COMPDYN 2011
III ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis, V. Plevris (eds.)
Corfu, Greece, 25–28 May 2011

A HIERARCHY OF TIMOSHENKO BEAM THEORIES

S. V. Sorokin[1], **C. J. Chapman**[2]

[1] Department of Mechanical and Manufacturing Engineering, Aalborg University
Pontoppidanstraede 101, DK 9220 Aalborg, Denmark
e-mail: svs@m-tech.aau.dk

[2] Department of Mathematics, University of Keele
Keele, Staffordshire, ST5 5BG, United Kingdom
c.j.chapman@maths.keele.ac.uk

Keywords: Bending Wave, Dispersion Relation, Finite Products, Beam, Timoshenko Theory.

Abstract. *This paper shows that conventional Timoshenko theory for bending waves is a member of a two-parameter family (m; n) of approximations to the exact equations of linear elasticity. Higher members of the family are shown to represent the exact dispersion relation with extraordinary accuracy; in particular, an arbitrary number of branches can be captured accurately over their entire length, i.e. up to arbitrarily high frequencies and wavenumbers. The theory admits a rational accuracy analysis, and resolves certain controversies about the validity of higher-branch approximations. The paper demonstrates conclusively that Timoshenko theory is a completely rational theory, thus ending decades of doubt on the matter. The standard Euler-Bernoulli theory is a lower member of the two-parameter family. Especially useful is Timoshenko (1; 2) theory, which extends conventional Timoshenko (0; 1) theory by capturing the first four branches of the exact dispersion relation rather than merely the first two.*

1 INTRODUCTION

In the hierarchy of bending-wave theories for plates and beams, the first and simplest theory is Euler-Bernoulli theory, valid at the lowest frequencies and wave numbers; next comes Timoshenko theory, with or without various correction factors, valid to higher frequencies and wavenumbers; and finally comes exact linear theory, based on the full equations of linear elasticity, valid to any frequency and wavenumber for which linear theory still holds. The aim of this paper is to fill the very large gap between these last two theories, but use only polynomials. The gap is worth filling, because the dispersion relations obtained from exact linear theory are transcendental (when they can be obtained analytically at all), i.e. do not have the simple polynomial form of the earlier two theories, and engineers concerned with bending waves have overwhelmingly preferred the simplicity of polynomial theories

We shall fill the gap by means of a hierarchy of Timoshenko-type theories, in which the simplicity and accuracy of Timoshenko-type polynomial dispersion relations are maintained at very high frequencies and wavenumbers. At first sight, the quest for such dispersion relations would appear to be a hopeless undertaking. One reason is that the conventional Timoshenko dispersion relation is invariably derived from a kinematic hypothesis about the shape of frequencies. Another reason is that any truncated Taylor-series expansion of an exact dispersion relation must likewise fail at higher frequencies, because of the known finite radius of convergence of such an expansion. Thus strategies based on more elaborate kinematic hypotheses, or on ever-longer Taylor-series truncations, can at best succeed in a limited range of frequency and wavenumber.

The fundamental idea of this paper is to separate completely the derivation of a Timoshenko-type dispersion relation from any dependence on a kinematic hypothesis or a Taylor-series approximation. Instead, we use two families of finite-product polynomials, namely 'sine-based' polynomials S_m of degree m, and 'cosine-based' polynomials C_n of degree n. These polynomials are products of factors corresponding to a specified finite number of roots of a sine or cosine function. The roots correspond to the cut-on frequencies of different types of modes. The result is a two-parameter family (m, n) of polynomial approximations to the exact dispersion relation, of which the member (m, n)= (0, 1) is a conventional Timoshenko approximation.

If an analytical expression is available for the exact dispersion relation, the method is trivial to implement: for any (m, n), the corresponding finite-product approximation may be written down; and numerical computation of all of its roots, real or complex, is instantaneous on any computer, because of the universal availability and reliability of software for calculating the roots of polynomials. If, instead of an analytical expression, a numerical code is available for calculating the dispersion relation, the method may be implemented by first computing the cut-on frequencies of the low-order modes; these frequencies are then used to determine the appropriate finite-product polynomials.

In this paper, we implement the method in complete detail for the canonical problem of the subject, namely bending waves in a planar elastic layer, for which exact linear theory leads to the Rayleigh-Lamb dispersion relation. The numerical accuracy of the approximations obtained can be displayed explicitly in plots comparing the exact and approximate dispersion relations. It reveals the almost incredible accuracy of the above polynomials in representing the exact dispersion relation for small values of (m, n). Particularly accurate approximations are obtained when n = m+1. Hence a sequence of approximations is obtained which may be referred to as the Timoshenko (0, 1) dispersion relation, the Timoshenko (1, 2) dispersion relation, etc. This family of approximations approaches the exact Rayleigh-Lamb dispersion relation arbitrarily closely at arbitrarily high frequencies and wave numbers. The structure of these approximations, together with a detailed account of their numerical accuracy, is fully presented. The fact that such approximations can exist at all is far from obvious, but was demonstrated by Chapman & Sorokin [1]. Especially useful is the Timoshenko (1, 2) dispersion relation, which offers a massive extension of the range of validity of conventional Timoshenko (0, 1) dispersion relation, and provides ample accuracy for practically all antisymmetric plate waves and vibrations of engineering interest.

2 TERMINOLOGY

We use the term Timoshenko-type theory for any theory or approach which leads to a polynomial dispersion relation of a certain type (specified explicitly) which generalises the conventional Timoshenko dispersion relation to a higher order polynomial. It must be emphasized that the type of theory refers only to how the dispersion relation is derived from physical principles and approximations, not to the final form of the dispersion relation itself. In every case, the dispersion relation is of the same form, namely a member of a specific two-parameter family of dispersion relations labelled by the parameters (m, n). This codified scheme brings unity to what would otherwise be a ‘zoo’ of approximate methods. Two main aspects of the paper are, first, the demonstration that established approximations fit within the scheme, and second, that, when applied to slightly higher (m, n) than hitherto, the scheme gives new approximations displaying an extraordinary increase in accuracy and scope at almost no cost. Our use of the terms Timoshenko theory and Timoshenko dispersion relation is an extension of their familiar use, but is entirely logical in describing the results obtained in the paper.

One might wonder why such a simple method as ours has not been exploited already. The answer lies in Runge’s phenomenon, namely the fact that the polynomials S_m and C_n do not represent the underlying sine and cosines accurately on account of the high-amplitude oscillations which the polynomials display away from the centre of their range. Such oscillations are known to anyone who has tried polynomial interpolation with an equally spaced set of grid points. The crucial fact underlying this paper is that, in a homogeneous linear combination of products of sines and cosines, Runge’s phenomenon cancels out almost exactly if an appropriate choice is made of the number of factors in the sine-based and cosine-based polynomials. With such a choice, inaccuracy in the individual representations of the sines and cosines does not lead to inaccuracy in the resulting approximation to the dispersion relation. In this context the term homogeneous means that in every term of the exact dispersion relation the number of sines or cosines multiplied together is the same; for the Rayleigh-Lamb dispersion relation, this number is two.

The cancelling out of Runge’s phenomenon answers a long-standing objection to Timoshenko theory from devotees of ‘rational mechanics’. It has repeatedly been claimed that Timoshenko theory is ‘just an engineering approximation’, and in particular that the excellent performance of the Timoshenko dispersion relation in capturing the first thickness-shear branch near cut-on is spurious. Indeed, it has been stated that Timoshenko theory is a ‘low-frequency theory trying to be a high-frequency theory’. The key point, emphasized in this paper, is that the Timoshenko dispersion relation can be derived by a method independent of a kinematic hypothesis or a Taylor-series truncation. This method shows that the Timoshenko dispersion relation is an early member of a sequence of approximations which, because of the cancelling-out of Runge’s phenomenon, approaches the exact dispersion relation at high frequencies and wavenumbers with extraordinary rapidity. Accordingly, Timoshenko theory is completely rational. A mathematically rigorous justification of the above assertions is given in [1] for waves in a planar layer.

3 FINITE-PRODUCT TIMOSHENKO THEORY

The key idea in finite-product Timoshenko theory for a planar layer is to start with the exact dispersion relation of Rayleigh-Lamb theory, and immediately replace the sine and cosine terms by finite-product polynomials, chosen to have the same roots as the original sines and cosines in a finite region. After this step has been taken, all calculations are performed with polynomials, and the original transcendental dispersion relation is not used again, unless accuracy analysis is required. The lengths of the finite regions are at the choice of the investigator: the longer the regions, the higher is the polynomial order of the finite-product dispersion relation, and the greater is its region of accuracy in the frequency-wavenumber plane. This region of accuracy increases rapidly with polynomial order, and may be made arbitrarily large. In this sense, finite-product Timoshenko theory fills completely the gap between conventional Timoshenko theory and Rayleigh-Lamb theory, referred to in the Introduction.

The exact Rayleigh-Lamb dispersion relation for this problem is (see [2]):

$$L_3^4 S_1 C_2 + K^2 L_2^2 C_1 S_2 = 0. \tag{1}$$

Here $S_i \equiv S\left(L_i^2/4\right)$, $C_i \equiv C\left(L_i^2/4\right)$, $L_i^2 = \Omega_i^2 - K^2$, $i = 1,2$ and $L_3^2 = \frac{1}{2}\Omega_2^2 - K^2$. Furthermore, $\left(K, \Omega_1, \Omega_2\right) = \left(kh, \frac{\omega h}{c_1}, \frac{\omega h}{c_2}\right)$ with conventional notations for velocities of P- and S-waves.

The functions S and C are defined by $S\left(s^2\right) = s^{-1}\sin(s)$ and $C\left(s^2\right) = \cos(s)$. The use of S and C, rather than sine and cosine, helps avoid square roots in equations, where the square roots ultimately cancel out in pairs, and so maintains polynomial form. We define the finite products as follows:

$$S_{mi} = \prod_{m'=1}^{m}\left(1 - \frac{L_i^2}{4(m'\pi)^2}\right),\ C_{ni} = \prod_{n'=1}^{n}\left(1 - \frac{L_i^2}{(2n'-1)^2\pi^2}\right),\ i = 1,2 \tag{2}$$

If $m = 0$ or $n = 0$, the value of the corresponding finite product is defined as 1. Then the equation (1) is approximated by:

$$L_3^4 S_{m1} C_{n2} + K^2 L_2^2 C_{n1} S_{m2} = 0 \tag{3}$$

Although equation (3) is of simple form, it is divisible by $L_1^2 - L_2^2$. We introduce

$$I_{mn} = 4\frac{S_{m1}C_{n2} - C_{n1}S_{m2}}{L_1^2 - L_2^2} \tag{4}$$

Then equation (3) becomes ($\Omega_B = \frac{\omega h}{c_B}$, $c_B^2 = \frac{c_2^2\left(c_1^2 - c_2^2\right)}{3c_1^2}$)

$$-\frac{1}{3}\Omega_B^2 S_{m1} C_{n2} = K^2 L_2^2 I_{mn} \tag{5}$$

Equations (3) and (5) give the cut-on frequencies of the first m thickness-stretch modes and the first n thickness-shear modes. The reason is that these equations are satisfied if $K = 0$ and either $S_{m1} = 0$ or $C_{n2} = 0$. This is the motivation for using the finite-product method, since the exact Rayleigh-Lamb dispersion relation (1) is satisfied if $K = 0$ and either $S\left(L_1^2/4\right) = 0$ or $C\left(L_2^2/4\right) = 0$. The finite-product approximations (3) and (5) agree with the exact dispersion relation on a grid of points in the (K, Ω) plane, for which the above cut-on points form the boundary; details of the grid are given in [1].

The simplest way to introduce correction factors is to replace the term L_2^2 on the right-hand side of equation (5) by the quantity $L_{2\gamma\delta}^2$ defined by $L_2^2 = \gamma\Omega_2^2 - \delta K^2$:

$$-\frac{1}{3}\Omega_B^2 S_{m1} C_{n2} = K^2 L_{2\gamma\delta}^2 I_{mn} \qquad (6)$$

At this stage, the values of γ and δ are arbitrary. The replacement of L_2^2 by $L_{2\gamma\delta}^2$ is not made in any other term, thus C_{n2} and I_{mn} contain L_2^2 not $L_{2\gamma\delta}^2$. Hence, the grid is preserved.

4 TIMOSHENKO (0,1) THEORY

The finite product theory, presented above for arbitrary m and n, will now be analyzed in detail for $(m,n)=(0,1)$. In this case, the reduced equation (6) becomes:

$$\Omega_B^2 = \frac{1}{\pi^2}\Omega_B^2\Omega_2^2 - \frac{1}{\pi^2}\left(12\gamma + \frac{c_2^2}{c_B^2}\right)\Omega_2^2 K^2 + \frac{12\delta}{\pi^2}K^4 \qquad (7)$$

For equation (7), to recover the Bernoulli-Euler limit $\Omega_B^2 = K^4$, we must take $\delta = \frac{\pi^2}{12}$. Further analysis shows that γ is a re-parametrization of the standard stiffness correction factor κ via the relation

$$\gamma = \frac{\pi^2}{12\kappa} - \frac{c_2^2}{12c_B^2}(1-\kappa) \qquad (8)$$

For the conventional Timoshenko dispersion relation with $\kappa = \frac{\pi^2}{12}$, this formula gives

$$\gamma = 1 - \frac{c_2^2}{12c_B^2}\left(1 - \frac{\pi^2}{12}\right) \qquad (9)$$

In all these formulas, $\frac{c_2^2}{3c_B^2} = \frac{c_1^2}{c_1^2 - c_2^2}$ for both plane strain and plane stress.

5 TIMOSHENKO (1,2) THEORY

Then $(m,n)=(1,2)$, the definitions give: $S_{mi} = 1 - \frac{L_i^2}{4\pi^2}$, $C_{mi} = \left(1 - \frac{L_i^2}{\pi^2}\right)\left(1 - \frac{L_i^2}{9\pi^2}\right)$, $i = 1,2$.

Hence $I_{mn} = \frac{31}{9\pi^2}\left[1 - \frac{4}{31\pi^2}\left(L_1^2 + L_2^2\right) + \frac{4}{31\pi^2}L_1^2 L_2^2\right]$. In the reduced equation (7), we must take $\delta = \frac{3\pi^2}{31}$ to recover the Bernoulli-Euler limit $\Omega_B^2 = K^4$. Then the reduced equation becomes

$$\Omega_B^2\left(1 - \frac{L_1^2}{4\pi^2}\right)\left(1 - \frac{L_2^2}{\pi^2}\right)\left(1 - \frac{L_2^2}{9\pi^2}\right) = -K^2\left(\frac{31\gamma}{\pi^2}\Omega_2^2 - K^2\right)\left[1 - \frac{4}{31\pi^2}\left(L_1^2 + L_2^2\right) + \frac{4}{31\pi^2}L_1^2 L_2^2\right] \qquad (10)$$

This simple equation has a truly extraordinary range of numerical accuracy.

6 TIMOSHENKO (M, N) THEORY

We have seen that for the finite-product equation (7) or (10) to recover the Bernoulli-Euler limit $\Omega_B^2 = K^4$, the correction factor $\delta = \frac{3\pi^2}{31}$ must take value $\frac{\pi^2}{12}$ when $(m,n) = (0,1)$ and the value $\frac{3\pi^2}{31}$ when $(m,n) = (1,2)$. These are examples of the general result that, for arbitrary (m,n), the value of the correction factor is $\delta = \frac{1}{3(c_{n1} - s_{m1})}$, where $s_{m1} = \frac{1}{\pi^2}\sum_{m'=1}^{m}\frac{1}{m'^2}$ and $c_{n1} = \frac{4}{\pi^2}\sum_{n'=1}^{n}\frac{1}{(2n'-1)^2}$. Thus δ is always a rational multiple of π^2. For example, the next value in the sequence above is $\delta = \frac{300\pi^2}{3019}$ for $(m,n) = (2,3)$. Since $s_{m1} \to \frac{1}{6}$ as $m \to \infty$, and $c_{n1} \to \frac{1}{2}$ as $n \to \infty$, it follows that $\delta \to 1$ as m and n increase. The above results are consequences of series expansions

$$S_m = \prod_{m'=1}^{m}\left(1 - \frac{s}{(m'\pi)^2}\right) = 1 - s_{m1}s + s_{m2}s^2 - s_{m3}s^3 + \ldots \tag{11}$$

$$C_{ni} = \prod_{n'=1}^{n}\left(1 - \frac{4c}{(2n'-1)^2\pi^2}\right) = 1 - c_{n1}c + c_{n2}c^2 - c_{n3}c^3 + \ldots \tag{12}$$

$$I_{mn} = c_{n1} - s_{m1} - (c_{n2} - s_{m2})(s + c) + (c_{n3} - s_{m3})(s^2 + cs + c^2) + (s_{m1}c_{n2} - s_{m2}c_{n1})sc + \ldots \tag{13}$$

Expansion of the products in equations (11-12) gives $s_{m1} = \frac{1}{\pi^2}\sum_{m'=1}^{m}\frac{1}{m'^2}$ and $c_{n1} = \frac{4}{\pi^2}\sum_{n'=1}^{n}\frac{1}{(2n'-1)^2}$ (see above). Then substitution of equations (11-13), with appropriate arguments, into the finite-product equation (6) determines the correction factor as $\delta = \frac{1}{3(c_{n1} - s_{m1})}$.

7 BOUNDARY CONDITIONS

We have considered the planar elastic layer as a waveguide, i.e. as having indefinite length. For more general problems, boundary conditions at the ends must be included. The question then arises of solving a complete boundary-value problem if the Timoshenko (m, n) dispersion relation is to be used. Within the approach of this paper, this would be tackled by first formulating the problem with the exact equations of linear elasticity (either directly or using Hamilton’s principle with the exact Lagrangian), and then using the Timoshenko (m, n) field structure to reduce the problem to finite-dimensional modal form. This is an important direction for further work.

8 CONCLUSIONS

The method just described provides an incentive to re-examine certain classical problems for which analytical dispersion relations have been derived, but for which reasonable polynomial approximations have not been found. For example, the exact dispersion relation for elastic waves in a cylindrical shell contains many combinations of Bessel functions, quite beyond the range of analysis 'by hand'. Yet the finite-product method can be applied, yielding a family of polynomial approximations readily obtained with a symbolic mathematical software package. A theoretical task would be to relate the families of approximations arising in such problems to the wave-hierarchy theories of Whitham [3].

REFERENCES

[1] C.J. Chapman, S.V. Sorokin, The finite-product method in the theory of waves and stability. *Proceedings of the Royal Society of London* **A** 466, 471-491, 2010

[2] J.D. Achenbach, *Wave propagation in elastic solids*. Amsterdam: North-Holland, 1973.

[3] G.B. Whitham, *Linear and nonlinear waves.* New York: Wiley, 1974.