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ENHANCED MONTE CARLO FOR RELIABILITY-BASED DESIGN AND CALIBRATION

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Abstract. *This paper extends the recently developed enhanced Monte Carlo approach to the problem of reliability-based design. The objective is to optimize a design parameter(s) so that the system, represented by a set of failure modes or limit states, achieves a target reliability. In a large majority of design and/or calibration contexts, the design parameter α itself can be used to parameterize the system safety margin $M(\alpha)$. The lower tail of this random variable behaves in a regular way and is therefore amenable to straightforward parametric analysis. In contrast to the original Naess et al. method [1], the intention is to estimate the value α_T that corresponds to a (very) small target system failure probability p_{fT}. Monte Carlo sampling occurs at a range of values for α that result in larger failure probabilities, and so the design problem essentially amounts to a statistical estimation of a high quantile. Bounds for α_T can easily be constructed. Several examples of the approach are given in the paper.*

1 INTRODUCTION

A new Monte Carlo (MC) based method for estimating system reliability was recently developed in [1]. The aim of this method is to reduce computational cost while maintaining the advantages of crude MC simulation, specifically, its ease in dealing with complex systems. The key idea is to exploit the regularity of tail probabilities to enable an approximate prediction of far tail failure probabilities based on small Monte-Carlo sample results obtained for much more moderate levels of reliability. The motivation behind this approach is that systems with multiple and complex failure modes or limit states are often exceedingly difficult to analyze using traditional methods of structural reliability. While direct MC does not suffer from this problem, it is computationally burdensome for small probabilities. Hence originates the idea of sampling in a different less reliable range and performing a statistical extrapolation unto the tail. A similar but somewhat different idea is presented in [2].

The fundamentals of the method proposed in [1] are as follows. A safety margin $M = G(X_1, \ldots, X_n)$ expressed in terms of n basic variables, is extended to a parameterized class of safety margins using a scaling parameter λ ($0 \leq \lambda \leq 1$):

$$M(\lambda) = M - (1 - \lambda)\mu_M \,. \tag{1}$$

The failure probability is then assumed to behave as follows:

$$p_f(\lambda) = \text{Prob}\big(M(\lambda) \leq 0\big) \underset{\lambda \to 1}{\approx} q(\lambda) \exp\big\{ - a(\lambda - b)^c \big\} \,, \tag{2}$$

where the function $q(\lambda)$ is slowly varying compared with the exponential function $\exp\{-a(\lambda - b)^c\}$. It may be pointed out that the assumed behaviour of the failure probability applies to any safety margin for which FORM or SORM approximations can be used, but actually its range of applicability is much wider than that.

Clearly, the target failure probability $p_f = p_f(1)$ can be obtained from values of $p_f(\lambda)$ for $\lambda < 1$. It is now far easier to estimate the (larger) failure probabilities $p_f(\lambda)$ for $\lambda < 1$ accurately than the target value itself, since they are larger and hence require less simulations. Fitting the parametric function given by Eq. (2) for $p_f(\lambda)$ to the estimated values would then allow us to provide an estimate of the target value by extrapolation. The viability of this approach is demonstrated by both analytical and numerical examples in [1] and [3].

In the next sections, the Naess et al. [1] approach is extended to reliability-based design and calibration.

2 USING ENHANCED MONTE CARLO TO OPTIMIZE A DESIGN PARAMETER

First consider a typical component design, the reliability of which is governed by the safety margin:

$$M(\alpha) = G(X_1, \ldots, X_n; \alpha) \tag{3}$$

with

$$p_f(\alpha) = \text{Prob}(M(\alpha) \leq 0) \,, \tag{4}$$

where α acts as a design factor which "controls" the reliability of the component. The objective is now to determine the (assumed to be unique) value of $\alpha = \alpha_T$ that corresponds to a specified (target) component failure probability p_{fT}, i.e.:

$$\alpha_T : \ \text{Prob}(M(\alpha_T) \leq 0) = p_{fT} \,, \tag{5}$$

This assumes that the function $p_f(\alpha)$ is a monotonic function, that is, that the safety of the system either strictly increases or strictly decreases as the design factor α increases and approaches α_T. In practical design situations, α may represent a safety factor, a partial load or resistance factor, or some exceedance level and, the condition of monotonicity is generally speaking satisfied, unless the problem relates to a poor or an unfeasible design.

A more general situation, typical in the context of calibration of design specifications, consists of having the safety margin controlled by a design check function $c(\alpha) = c(x_{1c}, \ldots, x_{nc}; \alpha)$ involving characteristic values x_{ic} of each basic variable X_i. Admissible design are such that $c(\alpha) \leq 0$. Minimal acceptable designs are marked by $c(\alpha) = 0$, an assumption which is made throughout this paper. Often the design check function $c(\alpha)$ is selected to be the same mathematical function as G but this is not required - all that matters is that the resulting safety margin $M(\alpha) = G\big(X_1, \ldots, X_n | c(x_{1c}, \ldots, x_{nc}; \alpha) = 0\big)$ is monotonic with respect to α in its approach to the target α_T. Hence the objective is to determine α_T as follows:

$$\alpha_T : \text{ Prob}\Big(G\big(X_1, \ldots, X_n | c(x_{1c}, \ldots, x_{nc}; \alpha_T) = 0\big) \leq 0\Big) = p_{fT} \,. \tag{6}$$

Typically, p_{fT} is a very small target probability and hence the behavior of p_f as a function of of α is similar to a deep tail estimation problem so that it is reasonable to assume that:

$$p_f(\alpha) \underset{\alpha \to 1}{\approx} q(\alpha) \exp\big\{ - a(\alpha - b)^c \big\} \,, \tag{7}$$

where $q(\alpha)$ is slowly varying compared to the exponential expression.

To illustrate this premise, consider a basic load and resistance safety margin $M(\alpha) = R(\alpha) - S$ controlled by a design check function $c(\alpha) = (r_c(\alpha)/\alpha) - s_c$, where r_c and s_c are characteristic values of a resistance R and a load S, and α acts as a partial resistance factor ($\alpha > 1$). Assume the load S is Weibull distributed with exponent d and scale parameter s_0, then the characteristic load s_c at its $(1-\theta)$ quantile, is equal to $s_c = (-\ln \theta)^{1/d} s_0 = k s_0$ where k is a known positive constant > 1. First consider the limiting case where the variance of R is zero, $\sigma_R^2 = 0$, hence $r_c(\alpha) = \alpha s_c$ such that $p_f(\alpha) = \text{Prob}(M(\alpha) \leq 0) = \exp\big(- (\alpha k)^d \big)$ which is fully consistent with Eq. (7) above. If the variance of $R(\alpha)$ now increases, then the mean resistance will shift even further down the tail since $r_c(\alpha)$ is a small quantile of R. But, the function $p_f(\alpha)$ will only be slightly "contaminated" by a much slower varying function of α; however, and this is certainly valid in the tail area as $\alpha \to \alpha_T$, the general form in Eq. (7) will persist and it is amenable to be fitted to data pairs $(p_f(\alpha), \alpha)$ obtained for (much) higher failure probabilities.

Once a satisfactory fit is achieved, the target value α_T corresponding to p_{fT} needs to be estimated, a problem which is similar to a high quantile estimation.

3 EXTENSION TO SYSTEM RELIABILITY

Using Monte Carlo methods for system reliability analysis has several attractive features, the most important being that the failure criterion is relatively easy to check almost irrespective of the complexity of the system. In order to limit the amount of computational effort that may be involved, it is useful to extend the above approach to systems.

Let $M_j(\alpha) = G_j(X_1, \ldots, X_n, \alpha)$, $j = 1, \ldots, m$ be a set of m given safety margins expressed in terms of n basic variables and a single design parameter α. The series system reliability expressed in terms of the failure probability can then be written as,

$$p_f(\alpha) = \text{Prob}\big(\bigcup_{j=1}^{m} \{M_j(\alpha) \leq 0\} \big) \,, \tag{8}$$

while for the parallel system,

$$p_f(\alpha) = \text{Prob}\Big(\bigcap_{j=1}^{m}\{M_j(\alpha) \le 0\}\Big)\,. \tag{9}$$

In general, any system can be written as a series system of parallel subsystems. The failure probability would then be given as,

$$p_f(\alpha) = \text{Prob}\Big(\bigcup_{j=1}^{l}\bigcap_{i\in C_j}\{M_i(\alpha) \le 0\}\Big)\,, \tag{10}$$

Here each C_j is a subset of $1,\ldots,m$, for $j = 1,\ldots,l$. The C_js denote the index sets defining the parallel subsystems.

We then make the assumption that $p_f(\alpha)$ can also be represented as in Eq. (7) for the system reliability problems. Again, the objective is to determine the value α_T that achieves a stated overall system reliability.

4 IMPLEMENTATION

The method to be described in this section is based on the assumption expressed by Eq. (7). For practical applications it is implemented in the following form:

$$p_f(\alpha) \approx q(\alpha)\exp\big\{-a(\alpha-b)^c\big\}\,, \text{for } \alpha_0 \le \alpha \le \alpha_T\,, \tag{11}$$

for a suitable value of α_0. An important part of the method is therefore to identify a suitable range for α so that the right hand side of Eq. (7) represents a good approximation of $p_f(\alpha)$ for $\alpha \in [\alpha_0, \alpha_T]$.

For a sample of size N of the vector of basic random variables $\mathbf{X} = (X_1,\ldots,X_n)$, let $N_f(\alpha)$ denote the number of outcomes of the random vector in the failure domain of $M(\alpha)$. The estimate of the failure probability is then

$$\hat{p}_f(\alpha) = \frac{N_f(\alpha)}{N}\,. \tag{12}$$

The coefficient of variation C_v of this estimator is

$$C_v(\hat{p}_f(\alpha)) = \sqrt{\frac{1-\hat{p}_f(\alpha)}{\hat{p}_f(\alpha)N}}\,. \tag{13}$$

A fair approximation of the 95% confidence interval for the value $\hat{p}_f(\alpha)$ can be obtained as $\text{CI}_{0.95} = (C^-(\alpha),\, C^+(\alpha))$, where

$$C^{\pm}(\alpha) = \hat{p}_f(\alpha)(1 \pm 1.96 C_v(\hat{p}_f(\alpha))\,. \tag{14}$$

Assuming now that we have obtained empirical Monte Carlo estimates of the failure probability, the problem then becomes one of optimal use of the information available. By plotting $\log|\log\hat{p}_f(\alpha)/q(\alpha)|$ versus $\log(\alpha-b)$, it is expected that an almost perfectly linear tail behavior will be obtained according to Eq. (11). Recalling that the function $q(\alpha)$ was assumed to be slowly varying compared with the exponential function $\exp\{-a(\alpha-b)^c\}$ for values of α close to α_T, it is now tentatively proposed to replace $q(\alpha)$ by a suitable constant value, q say, for tail

values of α, say $\alpha > \alpha_1 (\geq \alpha_0)$. Hence, we will investigate the viability of the following simpler version of Eq. (11):

$$p_f(\alpha) \approx q \exp\left\{ -a(\alpha - b)^c \right\}, \text{for } \alpha_1 \leq \alpha \leq \alpha_T, \tag{15}$$

for a suitable choice of α_1.

The problem of finding the optimal values of the parameters a, b, c, q is carried out by optimizing the fit on the log level by minimizing the following mean square error function with respect to all four arguments [4],

$$F(a, b, c, q) = \sum_{j=1}^{M} w_j \left(\log \hat{p}_f(\alpha_j) - \log q + a(\alpha_j - b)^c \right)^2, \tag{16}$$

where $\alpha_1 < \ldots < \alpha_M$ denotes the set of α values where the failure probability is empirically estimated. The w_j denote weight factors that put more emphasis on the more reliable data points, alleviating the heteroscedasticity of the estimation problem at hand. The choice of weight factor is to some extent arbitrary. In this paper, we use $w_j = \left(\log C^+(\alpha_j) - \log C^-(\alpha_j) \right)^{-\theta}$ with the values θ = 1 and 2, combined with a Levenberg-Marquardt least squares optimization method [5]. Note that the form of w_j puts some restriction on the use of the data. Usually, there is a level α_j beyond which w_j is no longer defined. Hence, the summation in Eq. (16) has to stop before that happens. Also, the data should be preconditioned by establishing the tail marker α_1 in a sensible way.

Although the Levenberg-Marquardt method as described above generally works well, it may be simplified by exploiting the structure of F. It is realized by scrutinizing Eq. (16) that if b and c are fixed, the optimization problem reduces to a standard weighted linear regression problem. That is, with both b and c fixed, the optimal values of a and $\log q$ are found using closed form weighted linear regression formulas in terms of w_j, $y_j = \log \hat{p}_f(\alpha_j)$ and $x_j = (\alpha_j - b)^c$.

It is obtained that the optimal values of a and q are given by the relations,

$$a^*(b, c) = -\frac{\sum_{j=1}^{M} w_j (x_j - \overline{x})(y_j - \overline{y})}{\sum_{j=1}^{M} w_j (x_j - \overline{x})^2}, \tag{17}$$

and

$$\log q^*(b, c) = \overline{y} + a^*(b, c)\overline{x}, \tag{18}$$

where $\overline{x} = \sum_{j=1}^{M} w_j x_j / \sum_{j=1}^{M} w_j$, with a similar definition of $\overline{y}$.

The Levenberg-Marquardt method may now be used on the function $\tilde{F}(b, c) = F(a^*(b, c), b, c, q^*(b, c))$ to find the optimal values b^* and c^*, and then the corresponding a^* and q^* can be calculated from Eq. (17) and (18).

For estimation of the confidence interval for the predicted target quantile α_T provided by the optimal curve, the empirical confidence band is reanchored to the optimal curve. The range of fitted curves that stay within the reanchored confidence band will determine an optimized confidence interval of the predicted value.

5 NUMERICAL EXAMPLES

The examples in the following two sections all have simple explicit limit state functions in terms of the basic random variables. The computational issue is therefore minor and no effort has been made to investigate the possibility of implementing more effective sampling strategies.

If the proposed method were to be used in combination with computationally demanding procedures involving e.g. a FE method for calculating the sample, it would be necessary in general to use more effective sampling strategies than the brute force procedure used here.

5.1 Component load factor calibration

In this first example, the 10-bar truss structure shown in Fig. 1 is studied. An enhanced Monte-Carlo reliability analysis of this truss is given in [1]. Here a load factor for a transversal load P is calibrated in order to achieve a target reliability of (10^{-6}) with respect to the horizontal sway of the truss. The ten truss members are cut from three different aluminum rods with cross-sectional areas A_1, A_2 and A_3, as shown in Fig. 1. The structure is subjected to external loads P as shown in Fig. 1. The horizontal displacement D at the upper right hand corner of the truss structure can be written as [6]:

$$\begin{aligned} D = & \frac{BPL}{A_1 A_3 E}\Big\{\frac{4\sqrt{2}A_1^3(24A_2^2+A_3^2)+A_3^3(7A_1^2+26A_2^2)}{D_T} \\ & + 4A_1A_2A_3\frac{20A_1^2+76A_1A_2+10A_3^2}{D_T} \\ & + 4\sqrt{2}A_1A_2A_3^2\frac{25A_1+29A_2}{D_T}\Big\} \end{aligned} \tag{19}$$

where $D_T = 4A_2^2(8A_1^2+A_3^2)+4\sqrt{2}A_1A_2A_3(3A_1+4A_2)+A_1A_3^2(A_1+6A_2)$ and E is Young's modulus. The random variable B accounts for model uncertainties. It is assumed that A_1, A_2, A_3, B, P, E are independent basic random variables. Their properties are summarized in Table 1. Also shown are the characteristic values used in the design check Eq. (21).

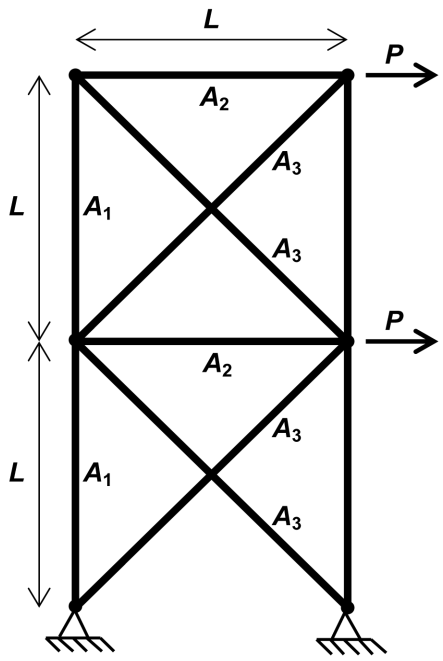

Figure 1: Ten-bar truss structure

The safety margin

$$M(\alpha) = d_0 - D\big(A_1, A_2, A_3, B, E, P(\alpha)\big)\,, \tag{20}$$

	Mean value	Coef. of var.	Prob. distr.	Char. value in (20)
A_1	10^{-2} m^2	0.05	Normal	1% quantile
A_2	$1.5 \cdot 10^{-3}$ m^2	0.05	Normal	1% quantile
A_3	$6.0 \cdot 10^{-3}$ m^2	0.05	Normal	1% quantile
B	1.0	0.10	Normal	mean
E	$6.9 \cdot 10^4$ MPa	0.05	Lognormal	1% quantile
P	based on Eq. (21)	0.10	Gumbel	95% quantile
d_0	0.1 m	-	-	-
L	9.0 m	-	-	-

Table 1: Basic variables.

and the design check constraint is

$$c(\alpha) = d_0 - D\big(A_{1c}, A_{2c}, A_{3c}, B_c, E_c, \alpha P_c(\alpha)\big)\,, \tag{21}$$

where α represents the transversal load factor.

Figs. 2 and 3 show the optimized fitted parametric curve to the empirical data in a log plot for sample size 10^5 and for weighted regression coefficients θ = 2 and θ = 1, respectively. The difference between the two tail extrapolations is minimal. Applying the proposed procedure with a sample of size 10^5 gives the estimated value for α_T with the 95% confidence interval shown in Table 2 for both θ = 2 and 1. Note that a crude Monte Carlo simulation verification of $(\alpha_T = 1.46, p_{fT} = 10^{-6})$ using $3 \cdot 10^9$ samples to within 2.5% at 95% confidence requires a computation time of about 24h on a laptop computer. The CPU time for the results shown in Table 2 was only about 40 seconds on a standard laptop.

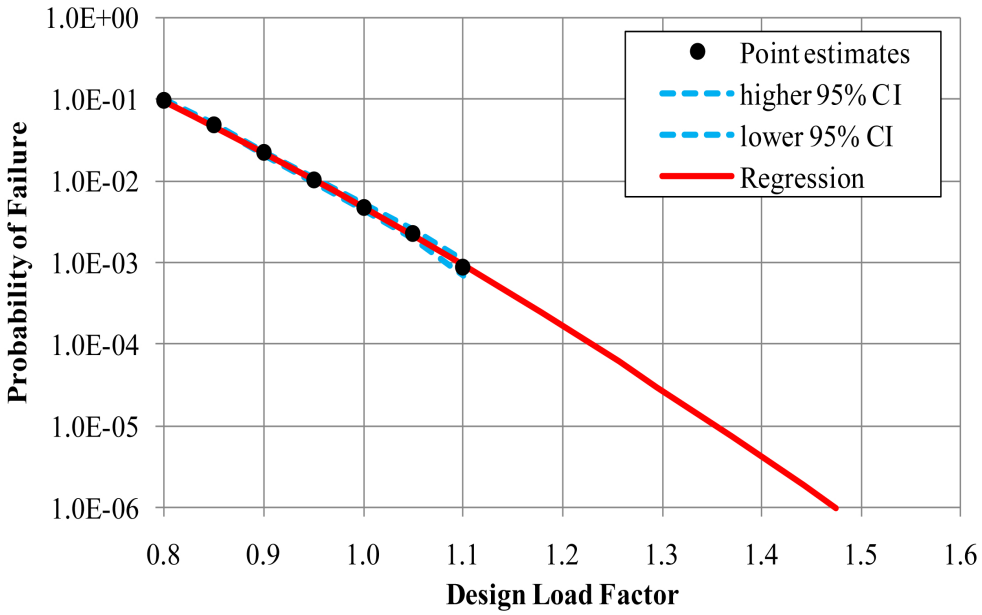

Figure 2: Sample size 10^5 — weighted regression θ = 2.

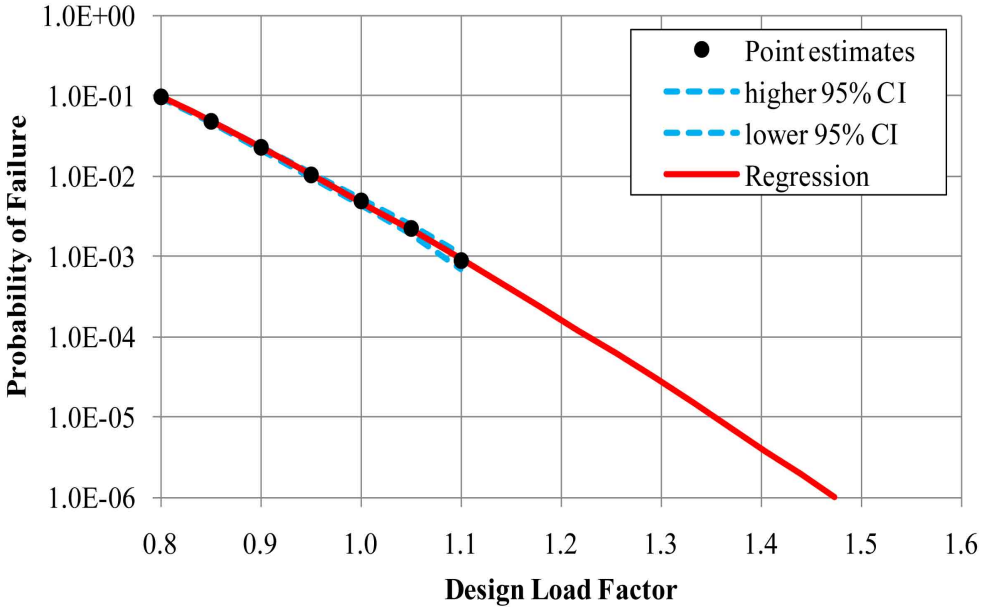

Figure 3: Sample size 10^5 — weighted regression θ = 1.

	$\theta = 2$	$\theta = 1$
higher 95% CI	1.48	1.48
α_T	1.47	1.47
lower 95% CI	1.46	1.46

Table 2: Optimal load factor α_T corresponding to $p_{fT} = 10^{-6}$ using sample size 10^5.

5.2 Design resistance safety factor in a series system

This example concerns the maximum internal forces in the members of a statically determinate 13-member truss structure subjected to external loading. The structure is shown in Fig. 4, which also displays the numbering of the truss elements from 1 to 13. The external loads P_1, P_2, P_3 which are acting on the structure as shown in Fig. 4, are modelled as independent Gaussian variables. The capacity for axial stress of truss element number j is expressed as $R_j = \sigma_{yj} A_j$ where σ_{yj} = the yield stress (MPa) and A_j = the cross-sectional area of this element (cm^2), and α is a resistance safety factor > 1 used as a division factor in the design check equation below. It is assumed that $A_1 = A_7 = 18.7$, $A_2 = A_8 = 13.1$, $A_3 = A_9 = A_{12} = A_{13} = 11.7$, $A_4 = A_{10} = 11.3$, $A_5 = A_{11} = 3.3$, $A_6 = 8.0$. The 13 yield stresses are assumed to be independent Gaussian variables. The 16 basic random variables in this problem are listed in Table 3.

	Mean Value	Coef. of Var.	Prob. distr.	Char. value in Eq. (22)
$P_j\ ,\ j = 1, 2, 3$	89 kN	0.15	Normal	99% quantile
$\sigma_{yj}\ ,\ j = 1, \ldots, 13$	based on Eq. (23)	0.15	Normal	5% quantile
L	2.54 m	-	-	-

Table 3: The 16 basic variables.

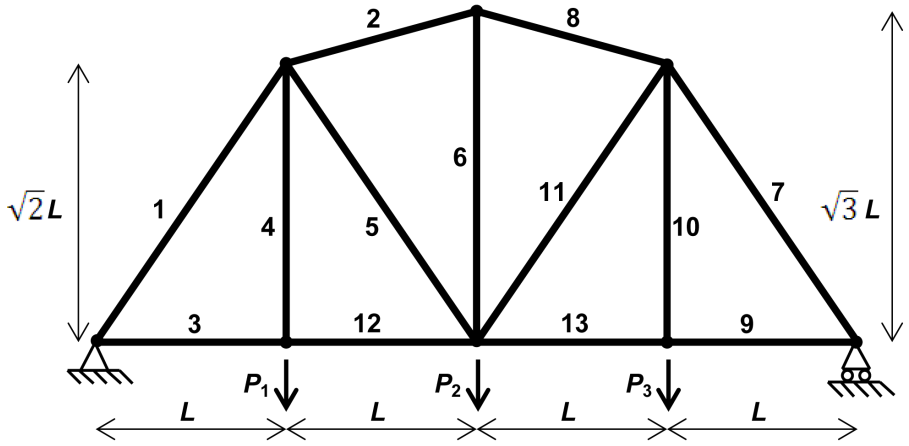

Figure 4: Truss bridge example.

$$
\begin{aligned}
M_1 &= R_1 - 0.9186P_1 - 0.6124P_2 - 0.3062P_3 \\
M_2 &= R_2 - 0.3029P_1 - 0.6058P_2 - 0.3029P_3 \\
M_3 &= R_3 - 0.5303P_1 - 0.3535P_2 - 0.1768P_3 \\
M_4 &= R_4 - P_1 \\
M_5 &= R_5 + 0.4186P_1 - 0.3876P_2 - 0.1938P_3 \\
M_6 &= R_6 - 0.1835P_1 - 0.3670P_2 - 0.1835P_3 \\
M_7 &= R_7 - 0.3062P_1 - 0.6124P_2 - 0.9186P_3 \\
M_8 &= R_8 - 0.3029P_1 - 0.6058P_2 - 0.3029P_3 \\
M_9 &= R_9 - 0.1768P_1 - 0.3535P_2 - 0.5303P_3 \\
M_{10} &= R_{10} - P_1 \\
M_{11} &= R_{11} - 0.1938P_1 - 0.3876P_2 + 0.4186P_3 \\
M_{12} &= R_{12} - 0.5303P_1 - 0.3536P_2 - 0.1768P_3 \\
M_{13} &= R_{13} - 0.1768P_1 - 0.3536P_2 - 0.5303P_3
\end{aligned}
\tag{22}
$$

The 13 design check equations have the same mathematical set of 13 equations except that the deterministic characteristic values of Table 3 are used and a resistance safety factor is involved. The most severe constraint is the compressive stress in members 1 and 7 which therefore governs the design of the system as a whole:

$$
\frac{\sigma_{yc}(\alpha)A_1}{\alpha} - 1.8372P_c = 0\,, \tag{23}
$$

The objective is to find the value α_T such that the series system failure probability given by Eq. (8) is equal to a target $p_{fT} = 10^{-5}$. The log plot of $p_f(\alpha)$ versus α is shown in Figures 5 and 6 for θ = 1 and for samples of size 10^4 and 10^5, respectively. The estimated α_T corresponding to $p_f = 10^{-5}$ together with their CIs are shown in Table 4. A Winbugs script runs the entire analysis in under 1min for 10^5 samples. As a contrast, crude Monte Carlo simulation with $5 \cdot 10^9$ samples confirms $(\alpha_T = 1.89, p_f = 10^{-5})$ for the series system accurate to within about 0.5% with 95% confidence, but requires a computation time of about 24h on a laptop computer.

	$N = 10^4$	$N = 10^5$
higher 95% CI	1.95	1.92
α_T	1.85	1.88
lower 95% CI	1.69	1.80

Table 4: Optimal resistance safety factor α_T corresponding to a system $p_{fT} = 10^{-5}$ using sample size 10^4 and 10^5 with $\theta = 1$.

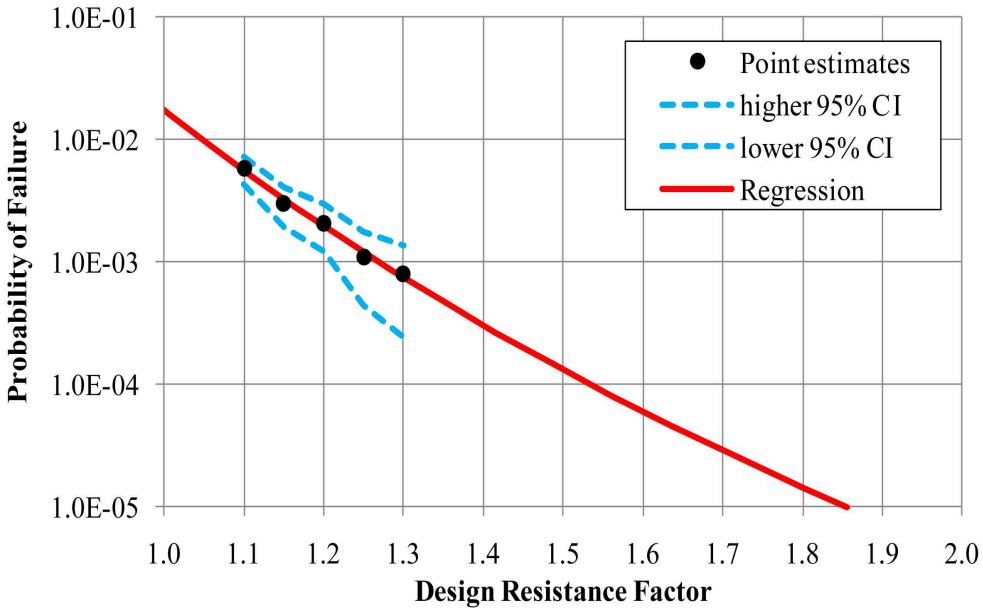

Figure 5: Sample size 10^4 — weighted regression with $\theta = 1$.

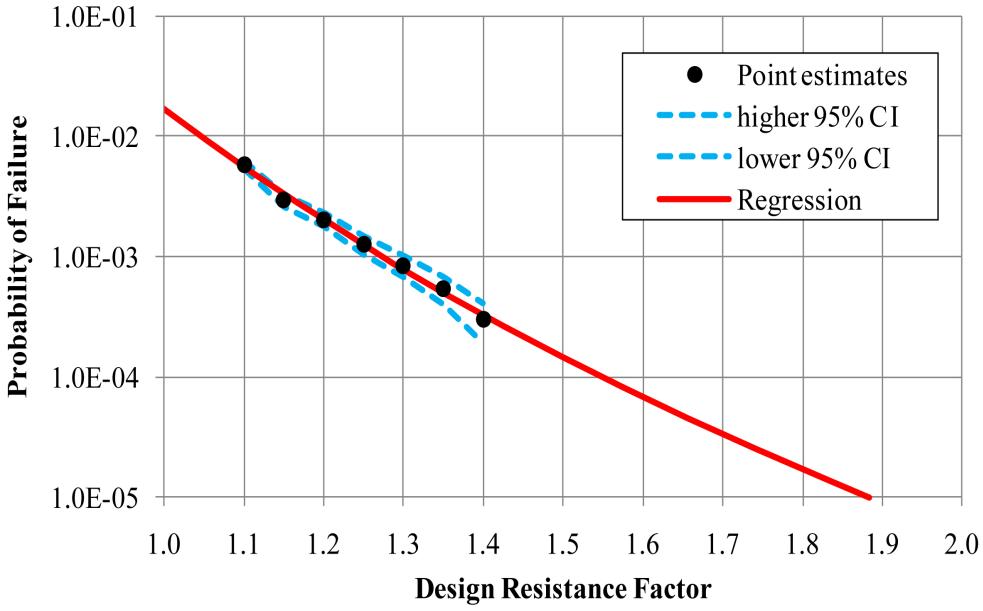

Figure 6: Sample size 10^5 — weighted regression with $\theta = 1$.

6 CONCLUSIONS

In this paper, we have described a Monte Carlo based method for a reliability-based calibration of design parameters such as load/resistance factors, safety factors or specification levels of structural systems. It has been shown that the method may provide good estimates of design factors for structural systems with a moderate computational effort. It has been pointed out that the use of Monte Carlo methods for system reliability analysis has several very attractive features, the most important being that the failure criterion is usually relatively easy to check almost irrespective of the complexity of the system and the number of basic random variables.

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