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MODELLING MOVING ONE-DIMENSIONAL WAVEGUIDES USING WAVES AND FINITE ELEMENT ANALYSIS

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Abstract. *One approach to the numerical analysis of complex waveguides is the Wave Finite Element (WFE) method. In this method conventional Finite Elements (FEs) are used to discretise a small segment of a waveguide. The FE model of just this small part of the structure is post-processed using periodicity conditions, and an eigenproblem is then solved to predict dispersion characteristics and wavemodes. Once the wave characteristics are predicted, free vibration and response of the structure as a whole can be modelled in terms of these waves. This paper presents an extension of the method to moving one-dimensional waveguides. In particular an axially moving beam is considered. The FE formulation of a moving beam element is developed and the WFE method is applied to find the wave properties of such a beam. Natural frequency are obtained using the Phase Closure Principle and the Dynamic Stiffness Matrix, both formulated in terms of wavemodes and dispersion relation obtained from the WFE eigenproblem. The analytical equation of transverse motion of the travelling beam is also solved in terms of propagating and decaying waves, and the frequency equation is obtained using the phase closure principle. Numerical results are shown.*

1 INTRODUCTION

The research on the dynamics of moving media has had a renewed interest in the last few decade giving rise to large scientific production. There are a number of applications which involve moving structures and fluids, and the necessity to improve their performance has motivated the development of numerical and analytical methods to investigate the dynamics of mechanical models involving transport of mass. Studies on the dynamics of axially moving structures have been made by several authors, and many original and interesting studies have been produced. To cite just a few, the works [1, 2, 3, 4] can be listed.

In this paper the dynamics of a travelling one-dimensional waveguides is investigated using a Wave Finite Element method [5, 6]. The method is an FE post-processing technique for obtaining numerical prediction of wave characteristics. The method involves conventional FE analysis of a small segment of the structure. Typically this consists of just a single finite element, or a stack of elements meshed through the cross-section. The FE matrices of this segment are then post-processed using periodicity conditions, resulting in an eigenproblem, whose solutions yield the dispersion curves, frequency evolution of the wavenumber, and wavemodes. Once the dispersion characteristics are known, existing wave propagation methods can be applied to take into account boundary conditions, calculate natural frequencies, forced response or determine the systems stability [7].

An axially moving beam, modelled using the classical linear theory, is studied in the present paper. The FE element formulation of a linear moving beam element is derived following [8], and application of the WFE method is briefly described. Free vibrations of the moving beam are predicted using the Phase Closure Principle [9] and the Dynamic Stiffness Matrix [10] formulated in terms of wavemodes. These are applied once the wave characteristics are known from the WFE model. An analytical wave approach is also presented to show some characteristics of the elastic waves propagating in such a beam. Numerical results are shown and the accuracy of the proposed approach is discussed. The theory used in this paper is linear, therefore results are consistent with ‘low’ axial velocity and small deformation. However, this linear model well approximates the behaviour of many real cases, and it can be helpful for predicting instability region, where nonlinear terms become significant [11]. It must be pointed out that wave propagation in a travelling beam were also investigated by Chakraborty and Mallik in [12], and a Dynamic Stiffness Formulation of a moving beam was given by Banerjee and Gunawardana [13]. Both these works involve an analytical formulation of the equation of motion, while the present work concerns the application of a numerical technique based on FE analysis.

Although for the case of a moving beam there are no practical advantages in calculating dispersion curves and natural frequencies using the proposed method, this work shows that the WFE method can provide results for moving waveguides. This is of particular interest in more complicated cases where analytical formulation of the problem can be difficult and computational cost using standard numerical approaches can be very large.

2 EQUATION OF MOTION AND FE FORMULATION OF A MOVING BEAM ELEMENT

A prismatic axially moving beam is considered. The beam is modelled using an Eulero–Bernoulli beam with constant mass per unit length ρ and constant flexural rigidity EI. The beam moves with axial speed $v(t)$, and its transverse displacement, measured by a stationary observer, is denoted by $y(x,t)$, figure 1. External excitation are not taken into account.

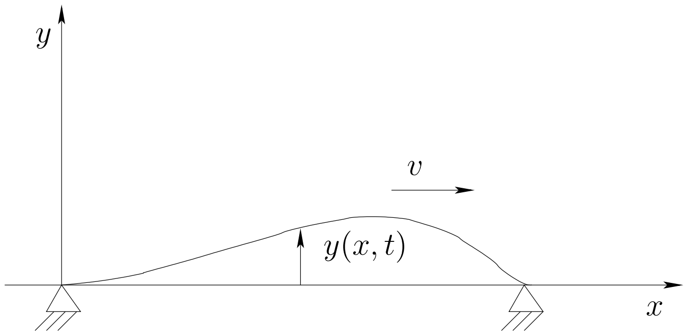

Figure 1: Schematic model of an axially moving beam.

Time differentiation of the displacement yields

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{\partial y}{\partial t} + v\frac{\partial y}{\partial x}; \\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} &= \frac{\partial^2 y}{\partial t^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + v^2\frac{\partial^2 y}{\partial x^2} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial x}. \end{aligned} \tag{1}$$

Accordingly, the analytical equation of transverse motion is

$$EI\frac{\partial^4 y}{\partial x^4} + \rho\left(\frac{\partial^2 y}{\partial t^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + v^2\frac{\partial^2 y}{\partial x^2} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial x}\right) = 0. \tag{2}$$

In order to apply the WFE approach, a small segment of length L is taken and discretised using FE elements. Therefore a moving beam FE element is developed. The kinetic and potential energy of the element are

$$\begin{aligned} T &= \frac{1}{2}\rho\int_0^L \left(\frac{\partial^2 y}{\partial t^2} + 2v\frac{\partial^2 y}{\partial x \partial t} + v^2\frac{\partial^2 y}{\partial x^2}\right)\mathrm{d}x; \\ V &= \frac{1}{2}EI\int_0^L \left(\frac{\partial^2 y}{\partial x^2}\right)^2 \mathrm{d}x. \end{aligned} \tag{3}$$

The transverse displacement of the beam element is modelled by standard cubic shape function $N(x)$ [14] and nodal displacements $\mathbf{q}$, that is

$$y = N(x)\mathbf{q}. \tag{4}$$

Substituting this equation in (3), the Lagrangian of the moving element is given by [8]

$$L = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{M}\dot{\mathbf{q}} + \mathbf{\frac{1}{2}q^T K_1 q} - \mathbf{\frac{1}{2}q^T K_2 q} + \mathbf{\frac{1}{2}q^T C_1 \dot{q}} + \mathbf{\frac{1}{2}\dot{q}^T C_2 q}, \tag{5}$$

with

$$\begin{aligned} &\mathbf{M} = \rho\int_{\mathbf{0}}^{\mathbf{L}}\mathbf{N^T N}\,\mathrm{d}\mathbf{x}; \quad \mathbf{K_1} = \rho\mathbf{v^2}\int_{\mathbf{0}}^{\mathbf{L}}\mathbf{N'^T N'}\,\mathrm{d}\mathbf{x}; \quad \mathbf{K_2} = \mathbf{EI}\int_{\mathbf{0}}^{\mathbf{L}}\mathbf{N''^T N''}\,\mathrm{d}\mathbf{x}; \\ &\mathbf{C_1} = \rho\mathbf{v}\int_{\mathbf{0}}^{\mathbf{L}}\mathbf{N'^T N}\,\mathrm{d}\mathbf{x}; \quad \mathbf{C_2} = \rho\mathbf{v}\int_{\mathbf{0}}^{\mathbf{L}}\mathbf{N^T N'}\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{6}$$

where $'$ denotes differentiation with respect to x. The FE equation of motion can be obtained using Euler–Lagrangian equations. Therefore the FE equations of motion of a moving beam element are

$$\mathbf{M\ddot{q}} + \left(\mathbf{C_2} - \mathbf{C_2^T}\right)\dot{\mathbf{q}} + \left(\dot{\mathbf{C}}_\mathbf{2} - \mathbf{K_2} + \mathbf{K_1}\right)\mathbf{q} = \mathbf{f}, \tag{7}$$

where $\mathbf{f}$ are nodal forces.

Although in this section a general formulation considering time dependent velocity is given, in the following sections a constant axial velocity is assumed, that is $\frac{\partial v}{\partial t} = 0$ in equation (2) and $\dot{\mathbf{C}}_2 = \mathbf{0}$ in equation (7).

3 WAVE MOTION AND FREE VIBRATION

In this section the dynamics of the travelling beam is described using a wave approach. Solution of equation (2) are written in terms of harmonic waves $y(x,t) = Ae^{i(\omega t - kx)}$ which, substituted into (2), gives the dispersion relation between wavenumbers and frequencies:

$$a^2k^4 - v^2k^2 + 2vwk - w^2 = 0, \tag{8}$$

where $a = \sqrt{\frac{EI}{\rho}}$. The roots of this equation are

$$\begin{aligned} k_d &= -\frac{1}{2}\frac{v}{a} + \frac{1}{2}\sqrt{\frac{v^2}{a^2} + 4\frac{w}{a}}; \qquad k_u = \frac{1}{2}\frac{v}{a} + \frac{1}{2}\sqrt{\frac{v^2}{a^2} + 4\frac{w}{a}}; \\ k_{de} &= \frac{1}{2}\sqrt{4\frac{w}{a} - \frac{v^2}{a^2}} + i\frac{1}{2}\frac{v}{a}; \qquad k_{ue} = \frac{1}{2}\sqrt{4\frac{w}{a} - \frac{v^2}{a^2}} - i\frac{1}{2}\frac{v}{a}, \end{aligned} \tag{9}$$

and the solution of the homogeneous equation (2) can be written as a linear combination of these four flexural waves

$$y(x,t) = A_d e^{-ik_d x}e^{iwt} + A_u e^{ik_u x}e^{iwt} + A_{de}e^{-k_{de}x}e^{iwt} + A_{ue}e^{k_{ue}x}e^{iwt}. \tag{10}$$

For $0 < v < 2\sqrt{\omega a}$ solutions k_{de} and k_{ue} represent attenuating waves, therefore equations (9) represent one positive-going and one negative-going propagating waves, k_d and k_u respectively, and one positive-going and one negative-going decaying waves, k_{de} and k_{ue} respectively. On the other side, for $v > 2\sqrt{\omega a}$ solutions k_{de} and k_{ue} are real numbers representing two positive-going propagating waves. Hence, when $v > 2\sqrt{\omega a}$ there are four propagating waves, three positive-going waves, k_d, k_{de} and k_{ue}, and one negative-going wave, k_u. However, $v_g = 2\sqrt{\omega a}$ is the group velocity of a disturbance propagating in the beam, and it is expected that instability occurs when the axial velocity is faster than the energy velocity in the medium considered. Moreover, the axial speed is often much smaller than the speed of any travelling disturbance in the beam. Therefore the inequality $0 < v < 2\sqrt{\omega a}$ is assumed in the present paper.

Using the same formalism proposed in [9], the wave amplitude of the positive and negative going waves is given by $\mathbf{A}^+ = [A_d, A_{de}]^T$ and $\mathbf{A}^- = [A_u, A_{ue}]^T$, while the transfer matrices $\mathbf{F}^+$ and $\mathbf{F}^-$ describe the propagation from one point to another through appropriate wavenumber

$$\mathbf{F}^+(x) = \begin{bmatrix} e^{-ik_d x} & 0 \\ 0 & e^{-k_{de}x} \end{bmatrix}, \qquad \mathbf{F}^-(x) = \begin{bmatrix} e^{ik_u x} & 0 \\ 0 & e^{k_{ue}x} \end{bmatrix}. \tag{11}$$

To evaluate the free vibration, the interaction of both propagating and decaying waves with discontinuities is considered, and wave amplitudes and phases after reflection and transmission are arranged in reflection and transmission matrices. The elements of these matrices are obtained imposing equilibrium conditions [9]. In this paper we consider boundary conditions at the ends of the beam $A(x = 0)$ and $B(x = L_{tot})$, where L_{tot} is the total length of the beam, and the reflection matrices at the boundaries are denoted by $\mathbf{R}_A$ and $\mathbf{R}_B$.

The wave amplitudes at each ends of the beam are denoted by $\mathbf{A}^+$, $\mathbf{A}^-$, $\mathbf{B}^+$ and $\mathbf{B}^-$. If the incident waves impinging upon the right end at $A(x=0)$ are the negative-going waves $\mathbf{A}^-$, then reflected waves have amplitude $\mathbf{A}^+ = \mathbf{R}_A\ \mathbf{A}^-$. Similarly if the incident waves impinging upon the left end are the positive-going waves of amplitude $\mathbf{B}^+$, reflected waves are $\mathbf{B}^- = \mathbf{R}_B\ \mathbf{B}^+$. Considering the wave reflection matrices and using the transfer matrices, the following relations can be written

$$\mathbf{A}^+ = \mathbf{R}_A\ \mathbf{A}^-;\ \ \mathbf{B}^- = \mathbf{R}_B\ \mathbf{B}^+;\ \ \mathbf{B}^+ = \mathbf{F}^+(L_{tot})\ \mathbf{A}^+;\ \ \mathbf{A}^- = \mathbf{F}^-(-L_{tot})\ \mathbf{B}^-. \tag{12}$$

Combing equations in (12), the characteristic equation is obtained

$$\left[\mathbf{R}_A\ \mathbf{F}^-(-L)\ \mathbf{R}_A\ \mathbf{F}^+(L) - \mathbf{I}\right]\ \mathbf{A}^+ = 0. \tag{13}$$

For non trivial solution, the determinant of the coefficients $\mathbf{A}^+$ must be zero, that is

$$\left| det\left(\mathbf{R}_A\ \mathbf{F}^-(-L_{tot})\ \mathbf{R}_A\ \mathbf{F}^+(L_{tot}) - \mathbf{I}\right)\right| = 0. \tag{14}$$

Solutions of equations (14) and (13) give the natural frequencies and natural modes.

4 WAVE FINITE ELEMENT MODEL

In this section the WFE approach is briefly described. The FE equation of motion (7) of the moving beam element is considered with $\dot{\mathbf{C}}_2 = \mathbf{0}$. Assuming time harmonic behaviour, the equation of motion is

$$\left(-\omega^2\mathbf{M} + i\omega\mathbf{C}_{eq} + \mathbf{K}_{eq}\right)\mathbf{q} = \mathbf{f}, \tag{15}$$

where $\mathbf{C_{eq}} = \mathbf{C_2} - \mathbf{C_2^T}$ and $\mathbf{K_{eq}} = \mathbf{K_2} - \mathbf{K_1}$.

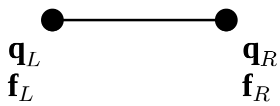

Figure 2: WFE model and node numbering.

The nodal degrees of freedom (DOFs) $\mathbf{q}$ and the nodal forces $\mathbf{f}$ are partitioned into left and right, figure 2, that is

$$\mathbf{q} = [\mathbf{q}_L^T\ \ \mathbf{q}_R^T]^T; \tag{16}$$

T denoting the transpose, with a similar expression for the nodal forces $\mathbf{f}$. Under the passage of a wave the nodal DOFs are related by periodicity conditions [15]

$$\mathbf{q}_R = \lambda\mathbf{q}_L, \tag{17}$$

where $\lambda = e^{-ikL}$ and k is the wavenumber, while equilibrium at left side of the segment implies

$$[\mathbf{I}\quad \lambda^{-1}\mathbf{I}]\mathbf{f} = \mathbf{0}. \tag{18}$$

By substituting equation (17) in equation (15), and premultiplying both sides of equation (15) by the matrix in equation (18), the equation of free wave motion takes the form

$$[\overline{\mathbf{K}_{eq}}(kL) - \omega^2\overline{\mathbf{M}}(kL) + i\omega\overline{\mathbf{C}_{eq}}(kL)]\mathbf{q}_L = \mathbf{0}. \tag{19}$$

Solutions of equation (19) yield the dispersion curves and the wavemode shapes. This involves solving a quadratic polynomial eigenvalue problem, which is recast as the standard linear eigenvalue problem [5, 6]. Associated with eigenvalues, viz. wavenumbers, are eigenvectors Φ_{qj}, which represent wavemodes, and vectors

$$\Phi_{fj} = \left[\overline{\mathbf{K}_{eq}}(kL) - \omega^2\overline{\mathbf{M}}(kL) + i\omega\overline{\mathbf{C}_{eq}}(kL)\right]\Phi_{qj}. \tag{20}$$

These can be partitioned into positive-going and negative-going waves, which are denoted by $(\lambda_j^+, \Phi_{qj}^+, \Phi_{fj}^+)$ and $(\lambda_j^-, \Phi_{qj}^-, \Phi_{fj}^-)$ respectively. Positive going waves are typically characterised by [7]

$$|\lambda_j^+| <= 1; \quad Real[i\omega \mathbf{f}_L \mathbf{q}_L] < 0 \ \text{ if } \ |\lambda_j^+| = 1. \tag{21}$$

Note that if there are n degrees of freedom for node L and R, then there are $2n$ eigenvalues and the eigenvectors matrix Φ_q and Φ_f are $n \times 2n$.

5 FREE VIBRATION USING WFE DISPERSION RELATION AND WAVEMODES

Free vibration and dynamic response can be obtained using wave-based methods, in particular when the analysis focus on disturbance propagation [7]. In this section natural frequencies are predicted using wavemodes and dispersion curves obtained from the WFE eigenproblem (19). In the following it is assumed that nodal forces and displacements are described in terms of wave amplitudes $\mathbf{A}$ using wavemodes as a basis, that is

$$\mathbf{q}_L = \Phi_q\mathbf{A}; \quad \mathbf{f}_L = \Phi_f\mathbf{A}. \tag{22}$$

In practise displacements are often expanded onto a reduced basis and only $m < 2n$ pairs of waves - positive and negative going waves - are retained, so that Φ_q and Φ_f are $n \times m$ matrices and λ is an array of size m [7]. Although there is not a rigorous criterium, this reduced basis is supposed to include all the wavemodes which mostly contributes to the structure response. Typically all the propagating waves are retained together with the attenuating and evanescent waves which either decay not very rapidly with distance or are becoming propagating in the frequency band of interest.

5.1 Dynamic Stiffness Formulation

The dynamic stiffness matrix [10] of the beam is developed using wavemodes and dispersion curves, and subsequently used to investigate free vibration characteristics. If $N = L_{tot}/L$, where L_{tot} is the total length of the beam and N is typically an entire number, the total displacement at $A(x = 0)$ and $B(x = NL)$ can be expanded as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(NL) \end{bmatrix} = \begin{bmatrix} \Phi_q \\ \Phi_q\wedge^N \end{bmatrix}\mathbf{A} = \mathbf{D}_q\mathbf{A}. \tag{23}$$

In equation (23) the diagonal matrix $\wedge = diag(\lambda_1, \ldots, \lambda_m)$ contains the WFE eigenvalues, and the matrix $\mathbf{D}_q$ is a rectangular $2n \times m$ matrix. A similar expression is given for nodal forces

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}(0) \\ \mathbf{f}(NL) \end{bmatrix} = \begin{bmatrix} \Phi_f \\ \Phi_f\wedge^N \end{bmatrix}\mathbf{A} = \mathbf{D}_f\mathbf{A}. \tag{24}$$

Since wavemodes are assumed to be linearly independent, the matrix $\mathbf{D}_q$ is a full column rank matrix, i.e. $rank(\mathbf{D}_q) = m$, and the left pseudoinverse $\mathbf{D}_q^\dagger$ can be calculated. Hence the wave amplitude can be obtained from

$$\mathbf{A} = \mathbf{D}_q^\dagger\mathbf{Q}. \tag{25}$$

Substituting equation (25) in equation (24) yields

$$\mathbf{F} = \mathbf{D}_f \mathbf{D}_q^{\dagger} \mathbf{Q} = \overline{\mathbf{D}}\mathbf{Q}, \tag{26}$$

where $\overline{\mathbf{D}}$ is the dynamic stiffness matrix. Once the dynamic stiffness matrix is obtained, free vibration are calculated in the usual way as it is done in the FE method. In particular resonances occur when $|det(\overline{\mathbf{D}}')| = 0$, where $\overline{\mathbf{D}}'$ is obtained from $\overline{\mathbf{D}}$ after applying boundary conditions. A numerical procedure is used in the present work. Therefore the determinant is evaluated numerically for each frequency, and resonances occur when $|det(\overline{\mathbf{D}}')|$ shows stationary values. It is of worth to point out that the present formulation can also be used to evaluated the forced response of the structure.

5.2 Phase Closure Principle

Following section 3, consider a beam with some boundary conditions at the end $A(x = 0)$ and $B(x = L_{tot})$. Again wave amplitudes at each end are denoted by $\mathbf{A}^+$, $\mathbf{A}^-$, $\mathbf{B}^+$ and $\mathbf{B}^-$ and the boundary conditions have reflection matrices $\mathbf{R}_A$ and $\mathbf{R}_B$. According to [7], the boundary conditions can be written as $\mathbf{Eq} + \mathbf{Gf} = \mathbf{0}$. Complicated boundary conditions can be easily introduced using this formulation. Since displacements and forces can be expanded as $\mathbf{q} = \Phi_q^+\mathbf{A}^+ + \Phi_q^-\mathbf{A}^-$ and $\mathbf{f} = \Phi_f^+\mathbf{A}^+ + \Phi_f^-\mathbf{A}^-$, reflection matrices are

$$\begin{aligned} \mathbf{R}_B &= -(\mathbf{E}_B\Phi_f^- + \mathbf{G}_B\Phi_q^-)^{\dagger}(\mathbf{E}_B\Phi_f^+ + \mathbf{G}_B\Phi_q^+); \\ \mathbf{R}_A &= -(\mathbf{E}_A\Phi_f^+ + \mathbf{G}_A\Phi_q^+)^{\dagger}(\mathbf{E}_A\Phi_f^- + \mathbf{G}_A\Phi_q^-). \end{aligned} \tag{27}$$

Equation (12) can be rewritten as $\mathbf{B}^+ = [\wedge^+]^N \mathbf{A}^+$, $\mathbf{A}^- = [\wedge^-]^N \mathbf{B}^-$, $\mathbf{B}^- = \mathbf{R}_B\mathbf{B}^+$ and $\mathbf{A}^+ = \mathbf{R}_A\mathbf{A}^-$, which solved give

$$[\mathbf{R}_A \left[\wedge^-\right]^N \mathbf{R}_B \left[\wedge^+\right]^N - \mathbf{I}]\mathbf{A}^+ = \mathbf{0}, \tag{28}$$

where $[\wedge^+] = diag(\lambda_1^+, \ldots, \lambda_m^+)$ and $[\wedge^-] = diag(\lambda_1^-, \ldots, \lambda_m^-)$ correspond to positive-going and negative-going waves. Therefore natural frequencies occur when

$$|det(\mathbf{R}_A \left[\wedge^-\right]^N \mathbf{R}_B \left[\wedge^+\right]^N - \mathbf{I})| \tag{29}$$

shows stationary values.

6 NUMERICAL EXAMPLE

In this section a numerical example is presented and WFE results are compared with those obtained using the wave approach described in section 3. The beam considered is an aluminum beam with cross section $h = 0.002$m. The element length for the WFE formulation is $L = 0.001$m. Figure 3 shows the complex dispersion curves obtained from equations (9) and (19), where $\Omega = L^2\omega/a$ is the non-dimensional frequency. It can be noticed that results agree very well. Table 1–3 show natural frequencies of the beam predicted for different values of the axial velocity. The beam is simply supported, and its total length is $L_{tot} = 0.4$m. Results obtained from equations (14), (26), and (28) are compared. It can be noticed that equation (28) gives more accurate results with respect to equation (26). Hence the Phase Closure Principle seems to be preferable for WFE application. Accuracy of the numerical WFE results increases with frequency. This is perhaps due to the short length of the FE element. It can also be noticed that natural frequencies decrease when the axial velocity increases. Divergence instability in

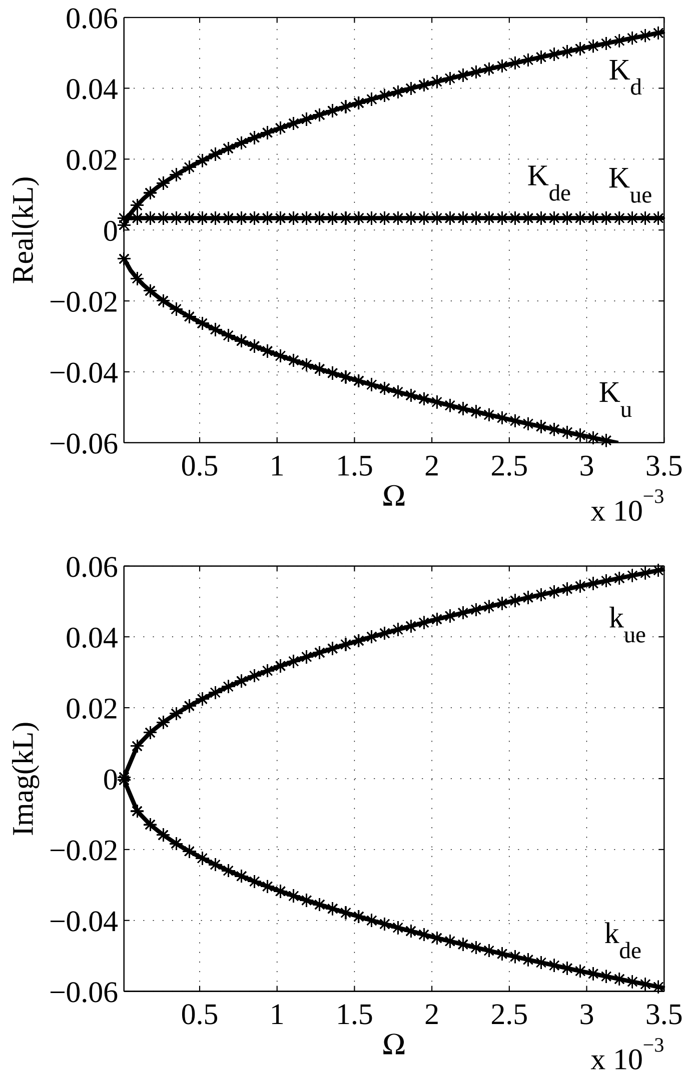

Figure 3: Complex valued dispersion curves, $v = 20$m/s. Analytical results from (9): ——- ; WFE results from (19): * * * *.

Natural frequency [Hz]						
Analytical solutions, Eq. (14)	28.9	116.8	263.3	468.4	732.1	1054.4
WFE Dynamic Stiffness Matrix Eq. (26)	28.5	116.6	263.2	468.3	732.0	1054.3
WFE Phase Closure Principle Eq. (28)	28.9	116.8	263.3	468.4	732.1	1054.4

Table 1: Simply supported beam, $v = 5$m/s.

Natural frequency [Hz]						
Analytical solutions, Eq. (14)	23.96	111.9	258.4	463.4	727.1	1049.4
WFE Dynamic Stiffness Matrix Eq. (26)	-	108.2	256.0	461.7	725.7	1048.3
WFE Phase Closure Principle Eq. (28)	-	112	258.4	463.4	727.1	1049.4

Table 2: Simply supported beam, $v = 20$m/s.

Natural frequency [Hz]						
Analytical solutions, Eq. (14)	17.3	105.2	251.7	456.7	720.5	1042.7
WFE Dynamic Stiffness Matrix Eq. (26)	-	96.0	246.2	452.8	717.3	1040.1
WFE Phase Closure Principle Eq. (28)	-	106.0	252.0	456.9	720.5	1042.8

Table 3: Simply supported beam, $v = 30$m/s.

fact occurs at a critical moving speed [11], where the linear theory breaks down. However, far from the critical speed, which for the present case is quite high, the method proposed offers a powerful alternative to analytical approaches, in particular for more complicated cases, such as pipe conveying fluid, which will be the subject of further analysis. Moreover, the linear theory can be the first step in the analsysis to provides useful information for a more refined nonlinear investigation.

It is of worth noting that, for a simply supported beam, a closed form approximate frequency neglecting evanescent waves in (14) is

$$\omega = \frac{n^2\pi^2}{L^2}a - \frac{v^2}{4a}. \tag{30}$$

Accuracy of equation (30) is related to the rate of decay of evanescent waves k_{de} and k_{ue}, thus it is expected that equation (30) is more accurate for higher frequency and low axial speed.

7 CONCLUSIONS

In the present paper the dynamics of a travelling one-dimensional waveguide was studied using a Wave Finite Element method. In particular an axially moving beam, modelled using the classical linear theory, was considered. The FE formulation of a moving beam element was developed, and a brief description of the WFE method was given. Natural frequencies were then calculated using the Dynamic Stiffness Matrix and the Phase Closure Principle, both formulated in terms of wave characteristics obtained from the WFE eigenproblem. An analytical wave approach was also presented to show some characteristics of the elastic waves propagating in such a beam. Numerical examples were shown. Results were compared with those obtained using the analytical approach, showing the accuracy of the proposed technique, in particular at higher frequency.

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