COMPDYN 2011
ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis, V. Plevris (eds.)
Corfu, Greece, 25-28 May 2011

CO-ROTATIONAL DYNAMIC FORMULATION FOR 2D BEAMS

Thanh-Nam Le[1,2], Jean-Marc Battini[2], Mohammed Hjiaj[1]

[1]Structural Engineering Research Group /LGCGM , INSA de Rennes, 20 Avenue des Buttes de Coësmes, 35708 Rennes Cedex 7, France
e-mail: Thanh-Nam.Le@insa-rennes.fr, mohammed.hjiaj@insa-rennes.fr

[2]Department of Civil and Architectural Engineering, KTH, Royal Institute of Technology, SE-10044 Stockholm, Sweden
email: Jean-Marc.Battini@byv.kth.se

Keywords: Corotational method, Nonlinear dynamic analysis, 2D beam element.

Abstract. *The corotational method is an attractive approach to derive non-linear finite beam elements. In a number of papers, this method was employed to investigate the non-linear dynamic analysis of 2D beams. However, most of the approaches found in the literature adopted either a lumped mass matrix or linear local interpolations to derive the inertia terms (which gives the classical linear and constant Timoshenko mass matrix), although local cubic interpolations were used to derive the elastic force vector and the tangent stiffness matrix. In this paper, a new corotational formulation for dynamic nonlinear analysis is presented. Cubic interpolations are used to derive both the inertia and elastic terms. Numerical examples show that the proposed approach is more efficient than using lumped or Timoshenko mass matrices.*

1 Introduction

The corotational approach is an attractive method to derive efficient nonlinear finite beam elements [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The main idea of the method can be summarized as follow: the motion of the beam element is decomposed into rigid body and pure deformational parts. A local coordinates system, which moves and rotates with the element's overall rigid body motion, is defined. The deformational part is measured in this local system.

Many different assumptions to represent the local deformations, giving different possibilities for the local element formulation, can be found in the literature. If linear interpolations are used for the local formulation, inertia corotational terms are easily derived and the classical linear and constant Timoshenko mass matrix is obtained. However, this assumption has the drawback that the local vertical displacements are zero along the element, which is not accurate, especially for flexible beams. If cubic interpolations are used for the local formulation, then the derivation of the inertia terms becomes very complicated. To avoid this complexity, Crisfield and al.[2] used the constant Timoshenko mass matrix, although they used local cubic interpolations to derive the elastic force vector and tangent stiffness matrix. The same approach was adopted in [6]. In [7, 8, 9], the authors used a constant lumped mass matrix without any attempt to check its accuracy. In [10], Behdinan and al. proposed a corotational dynamic formulation. However, the cubic shape functions were used to describe the global displacements, which is not consistent with the idea of the corotational method.

In this paper, a new corotational formulation is presented where cubic shape functions are adopted. In order to consider the bending shear deformations, the cubic shape functions of the Interpolation Interdependent Element (IIE) [12] are used to derive the local elastic force vector and local tangent stiffness matrix. It is shown that with some simplifications, the inertia terms can be derived. The new formulation provide accurate results with a minimum number of elements.

The paper is organized as follows: in Section 2 and 3 the corotational kinematic of a 2D beam element and the derivation of the elastic force vector and tangent stiffness matrix are presented. More details about that can be found in [13]. Section 4 and 5 are devoted to the derivation of the inertia terms. In Section 6, two examples are presented in order to assess the accuracy of the present dynamic formulation. Finally conclusions are given in Section 7.

2 Beam kinematics

The notations used are defined in Fig. 1. The coordinates for the nodes 1 and 2 in the global coordinate system (x,z) are (x_1, z_1) and (x_2, z_2). The vector of global displacements is defined by

$$\mathbf{q} = \begin{bmatrix} u_1 & w_1 & \theta_1 & u_2 & w_2 & \theta_2 \end{bmatrix}^{\mathrm{T}} \tag{1}$$

The vector of local displacements is defined by

$$\overline{\mathbf{q}} = \begin{bmatrix} \overline{u} & \overline{\theta}_1 & \overline{\theta}_2 \end{bmatrix}^{\mathrm{T}} \tag{2}$$

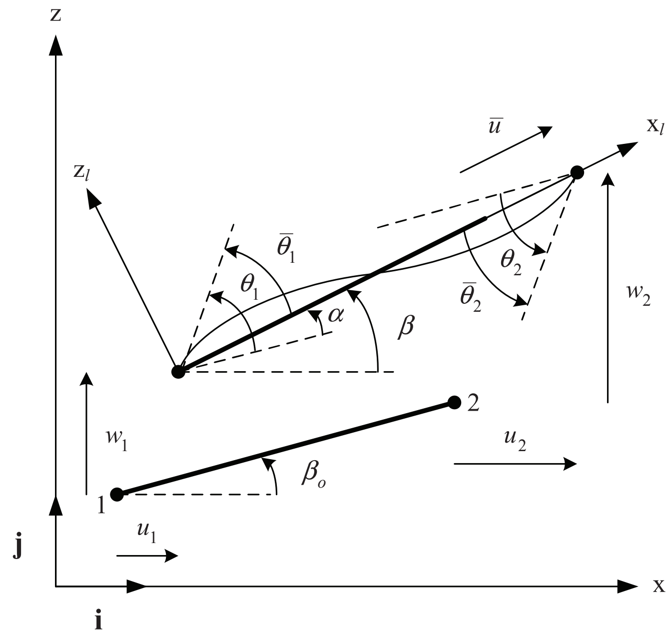

Figure 1: Beam kinematics 1.

The components of $\overline{\mathbf{q}}$ can be computed according to

$$\overline{u} = l_n - l_o \tag{3}$$

$$\overline{\theta}_1 = \theta_1 - \alpha = \theta_1 - \beta - \beta_o \tag{4}$$

$$\overline{\theta}_2 = \theta_2 - \alpha = \theta_2 - \beta - \beta_o \tag{5}$$

In (3), l_o and l_n denote the initial and current lengths of the element, respectively. The current angle of the local system with respect to the global system is denoted as β and is given by

$$c = \cos\beta = \frac{1}{l_n}(x_2 + u_2 - x_1 - u_1) \tag{6}$$

$$s = \sin\beta = \frac{1}{l_n}(z_2 + w_2 - z_1 - w_1) \tag{7}$$

The differentiation of equation (7) gives

$$\delta\beta = \frac{1}{l_n}\begin{bmatrix} s & -c & 0 & -s & c & 0 \end{bmatrix}\delta\mathbf{q} \tag{8}$$

The differentiation of equations (3) to (5) gives

$$\delta\overline{\mathbf{q}} = \mathbf{B}\,\delta\mathbf{q} \tag{9}$$

with

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \\ -s/l_n & c/l_n & 1 & s/l_n & -c/l_n & 0 \\ -s/l_n & c/l_n & 0 & s/l_n & -c/l_n & 1 \end{bmatrix} \tag{10}$$

3 Elastic force vector and tangent stiffness matrix

By equating the virtual work in the local and global systems, the relation between the local elastic force vector $\mathbf{f}_l$ and the global one $\mathbf{f}_g$ is obtained as

$$V = \delta\mathbf{q}^{\mathrm{T}}\,\mathbf{f}_g = \delta\overline{\mathbf{q}}^{\mathrm{T}}\,\mathbf{f}_l = \delta\mathbf{q}^{\mathrm{T}}\,\mathbf{B}^{\mathrm{T}}\,\mathbf{f}_l \tag{11}$$

The equation (11) must apply for any arbitrary $\delta\mathbf{q}$. Hence the global elastic force vector $\mathbf{f}_g$ is given by

$$\mathbf{f}_g = \mathbf{B}^{\mathrm{T}}\,\mathbf{f}_l \qquad \mathbf{f}_l = \begin{bmatrix} N & M_1 & M_2 \end{bmatrix}^{\mathrm{T}} \tag{12}$$

The global tangent stiffness matrix is defined by

$$\delta\mathbf{f}_g = \mathbf{K}_g\,\delta\mathbf{q} \tag{13}$$

By taking the differentiation of (12), it is obtained

$$\mathbf{K}_g = \mathbf{B}^{\mathrm{T}}\,\mathbf{K}_l\,\mathbf{B} + \frac{\mathbf{z}\,\mathbf{z}^{\mathrm{T}}}{l_n}N + \frac{1}{l_n^2}(\mathbf{r}\,\mathbf{z}^{\mathrm{T}} + \mathbf{z}\,\mathbf{r}^{\mathrm{T}})(M_1 + M_2) \tag{14}$$

where

$$\mathbf{r} = \begin{bmatrix} -c & -s & 0 & c & s & 0 \end{bmatrix}^{\mathrm{T}} \tag{15}$$

$$\mathbf{z} = \begin{bmatrix} s & -c & 0 & -s & c & 0 \end{bmatrix}^{\mathrm{T}} \tag{16}$$

The local elastic force vector $\mathbf{f}_l$ and local tangent stiffness matrix $\mathbf{K}_l$, which is defined by $\delta\mathbf{f}_l = \mathbf{K}_l\,\delta\overline{\mathbf{q}}$, depend on the definition of the local formulation. In this work, the shape functions of the IIE (Interdependent Interpolation Element) are used together with a shallow arch beam theory. The shallow arch longitudinal and shear strains are given by

$$\epsilon = \frac{1}{l_o}\int_{l_o}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\mathrm{d}x - \frac{\partial^2 w}{\partial x^2}\,z \tag{17}$$

$$\gamma = \frac{\partial w}{\partial x} - \vartheta \tag{18}$$

Using IIE's shape functions taken from [12], the axial displacement u, the vertical displacement w and the local rotation ϑ are given by

$$u = \frac{x}{l_o}\,\overline{u} \tag{19}$$

$$w = \varphi_1\overline{\theta}_1 + \varphi_2\overline{\theta}_2 \tag{20}$$

$$\vartheta = \varphi_3\overline{\theta}_1 + \varphi_4\overline{\theta}_2 \tag{21}$$

where

$$\varphi_1 = \mu\, x \left[6\,\Omega \left(1 - \frac{x}{l_o}\right) + \left(1 - \frac{x}{l_o}\right)^2 \right] \tag{22}$$

$$\varphi_2 = \mu\, x \left[6\,\Omega \left(\frac{x}{l_o} - 1\right) - \frac{x}{l_o} + \frac{x^2}{l_o^2} \right] \tag{23}$$

$$\varphi_3 = \mu \left(1 + 12\,\Omega - \frac{12\,\Omega\, x}{l_o} - \frac{4\,x}{l_o} + \frac{3\,x^2}{l_o^2} \right) \tag{24}$$

$$\varphi_4 = \mu \left(\frac{12\,\Omega\, x}{l_o} - \frac{2\,x}{l_o} + \frac{3\,x^2}{l_o^2} \right) \tag{25}$$

$$\Omega = \frac{E\, I}{G\, A\, K_s\, l_o} \tag{26}$$

$$\mu = \frac{1}{1 + 12\Omega} \tag{27}$$

$A,\ I$: Section's area and inertia moment

K_s : Shear correction coefficient.

For a rectangular cross-section, $K_s = \frac{5}{6}$

With $\Omega = 0$, the hermitian shape functions of the classical Bernoulli elements are obtained. The interest of IIE formulation is to keep the accuracy inherent to the cubic interpolation and with Ω to add the bending shear deformation.

4 Inertia force vector and mass matrix

The inertia force vector is calculated from the kinetic energy by using the Lagrange's equation of motion:

$$\mathbf{f}_K = \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\partial K}{\partial \dot{\mathbf{q}}} \right] - \left[\frac{\partial K}{\partial \mathbf{q}} \right] \tag{28}$$

The kinetic energy K of an element is given as

$$K = \frac{1}{2} \rho \Big\{ \int_{l_o} A(\dot{u}_{\mathrm{G}}^2 + \dot{w}_{\mathrm{G}}^2) \mathrm{d}l + \int_{l_o} I\, \dot{\theta}^2 \mathrm{d}l \Big\} \tag{29}$$

where

ρ : Mass density

u_{G} , w_{G}: Global displacements of the centroid of the cross-section

θ : Global rotation of the cross-section

The global position of the centroid of the cross-section is given by (see Fig. 2)

$$\mathrm{OG} = (x_1 + u_1)\, \mathbf{i} + (z_1 + w_1)\, \mathbf{j} + \frac{l_n}{l_o} x\, \mathbf{a} + w\, \mathbf{b} \tag{30}$$

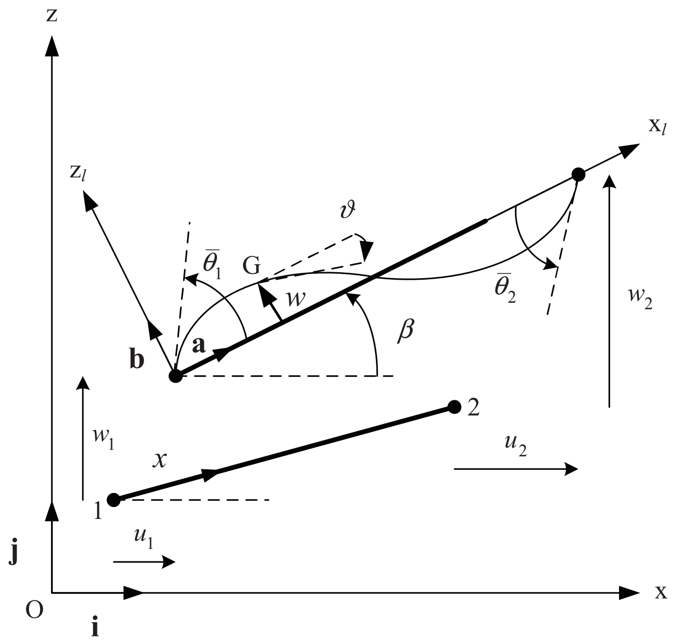

Figure 2: Beam kinematics 2.

with

$$\mathbf{a} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j} \tag{31}$$

$$\mathbf{b} = -\sin\beta\,\mathbf{i} + \cos\beta\,\mathbf{j} \tag{32}$$

After some algebraic manipulations, the velocities components can be derived

$$\dot{u}_{\mathrm{G}} = \dot{u}_1 + \frac{x}{l_o}(\dot{u}_2 - \dot{u}_1) - \dot{w}\sin\beta - w\dot{\beta}\cos\beta \tag{33}$$

$$\dot{w}_{\mathrm{G}} = \dot{w}_1 + \frac{x}{l_o}(\dot{u}_2 - \dot{u}_1) + \dot{w}\cos\beta - w\dot{\beta}\sin\beta \tag{34}$$

The global rotation of the cross section is given by

$$\dot{\theta} = \dot{\vartheta} + \dot{\alpha} = \dot{\vartheta} + \dot{\beta} \tag{35}$$

For the dynamic formulation, Ω is taken to 0. It is worth mentioning that assumption simplify the computations. Furthermore, extensive numerical studies performed by the authors have shown that this simplification does not modify the numerical results. The exact expression of the kinetic energy K can be obtained by substituting (33),(34) and (35) into (29), and by using (8) to calculate $\dot{\beta}$. K can be written as

$$K = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\,\mathbf{M}\,\dot{\mathbf{q}} \tag{36}$$

The local mass matrix $\mathbf{M}_l$ is defined by

$$\mathbf{M} = \mathbf{T}^{\mathrm{T}}\,\mathbf{M}_l\,\mathbf{T} \tag{37}$$

where $\mathbf{T}$ is rotation matrix.
Consequently, one obtains

$$K = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\,\mathbf{T}^{\mathrm{T}}\,\mathbf{M}_l\,\mathbf{T}\,\dot{\mathbf{q}} \tag{38}$$

At this point, two simplifications are introduced in the expression of the local mass matrix: the local displacement w is assumed small and therefore the terms containing w^2 are neglected; due to the assumption of the small deformation, the approximation $l_n = l_o$ is taken. With these simplifications, the local mass matrix is only function of $\overline{\theta}_1$ and $\overline{\theta}_2$ and is given by

$$\mathbf{M}_l = \mathbf{M}_{l\,1} + \mathbf{M}_{l\,2} \tag{39}$$

where $\mathbf{M}_{l\,1}$ is the mass matrix for local axial and vertical displacements, defined as

$$\mathbf{M}_{l\,1} = \frac{\rho\, A\, l_o}{420} \begin{bmatrix} 140 & m_1 & 0 & 70 & -m_1 & 0 \\ m_1 & 156 & 22l_o & m_2 & 54 & -13l_o \\ 0 & 22l_o & 4l_o^2 & 0 & 13l_o & -3l_o^2 \\ 70 & m_2 & 0 & 140 & -m_2 & 0 \\ -m_1 & 54 & 13l_o & m_2 & 156 & -22l_o \\ 0 & -13l_o & -3l_o^2 & 0 & 22l_o & 4l_o^2 \end{bmatrix}$$

with

$$m_1 = (21\overline{\theta}_1 - 14\overline{\theta}_2)$$
$$m_2 = (14\overline{\theta}_1 - 21\overline{\theta}_2)$$

$\mathbf{M}_{l\,2}$ is the mass matrix for rotation, defined as

$$\mathbf{M}_{l\,2} = \frac{\rho\, I}{30\, l_o} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3l_o & 0 & -36 & 3l_o \\ 0 & 3l_o & 4l_o^2 & 0 & -3l_o & -l_o^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3l_o & 0 & 36 & -3l_o \\ 0 & 3l_o & -l_o^2 & 0 & -3l_o & 4l_o^2 \end{bmatrix}$$

The differentiations of the kinetic energy can be computed as

$$\frac{\partial K}{\partial \dot{\mathbf{q}}} = \mathbf{M}\, \dot{\mathbf{q}} \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial K}{\partial \dot{\mathbf{q}}}\right] = \mathbf{M}\, \ddot{\mathbf{q}} + \dot{\mathbf{M}}\, \dot{\mathbf{q}} \tag{41}$$

$\mathbf{M}$ is function of $\beta, \overline{\theta}_1, \overline{\theta}_2$ which are dependent on the time:

$$\dot{\mathbf{M}} = \frac{\partial \mathbf{M}}{\partial \beta}\, \dot{\beta} + \frac{\partial \mathbf{M}}{\partial \overline{\theta}_1}\, \dot{\overline{\theta}}_1 + \frac{\partial \mathbf{M}}{\partial \overline{\theta}_2}\, \dot{\overline{\theta}}_2 \tag{42}$$

Using the notation $\dfrac{\partial \mathbf{M}}{\partial \beta} = \mathbf{M}_\beta; \dfrac{\partial \mathbf{M}}{\partial \overline{\theta}_1} = \mathbf{M}_{\overline{\theta}_1}; \dfrac{\partial \mathbf{M}}{\partial \overline{\theta}_2} = \mathbf{M}_{\overline{\theta}_2}$, the above equation can be rewritten in a more compact form

$$\dot{\mathbf{M}} = \mathbf{M}_\beta \Big(\frac{\mathbf{z}^{\mathrm{T}}}{l_n} \dot{\mathbf{q}}\Big) + \mathbf{M}_{\overline{\theta}_1} (\mathbf{b}_2^{\mathrm{T}} \dot{\mathbf{q}}) + \mathbf{M}_{\overline{\theta}_2} (\mathbf{b}_3^{\mathrm{T}} \dot{\mathbf{q}}) \tag{43}$$

The differentiation of K with respect to $\mathbf{q}$ is given by

$$\begin{aligned}\left\lfloor \frac{\partial K}{\partial \mathbf{q}} \right\rfloor &= \frac{\partial K}{\partial \beta}\,\frac{\partial \beta}{\partial \mathbf{q}} + \frac{\partial K}{\partial \bar{\theta}_1}\,\frac{\partial \bar{\theta}_1}{\partial \mathbf{q}} + \frac{\partial K}{\partial \bar{\theta}_2}\,\frac{\partial \bar{\theta}_2}{\partial \mathbf{q}} \\ &= \Big(\frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}_{\beta}\dot{\mathbf{q}}\Big)\frac{\mathbf{z}}{l_n} + \Big(\frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}_{\bar{\theta}_1}\dot{\mathbf{q}}\Big)\mathbf{b}_2 + \Big(\frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}_{\bar{\theta}_2}\dot{\mathbf{q}}\Big)\mathbf{b}_3 \end{aligned} \tag{44}$$

Substituting (43), (44) into (28), one obtains the expression of $\mathbf{f}_K$ as

$$\begin{aligned}\mathbf{f}_K = \mathbf{M}\,\ddot{\mathbf{q}} &+ \Big\{\mathbf{M}_{\beta}\Big(\frac{\mathbf{z}^{\mathrm{T}}}{l_n}\dot{\mathbf{q}}\Big) + \mathbf{M}_{\bar{\theta}_1}(\mathbf{b}_2^{\mathrm{T}}\dot{\mathbf{q}}) + \mathbf{M}_{\bar{\theta}_2}(\mathbf{b}_3^{\mathrm{T}}\dot{\mathbf{q}})\Big\}\dot{\mathbf{q}} \\ &- \Big(\frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}_{\beta}\dot{\mathbf{q}}\Big)\frac{\mathbf{z}}{l_n} - \Big(\frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}_{\bar{\theta}_1}\dot{\mathbf{q}}\Big)\mathbf{b}_2 - \Big(\frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}_{\bar{\theta}_2}\dot{\mathbf{q}}\Big)\mathbf{b}_3 \end{aligned} \tag{45}$$

The expression of $\mathbf{M}_{\beta}$ is given by

$$\mathbf{M}_{\beta} = \frac{\mathrm{d}\mathbf{T}^{\mathrm{T}}}{\mathrm{d}\beta}\,\mathbf{M}_l\,\mathbf{T} + \mathbf{T}^{\mathrm{T}}\,\mathbf{M}_l\,\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\beta} \tag{46}$$

where

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}\beta} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\mathbf{T} = \mathbf{I}_1\,\mathbf{T} \tag{47}$$

Hence,

$$\mathbf{M}_{\beta} = \mathbf{T}^{\mathrm{T}}(\mathbf{I}_1^{\mathrm{T}}\,\mathbf{M}_l + \mathbf{M}_l\,\mathbf{I}_1)\mathbf{T} = \mathbf{T}^{\mathrm{T}}\mathbf{M}_l^{\beta}\mathbf{T} \tag{48}$$

$\mathbf{M}_{\bar{\theta}_1}, \mathbf{M}_{\bar{\theta}_2}$ are calculated by

$$\mathbf{M}_{\bar{\theta}_1} = \mathbf{T}^{\mathrm{T}}\,\frac{\partial \mathbf{M}_l}{\partial \bar{\theta}_1}\,\mathbf{T} = \mathbf{T}^{\mathrm{T}}\mathbf{M}_{l,\bar{\theta}_1}\mathbf{T} \tag{49}$$

$$\mathbf{M}_{\bar{\theta}_2} = \mathbf{T}^{\mathrm{T}}\,\frac{\partial \mathbf{M}_l}{\partial \bar{\theta}_2}\,\mathbf{T} = \mathbf{T}^{\mathrm{T}}\mathbf{M}_{l,\bar{\theta}_2}\mathbf{T} \tag{50}$$

where

$$\mathbf{M}_{l,\bar{\theta}_1} = \frac{\rho A l_o}{60}\begin{bmatrix} 0 & 3 & 0 & 0 & -3 & 0 \\ 3 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ -3 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_{l,\bar{\theta}_2} = \frac{\rho A l_o}{60}\begin{bmatrix} 0 & -2 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

5 Non-linear equation of the motion

The non-linear equation of motion is

$$\mathbf{f}_K(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}}) + \mathbf{f}_g(\mathbf{q}) = \mathbf{p} \tag{51}$$

where

$\mathbf{f}_K$: Inertia force vector

$\mathbf{f}_g$: Elastic force vector

$\mathbf{p}$: Applied external loads

To solve (51), the differentiation of each terms must be calculated. The following notations are used

$$\mathbf{K}_g = \frac{\partial \mathbf{f}_g}{\partial \mathbf{q}} \tag{52}$$

$$\mathbf{M} = \frac{\partial \mathbf{f}_K}{\partial \ddot{\mathbf{q}}} \tag{53}$$

$$\mathbf{C}_K = \frac{\partial \mathbf{f}_K}{\partial \dot{\mathbf{q}}} \tag{54}$$

$$\mathbf{K}_K = \frac{\partial \mathbf{f}_K}{\partial \mathbf{q}} \tag{55}$$

The stiffness matrix $\mathbf{K}_g$ and the mass matrix $\mathbf{M}$ are defined in previous sections. Using (45), $\mathbf{C}_K$ can be computed as

$$\mathbf{C}_K = \dot{\mathbf{M}} + \mathbf{C}_1 - \mathbf{C}_1^{\mathrm{T}} \tag{56}$$

with

$$\mathbf{C}_1 = \mathbf{M}_\beta \left(\dot{\mathbf{q}}\,\frac{\mathbf{z}^{\mathrm{T}}}{l_n}\right) + \mathbf{M}_{\bar{\theta}_1}\left(\dot{\mathbf{q}}\,\mathbf{b}_2^{\mathrm{T}}\right) + \mathbf{M}_{\bar{\theta}_2}\left(\dot{\mathbf{q}}\,\mathbf{b}_3^{\mathrm{T}}\right) \tag{57}$$

The matrix $\mathbf{K}_K$ can be written as follow

$$\mathbf{K}_K = \mathbf{K}_1 + \mathbf{K}_2 - \mathbf{K}_3 \tag{58}$$

where

$$\mathbf{K}_1 = \mathbf{M}_\beta \ddot{\mathbf{q}}\frac{\mathbf{z}^{\mathrm{T}}}{l_n} + \mathbf{M}_{\bar{\theta}_1}\ddot{\mathbf{q}}\mathbf{b}_2^{\mathrm{T}} + \mathbf{M}_{\bar{\theta}_2}\ddot{\mathbf{q}}\mathbf{b}_3^{\mathrm{T}} \tag{59}$$

$$\begin{aligned}\mathbf{K}_2 = {} & \Big(\frac{\mathbf{z}^{\mathrm{T}}}{l_n}\dot{\mathbf{q}}\Big)\Big(\frac{\partial \mathbf{M}_\beta}{\partial \beta}\,\dot{\mathbf{q}}\frac{\mathbf{z}^{\mathrm{T}}}{l_n} + \frac{\partial \mathbf{M}_\beta}{\partial \bar{\theta}_1}\,\dot{\mathbf{q}}\mathbf{b}_2^{\mathrm{T}} + \frac{\partial \mathbf{M}_\beta}{\partial \bar{\theta}_2}\,\dot{\mathbf{q}}\mathbf{b}_3^{\mathrm{T}}\Big) + (\mathbf{b}_2^{\mathrm{T}}\dot{\mathbf{q}})\,\frac{\partial \mathbf{M}_{\bar{\theta}_1}}{\partial \beta}\,\dot{\mathbf{q}}\frac{\mathbf{z}^{\mathrm{T}}}{l_n} \\ & + (\mathbf{b}_3^{\mathrm{T}}\dot{\mathbf{q}})\,\frac{\partial \mathbf{M}_{\bar{\theta}_2}}{\partial \beta}\,\dot{\mathbf{q}}\frac{\mathbf{z}^{\mathrm{T}}}{l_n} - \Big(\mathbf{M}_\beta - \mathbf{M}_{\bar{\theta}_1} - \mathbf{M}_{\bar{\theta}_2}\Big)\dot{\mathbf{q}}\,\dot{\mathbf{q}}^{\mathrm{T}}\Big(\frac{\mathbf{r}\,\mathbf{z}^{\mathrm{T}} + \mathbf{r}\,\mathbf{z}^{\mathrm{T}}}{l_n^2}\Big)\end{aligned} \tag{60}$$

$$\begin{aligned}\mathbf{K}_3 = {} & \frac{1}{2}\Big[\Big(\dot{\mathbf{q}}^{\mathrm{T}}\,\frac{\partial \mathbf{M}_\beta}{\partial \beta}\,\dot{\mathbf{q}}\Big)\frac{\mathbf{z}\,\mathbf{z}^{\mathrm{T}}}{l_n^2} + \Big(\dot{\mathbf{q}}^{\mathrm{T}}\,\frac{\partial \mathbf{M}_\beta}{\partial \bar{\theta}_1}\,\dot{\mathbf{q}}\Big)\frac{\mathbf{z}}{l_n}\mathbf{b}_2^{\mathrm{T}} + \Big(\dot{\mathbf{q}}^{\mathrm{T}}\,\frac{\partial \mathbf{M}_\beta}{\partial \bar{\theta}_2}\,\dot{\mathbf{q}}\Big)\frac{\mathbf{z}}{l_n}\mathbf{b}_3^{\mathrm{T}} \\ & + \Big(\dot{\mathbf{q}}^{\mathrm{T}}\,\frac{\partial \mathbf{M}_\beta}{\partial \bar{\theta}_1}\,\dot{\mathbf{q}}\Big)\mathbf{b}_2\frac{\mathbf{z}^{\mathrm{T}}}{l_n} + \Big(\dot{\mathbf{q}}^{\mathrm{T}}\,\frac{\partial \mathbf{M}_\beta}{\partial \bar{\theta}_2}\,\dot{\mathbf{q}}\Big)\mathbf{b}_3\frac{\mathbf{z}^{\mathrm{T}}}{l_n} \\ & - \dot{\mathbf{q}}^{\mathrm{T}}\Big(\mathbf{M}_\beta - \mathbf{M}_{\bar{\theta}_1} - \mathbf{M}_{\bar{\theta}_2}\Big)\dot{\mathbf{q}}\Big(\frac{\mathbf{r}\,\mathbf{z}^{\mathrm{T}} + \mathbf{r}\,\mathbf{z}^{\mathrm{T}}}{l_n^2}\Big)\Big]\end{aligned} \tag{61}$$

The expressions of $\dfrac{\partial \mathbf{M}_\beta}{\partial \beta}$, $\dfrac{\partial \mathbf{M}_\beta}{\partial \bar{\theta}_1}$ and $\dfrac{\partial \mathbf{M}_\beta}{\partial \bar{\theta}_2}$ can be easily derived from (48), (49) and (50).

6 Numerical examples

Two numerical applications are presented in this section in order to assess the performance of the dynamic corotational formulation proposed in Sections 4 and 5. In particular, the accuracy of the new formulation is compared to the one of two formulations usually found in the literature, i.e. the lumped mass matrix and the Timoshenko mass matrix. These two constant mass matrices are given by

$$\mathbf{M}_{\text{Lumped}} = \frac{\rho\, A\, l_o}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & l_o^2/12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & l_o^2/12 \end{bmatrix}$$

$$\mathbf{M}_{\text{Timoshenko}} = \rho\, l_o \begin{bmatrix} A/3 & 0 & 0 & A/6 & 0 & 0 \\ 0 & A/3 & 0 & 0 & A/6 & 0 \\ 0 & 0 & I\, l_o/3 & 0 & 0 & I\, l_o/3 \\ A/6 & 0 & 0 & A/3 & 0 & 0 \\ 0 & A/6 & 0 & 0 & A/3 & 0 \\ 0 & 0 & I\, l_o/3 & 0 & 0 & I\, l_o/3 \end{bmatrix}$$

For all dynamic formulations, the elastic force vector and tangent stiffness matrix have been derived using the IIE shape functions in order to account for shear deformability. For each example, the three dynamic formulations are compared with a reference solution. This solution is obtained with a large number of elements and is identical for the three considered dynamic formulations. The reference solution has also been checked with Abaqus (Total Lagrangian formulation) and the same results have been obtained.
To solve the equation of motion, the Alpha method, which is presented in [14], is employed, with $\alpha = -0.01$. This moderate value of α gives a small numerical damping, which limits the influence of higher frequencies on the response. Damping is not considered.
For the presentation of the results, the following colors are used in all figures:

Reference solution — Lumped mass matrix
Timoshenko mass matrix — New formulation

6.1 Shallow arch

Consider a shallow, circular, elastic arch (see Fig. 3) of span $L = 10m$ with clamped ends. The radius R of the arch is equal to 10m with $\phi = 30°$. The shallow arch has a uniform rectangular cross-section and is subjected to a sinusoidal concentrated vertical force $P = P_o \sin(wt)$ at mid-span. The amplitude of the load P_o is taken equal to -80 MN and its frequency w is 1000 rad/s. The arch has cross-sectional area $A = 0.087$ m^2, modulus of elasticity $E = 210GPa$, second moment of area $I = 3.562 \cdot 10^{-3}$m^4 and mass per unit volume $\rho = 7850$kg/m^3. The time step size is chosen to be $\Delta t = 5 \cdot 10^{-5}$ s. In Fig. 4, the mid-span vertical displacement $v(t)$ history is depicted for the 3 different dynamic formulations as well as the reference solution, which has been obtained with 48 elements. Only 6 elements have been used for the

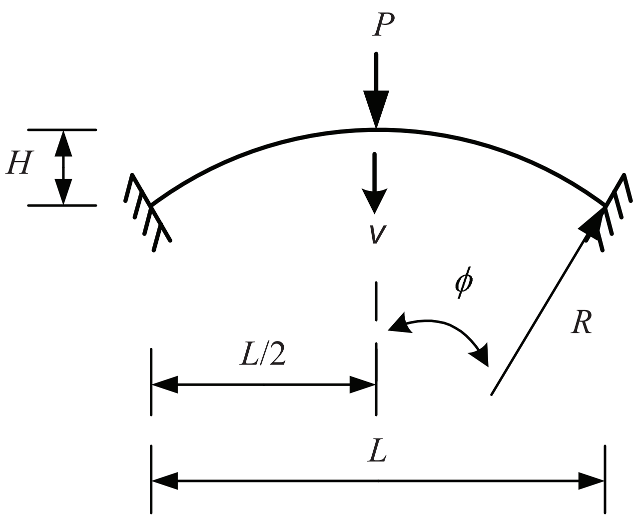

Figure 3: Shallow arch: geometrical data

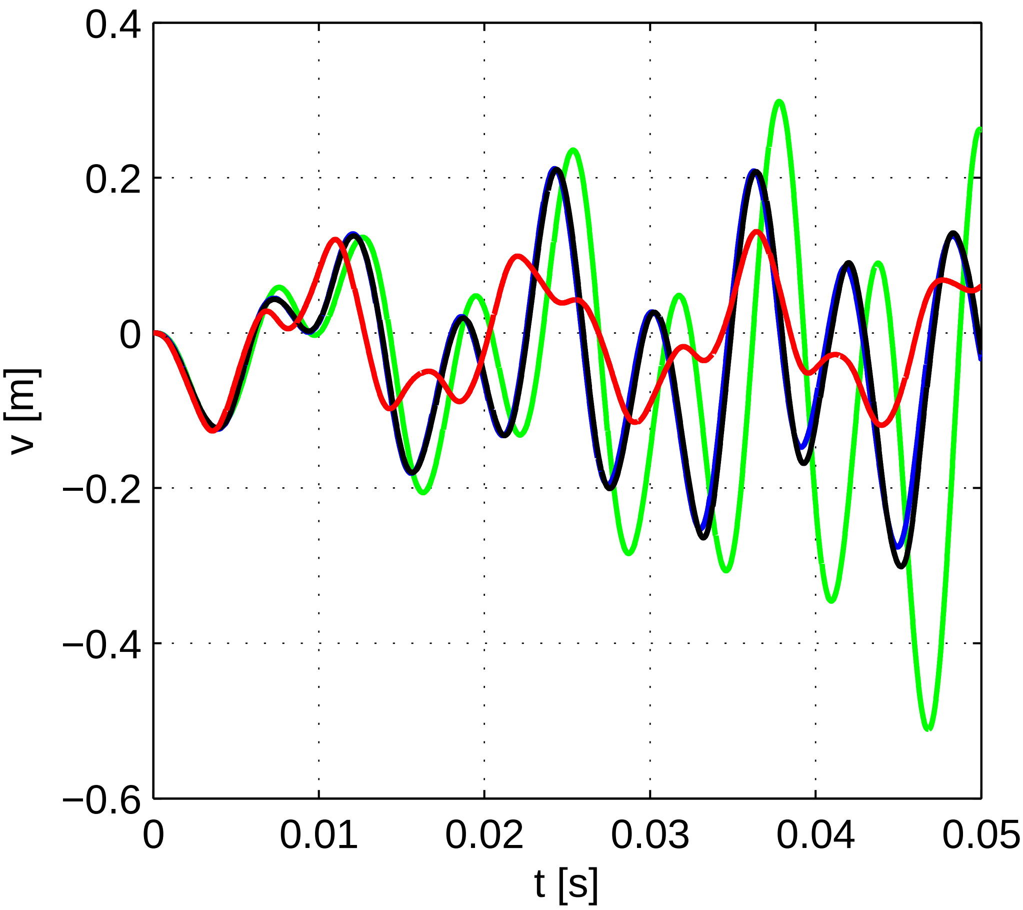

Figure 4: Shallow arc - Vertical displacement history

computations with the 3 formulations. It can be observed that the results obtained with the new approach are nearly identical to the reference solution. However, large discrepancies between the results obtained with the lumped and Timoshenko approaches and the reference solution can be observed. This indicates that the present formulation is able to capture the nonlinear dynamical behaviour of structures with minimal number of elements.

6.2 Lee's frame

A Lee's frame with uniform rectangular cross-section subjected to a suddenly applied constant load $P_o = 4.1$ MN is considered, (see Fig. 5). The frame and cross-section data (see Fig. 5) are : $L = 2.4\mathrm{m}$, $a = 0.2\mathrm{m}$ and $e = 0.3\mathrm{m}$. The members of the frame have modulus of elasticity $E = 210\mathrm{GPa}$ and mass per unit volume $\rho = 7850\mathrm{kg/m^3}$. The loading is defined as follows:

$$P = \begin{cases} 0 & \text{if } t \leq 0 \\ P_o & \text{if } t > 0 \end{cases}$$

The reference solution, obtained with 60 elements, and the results obtained with 10 elements

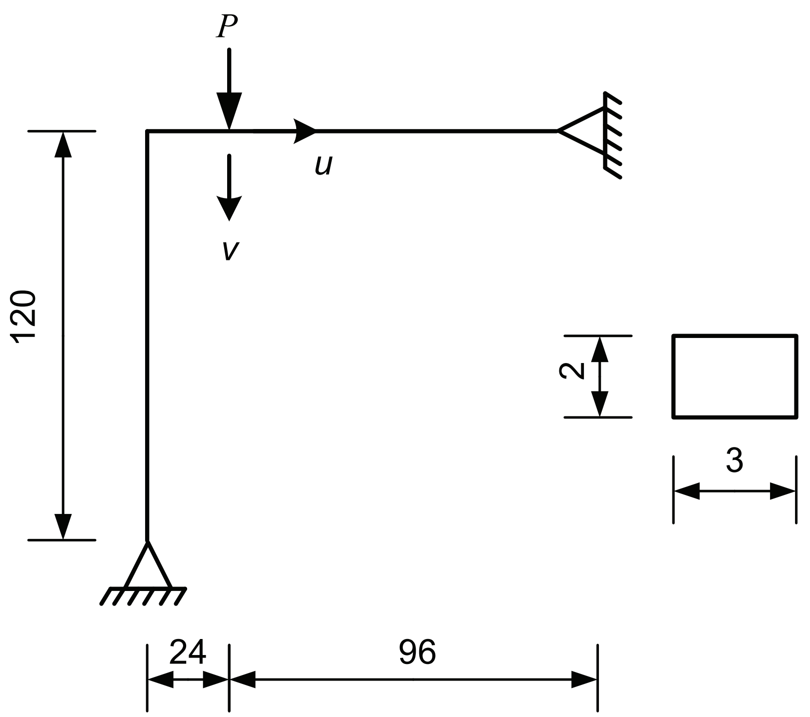

Figure 5: Lee's frame: geometrical data

are presented in Fig. 6. The time step size is $\Delta t = 2.5 \cdot 10^{-3}$ s. It can be noted that, with 10 elements, the results obtained with the new approach are in good agreement with the reference solution. However, the discrepancy between the results obtained with the lumped or Timoshenko approaches and the reference solution is not negligible.

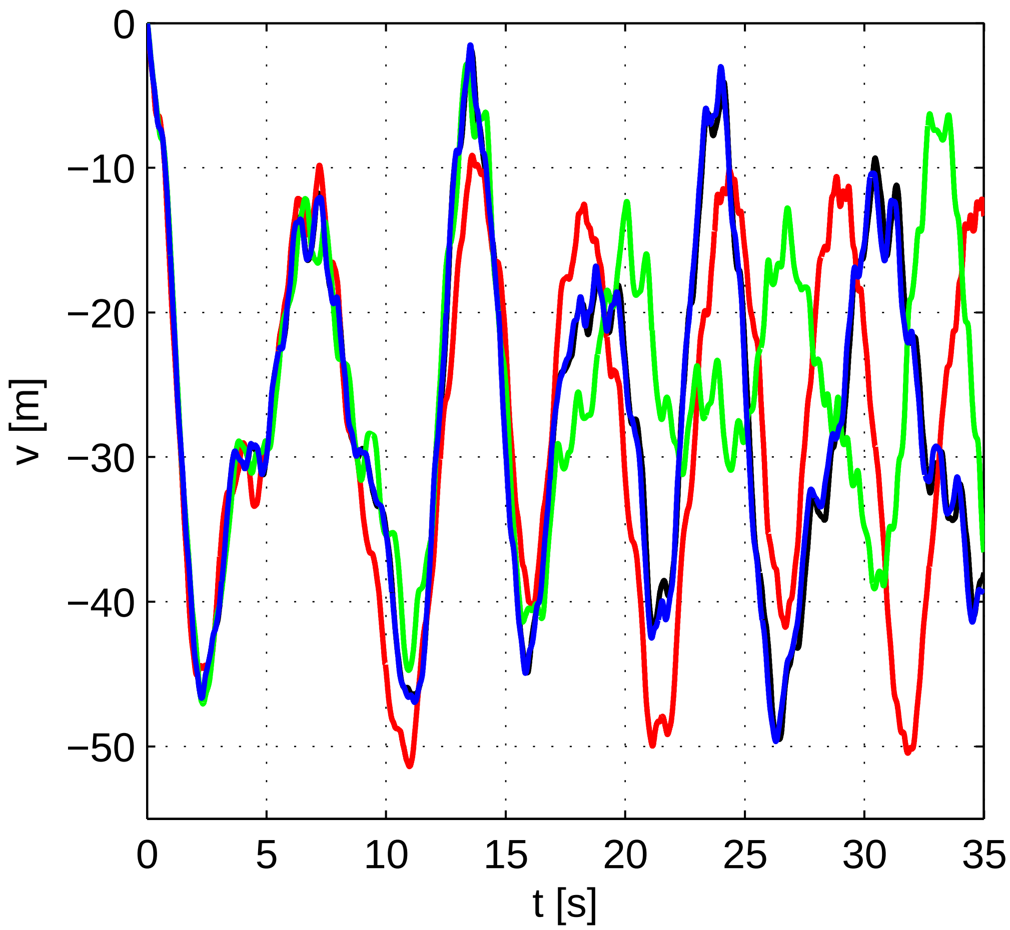

Figure 6: Lee's frame - Vertical displacement history

7 Conclusion

In this paper, a new dynamic formulation for corotational 2D beam has been presented. The new feature is that cubic interpolations are used to derive both inertia and elastic terms. The inertia terms are analytically derived by introducing some simplifications. Two numerical examples were implemented to compare this new formulation with linear Timoshenko and lumped mass matrices, which are often used in literature. The results show that the new formulation requires more computational time but allows to reduce significantly the number of elements. This advantage is due to a better representation of the local displacements in the inertia terms.

References

[1] M.A. Crisfield, J. Shi, *An energy conserving co-rotational procedure for non-linear dynamics with finite elements*, Nonlinear Dynamics, **9**, 37-52, 1994.

[2] M.A. Crisfield, U. Galvanetto, G. Jelenic, *Dynamics of 3-D co-rotational beams*, Computational Mechanics, **20**, 507-519, 1997.

[3] U. Galvanetto, M.A. Crisfield, *An energy conserving co-rotational procedure for dynamics of planar beam structures*, Int. J. Num. Methods. Engrg., **39**, 2265-2282, 1996.

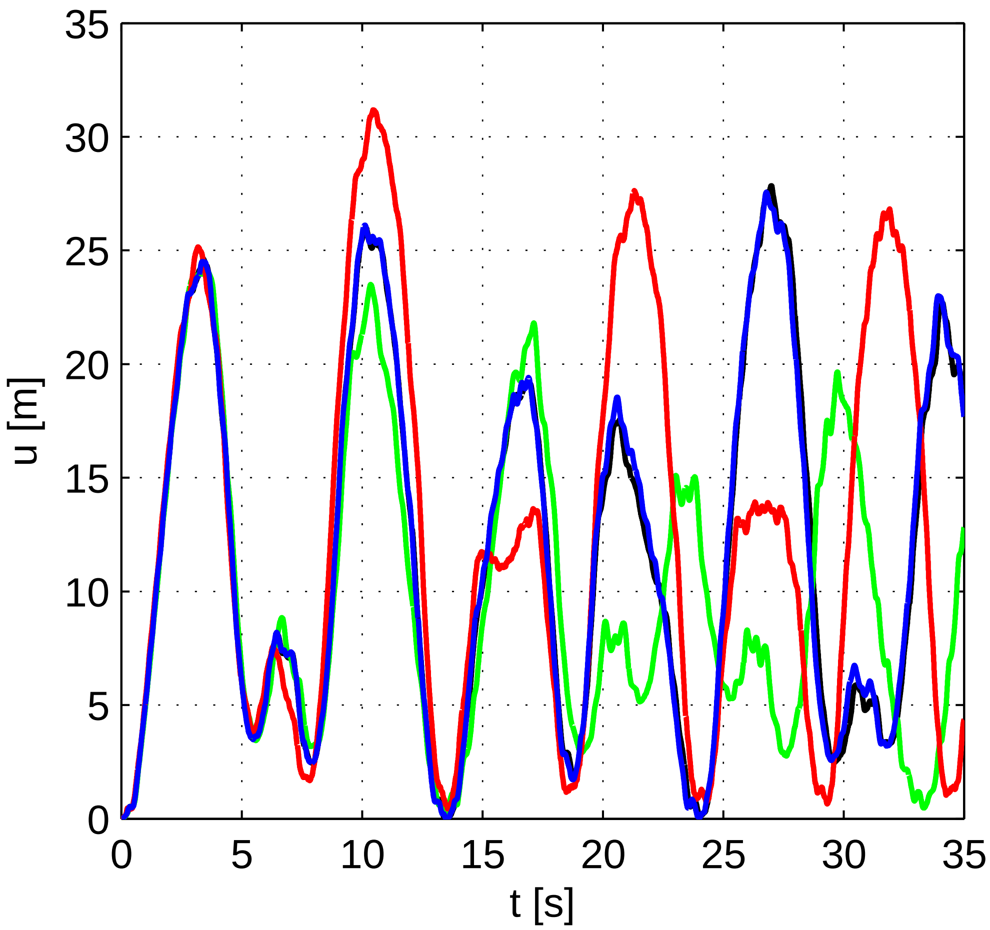

Figure 7: Lee's frame - Horizontal displacement

[4] K.M. Hsiao, R.T. Yang, *A co-rotational formulation for nonlinear dynamic analysis of curved Euler beam*, Compt.Struct. , **54**, No.6, 1091-1097, 1995.

[5] K.M. Hsiao, J.Y. Lin, W.Y. Lin, *A consisten co-rotational finite element formulation for geometrically nonlinear dynamic analysis of 3-D beams*, Comput. Methods Appl. Mech. Engrg., **169**, 1-18, 1999.

[6] M. Iura and S.N. Atluri, *Dynamic analysis of planar flexible beams with finite rotations by using inertial and rotating frames*, Comput. Struct., **55**, No.3, 453-462, 1995.

[7] C. Oran, A. Kassimali, *Large deformations of framed structures under static and dynamic loads*, Compt.Strut., **6**, 539-547, 1976.

[8] N. Masuda. T. Nishiwaki and M. Minaaawa, *Nonlinear dynamic analysis of frame structures*, Comput. Struct., **27**, No.1, 103-110, 1987.

[9] Q. Xue and J.L Meek, *Dynamic response and instability of frame structures*, Comput. Methods Appl. Mech. Engrg., **190**, 5233-5242, 2001.

[10] K. Behdinan, M.C Stylianou, B. Tabarrok, *Co-rotational dynamic analysis of flexible beams*, Comput. Methods Appl. Mech. Engrg., **154**, 151-161, 1998.

[11] H.A. Elkaranshawy and M.A. Dokainish, *Corotational finite element analysis of planar flexible multibody systems*, Comput. Struct., **54**, No.5, 881-890, 1995.

[12] J.N. Reddy, *On locking-free shear deformable beam finite elements*, Comput. Methods Appl. Mech. Engrg., **149**, 113-132, 1997.

[13] M.A. Crisfield, *Non-Linear Finite Element Analysis of Solids and Structures, Volume 1: Essentials*, 201-233. Wiley, Chischester, England, 1997.

[14] M.A. Crisfield, *Non-Linear Finite Element Analysis of Solids and Structures, Volume 2: Advanced Topics*, 455-456. Wiley, Chischester, England, 1997.