^{1}\rho.\sum_{i=1}^{p+1}\sum_{j=1}^{q+1}[N_i]^{T}.[N_j]|J|.r.d\xi.d\eta \tag{29}$$

is the element mass matrix

### 4.5 Numerical integration

The double integral appearing in the forms of the mass and stiffness matrices results in a numerical integration. For his implementation, one uses the Gauss quadrature expressed by:

$$\int_{0}^{1} f(x)d(x)=\sum_{i=1}^{N}W_i.f(x_i) \tag{30}$$

Where N is the integration points number. After having tested several integration points numbers, the number which is a compromise between the computing time and the precision is:

$$N_{int} = p+2 \tag{31}$$

## 5 RESULTS AND DISCUSSIONS

The convergence and comparison studies must be carried out to ensure the reliability of the results. The vibration frequency ω is expressed in terms of the frequency parameter

$$\Omega=\omega\cdot\sqrt{\frac{2\rho(1+\nu)}{E}} \tag{32}$$

### 5.1 Convergence

The bi-hierarchical finite element is used for the first time, to study the free vibration of a very thin cylindrical shell with free boundary conditions. The external and internal radius are respectively $R_e$=1 and $R_i$=0.99, the shell height is H=2. Table 1 show the convergence study of the first six modes with an increasing of the two hierarchical mode numbers p and q following respectively the radius and the axis directions for two and four elements.

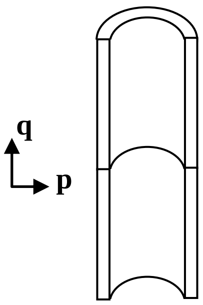
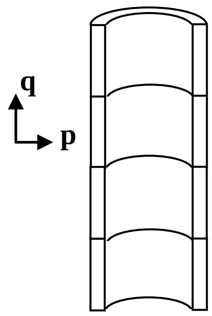
| Element number | q | p | DOF | Modes number | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | 1 | 2 | 3 | 4 | 5 | 6 |
| 2  | 1 | 1 | 18 | 0.014 | 0.018 | 1.923 | 1.930 | 3.012 | 4.007 |
| | 2 | 1 | 30 | 0.014 | 0.018 | 1057 | 1.347 | 1.882 | 2.614 |
| | 2 | 2 | 45 | 0.013 | 0.017 | 1.057 | 1.346 | 1.882 | 2.614 |
| | 4 | 2 | 81 | 0.013 | 0.017 | 0.996 | 1.300 | 1.463 | 1.561 |
| | 4 | 3 | 108 | 0.013 | 0.017 | 0.996 | 1.300 | 1.463 | 1.561 |
| | 6 | 3 | 156 | 0.013 | 0.017 | 0.996 | 1.300 | 1.461 | 1.531 |
| | 8 | 3 | 189 | 0.013 | 0.017 | 0.996 | 1.300 | 1.461 | 1.530 |
| | BCM | | | 0.013 | 0.017 | 0.996 | 1.300 | 1.465 | 1.529 |
| | ANSYS | | | 0.013 | 0.017 | 0.996 | 1.303 | 1.463 | 1.533 |
| 4  | 1 | 1 | 30 | 0.014 | 0.018 | 1.318 | 1.851 | 2.724 | 2.939 |
| | 2 | 1 | 54 | 0.014 | 0.018 | 0.998 | 1.310 | 1.529 | 1.576 |
| | 2 | 2 | 81 | 0.013 | 0.017 | 0.998 | 1.309 | 1.526 | 1.571 |
| | 4 | 2 | 153 | 0.013 | 0.017 | 0.996 | 1.300 | 1.461 | 1.531 |
| | 4 | 3 | 204 | 0.013 | 0.017 | 0.996 | 1.300 | 1.461 | 1.531 |
| | 6 | 3 | 300 | 0.013 | 0.017 | 0.996 | 1.300 | 1.461 | 1.530 |
| | 8 | 3 | 396 | 0.013 | 0.017 | 0.996 | 1.300 | 1.461 | 1.530 |
| | BCM | | | 0.013 | 0.017 | 0.996 | 1.300 | 1.465 | 1.529 |
| | ANSYS | | | 0.013 | 0.017 | 0.996 | 1.303 | 1.463 | 1.533 |

Table 1: Convergence and comparison study of frequency parameter Ω for a free cylindrical shell (Re =1, Ri=0.99, H=2)

The results are compared with those of the boundary collocation method [10] and those of ANSYS (20 harmonic and axisymmetric structural element solid with eight nodes). For the two idealization, two and four elements, an accuracy of three digits after the comma is reached for the first two modes for p=2 and q=2.

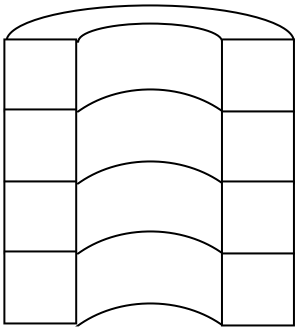
For modes 3, 4, 5; the convergence is reached for p=6 and q=3 for a 2 elements model, and for p=4 and q=2 for a 4 elements model. The sixth mode reaches the convergence for p=8 and q=3 for 2 elements, and p=6 and q=2 for 4 elements.

It is apparent that by increasing the number of elements, the hierarchical mode numbers p and q necessary to reach convergence are smaller. But the degrees of freedom number becomes more important. Indeed for mode 6 this number is of 300 for 4 elements whereas it is only of 189 for 2 elements.

The matrices size is smaller if one increases p and q rather than the elements number. Table (2) shows that the number of degrees of freedom can be easily decreased by modifying the values of p and q according to the shell geometry.

## 5.2 Comparison with other methods

In order to verify the accuracy of the bi-hierarchical finite element in solving the vibration of shells of revolution, a comparison study is conducted for different shells. A thick and a very thin cylindrical shells, a hollow cylindrical shell with varying thickness and a very thin plate. The results are compared with those of the boundary collocation method BCM, [10] and with those obtained by ANSYS where the harmonic and axisymmetric structural element solid with eight nodes is used.

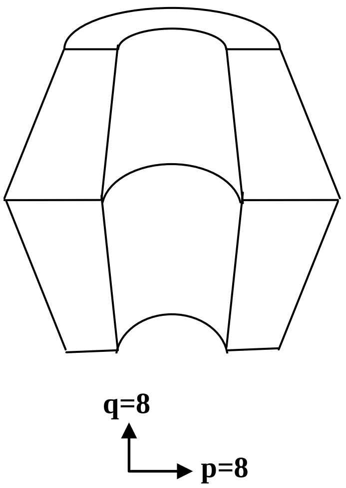



| | | Modes | | | | | |
|---|---|---|---|---|---|---|---|
| **circumferential wave's number** | **Method** | **1** | **2** | **3** | **4** | **5** | **6** |
| **0** | **Present** | 2.298 | 2.512 | 3.237 | 4.138 | 4.502 | 6.352 |
| | **BCM** | 2.299 | 2.513 | 3.237 | 4.137 | 4.500 | 6.347 |
| | **ANSYS** | 2.300 | 2.514 | 3.239 | 4.141 | 4.508 | 6.368 |
| **1** | **Present** | 2.124 | 2.716 | 3.305 | 3.448 | 3.892 | 4.645 |
| | **BCM** | 2.123 | 2.716 | 3.304 | 3.447 | 3.890 | 4.645 |
| | **ANSYS** | 2.125 | 2.718 | 3.307 | 3.452 | 3.895 | 4.650 |
| **2** | **Present** | 1.371 | 1.444 | 3.142 | 3.143 | 4.195 | 4.626 |
| | **BCM** | 1.371 | 1.441 | 3.139 | 3.140 | 4.194 | 4.622 |
| | **ANSYS** | 1.370 | 1.442 | 3.142 | 3.143 | 4.199 | 4.630 |
| **3** | **Present** | 2.820 | 3.079 | 3.762 | 4.285 | 5.531 | 5.716 |
| | **BCM** | 2.818 | 3.071 | 3.758 | 4.279 | 5.526 | 5.711 |
| | **ANSYS** | 2.818 | 3.074 | 3.761 | 4.286 | 5.538 | 5.727 |

Table 2: Comparison of frequency parameters $\Omega$ for free thick cylindrical shell (Re =1/3, Ri=1, H=4/3,p=8,q=8)

The comparison of the frequency parameters $\Omega$ with those of the boundary collocation method [10] and ANSYS (32 harmonic and axisymmetric structural element solid with eight nodes) for a free thick cylindrical shell is given in table 1. The inner radius is $R_i$=1/3, the outer radius is $R_{e=}$1and the shell height is H=4/3. Considering that the results accuracy is quickly reached for a low number of elements if one increases the number of hierarchical modes, the shell is idealized by only four bi-hierarchical finite elements with the same hierarchical modes numbers p and q according to the axial and radial directions. For this very thick cylindrical shell, p=q=8.

The results given in table 2 are in good agreement with those of the two other methods, as well for the axisymmetric case as for the non-axisymmetric case. For the high frequencies the results approach more those of the BCM that those of ANSYS. The reason is that the warping of cross-sections is important and thus the interpolation polynomials degree must be larger to describe these modes. It is easily and automatically possible by using this bi-hierarchical finite element.




| | | Modes | | | | | |
|---|---|---|---|---|---|---|---|
| circumferential wave's number | Method | 1 | 2 | 3 | 4 | 5 | 6 |
| 0 | Present | 2.006 | 2.193 | 2.366 | 2.844 | 3.273 | 3.880 |
| | BC M | 2.007 | 2.195 | 2.367 | 2.839 | 3.279 | 3.896 |
| | ANSYS | 2.007 | 2.193 | 2.367 | 2.846 | 3.275 | 3.889 |
| 1 | Present | 1.629 | 1.996 | 2.518 | 2.832 | 3.370 | 3.382 |
| | BCM | 1.627 | 2.002 | 2.518 | 2.835 | 3.370 | 3.384 |
| | ANSYS | 1.629 | 1.997 | 2.519 | 2.839 | 3.373 | 3.386 |
| 2 | Present | 0.644 | 0.761 | 1.853 | 2.466 | 2.967 | 3.309 |
| | BCM | 0.646 | 0.760 | 1.861 | 2.464 | 2.963 | 3.308 |
| | ANSYS | 0.644 | 0.761 | 1.855 | 2.468 | 2.976 | 3.312 |
| 3 | Present | 1.641 | 1.762 | 2.444 | 3.356 | 3.767 | 4.228 |
| | BCM | 1.641 | 1.755 | 2.449 | 3.348 | 3.761 | 4.228 |
| | ANSYS | 1.642 | 1.763 | 2.448 | 3.365 | 3.774 | 4.239 |

Tableau 3: Comparison of frequency parameters $\Omega$ for a free thick hollow cylindrical shell of varying thickness ($R_e$ varying , $R_i$=0.6, H=2,p=8,q=8)

Finally, a thick free hollow cylindrical shell with varying thickness is considered. Along the axis the inner radius is $R_i$=0.6 and the outer radius is varying from $R_e$=0,8 at the edges to $R_e$=1 at the middle height . The height is H=2. The shell being thick, one takes the same hierarchical modes numbers the p=q= 8. Each half cylindrical shell is idealized by one bi-hierarchical finite. Comparison Results with those of the BCM [10] and those of ANSYS (16 elements) are given in table 3. The same observation is made as for the two other examples. The results agree very well with those of the BCM and those of ANSYS, which shows the power of this element.

## 6 CONCLUSION

The bi-hierarchical finite element presented in this study is able to give accurate frequencies for shells of revolution of an unspecified shape. The results show clearly that this element can be easily used for the extreme cases of very thin or very thick shells of revolution. With this element one is not constrained any more to have the same number of hierarchical modes in the two main directions of the radial and axial shells of revolution. Indeed, if the shapes and dimensions change (thick or thin shells) the hierarchical modes number can change very easily. This element can also be used to idealize a composed shell, plate linked to cylindrical shell for example. Finally this bi-hierarchical finite element allows triple increase in the accuracy, finite elements number, and radial and axial hierarchical modes numbers.

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