COMPDYN 2011
3rd ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis, V. Plevris (eds.)
Corfu, Greece, 25–28 May 2011

STABILIZED HYBRID AND MIXED FINITE ELEMENT METHODS FOR HELMHOLTZ PROBLEMS

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Keywords: Hybrid methods, Stabilization, Mixed methods, Numerical pollution, Hybridization.

Abstract. *Stabilized hybrid and mixed finite element methods are proposed for solving Helmholtz problems in heterogeneous media. The methods are based on a hybridized dual mixed formulation in velocity (flux) and pressure fields stabilized by adding least squares residual of the governed equations. The local problems, in the velocity and pressure fields, are solved at element level to eliminate these variables in favor of the Lagrange multipliers, identified as the trace of the pressure on the element edges of the finite element mesh. A global system is assembled involving only the degrees of freedom associated with the Lagrange multipliers as usually in Hybrid methods. Polynomial bases are adopted to approximate the global problem in the Lagrange multipliers. Polynomial or special bases, such as plane-wave bases, can be also used to approximate the local problems at the element level. Numerical results are reported to illustrate the potential of the proposed formulation to efficiently solve Helmholtz problems in homogeneous or heterogeneous media at medium and high frequency regimes.*

1 INTRODUCTION

The linear model for propagation of acoustic waves in an ideal compressible fluid is governed by the wave equation

$$-\Delta\varphi + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{1}$$

where $\varphi(\mathbf{x}, t)$ represents small oscillations of the pressure and c is the velocity of the sound in the acoustic medium. Considering harmonic solutions in time with circular frequency ω, the pressure field is written as $\varphi(\mathbf{x}, t) = p(\mathbf{x})e^{-i\omega t}$, and the pressure amplitude p satisfies the Helmholtz equation

$$-\Delta p - k^2 p = 0, \tag{2}$$

where the parameter $k = \omega/c$, known as *wave number*, characterizes the oscillatory behavior of the solution φ.

Helmholtz problem has deserved especial attention in many physical applications associated with refraction and scattering of electromagnetic, elastic or sound waves, for example. From the numerical point of view k is a key parameter. It is well known that standard Galerkin finite element approximations deteriorate as k increases. For large values of k the solution p is highly oscillatory and, due to numerical dispersion and phase error, constructing finite element approximations for Helmholtz equation is a great challenge as reported in vast literature. See, for example, [4] and references therein. Several finite element methods have been developed to minimize the phase error. Stabilized finite element methods such as Galerkin Least-Squares(GLS) have been proposed with relative success [5]. An uniform nine node stencils with minimal pollution error, referred as QSFEM, is constructed in [6]. Variationally consistent finite element methods with stability properties equivalent to the QSFEM on uniform meshes have been proposed in [7] using a generalized GLS formulation, and in [8] using discontinuous Galerkin finite element methods. A Quasi Optimal Petrov-Galerkin (QOPG) finite element formulation for Helmholtz problem in two dimensions is introduced in [9] using polynomial weighting functions with the same support of the corresponding global test functions. The QOPG finite element formulation is naturally applied to non uniform and unstructured meshes. Generalized finite element methods using plane wave bases have been successfully developed in [3, 10, 11] to solve the Helmholtz equation with great accuracy when applied to problem with regular solutions.

Here, we consider stabilized hybrid and mixed finite element methods for solving Helmholtz problems in heterogeneous media with discontinuous wave number k. The proposed formulations is based on a hybridized dual mixed formulation in velocity (flux) and pressure fields stabilized by adding least squares residual of the governed equations. To simplify our presentation we consider as our model problem the Helmholtz equation

$$-\Delta p - k^2 p = f \text{ in } \Omega, \tag{3}$$

in a bounded domain $\Omega \subset R^2$ with a Lipschitz continuous and piecewise smooth boundary Γ subjected to Dirichlet boundary condition

$$p = g \ \text{ on } \ \Gamma = \partial\Omega. \tag{4}$$

The reminder of the paper is organized as follows. In Section 2 our model problem is presented in a mixed form and a stabilized dual mixed formulation in continuous spaces is presented. In Section 3 we review the classical dual hybrid mixed formulation. The stabilized dual hybrid mixed formulation is proposed in Section 4. The finite element approximation of the proposed dual hybrid mixed formulation is presented in Section 5. Numerical results are shown in Section 6 and some concluding remarks are presented in Section 7.

2 MIXED FORMULATION

Introducing the vector field

$$\mathbf{u} = -\nabla p$$

we can rewrite the Helmholtz equation (3) in the mixed form

$$\mathbf{u} + \nabla p = 0 \quad \text{in} \;\; \Omega$$

$$\mathrm{div}\mathbf{u} - k^2 p = f \quad \text{in} \;\; \Omega\,,$$

which will be used as the starting point to construct stabilized mixed and hybrid finite element approximations.

2.1 Dual mixed formulation

Defining the space $V = L^2(\Omega)$ of scalar functions with the corresponding inner product

$$(p, q) = \int_\Omega pq d\Omega \;\; \forall p, q \; \in V \tag{5}$$

and associated norm

$$||p||^2 = (p, p) \;\; \forall p \; \in V, \tag{6}$$

and the space $W = H(\mathrm{div})$ of vector functions:

$$H(\mathrm{div}) = \{\mathbf{v} \in [L^2(\Omega)]^2, \; \mathrm{div}\mathbf{v} \in L^2(\Omega)\} \tag{7}$$

with inner product

$$(\mathbf{u}, \mathbf{v})_{H(\mathrm{div})} = (\mathbf{u}, \mathbf{v}) + (\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v}) \;\; \forall \mathbf{u}, \mathbf{v} \; \in W \tag{8}$$

and associated norm

$$||\mathbf{u}||^2_{H(\mathrm{div})} = (\mathbf{u}, \mathbf{u}) + (\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{u}) \;\; \forall \mathbf{u} \; \in W \tag{9}$$

we present the dual mixed formulation of our model problem as

Find $[\mathbf{u}, p] \in W \times V$ such that

$$(\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\mathbf{v}) - (\mathrm{div}\mathbf{u}, q) - k^2(p, q) - (f, q) + \int_{\partial\Omega} g\mathbf{v} \cdot \mathbf{n} ds = 0 \;\; \forall\, [\mathbf{v}, q] \in W \times V \tag{10}$$

Dual mixed problems have been usually approximated using Raviart-Thomas (RT) or Brezzi-Douglas-Marini (BDM) spaces to get accurate approximations for the gradient of the pressure field. Stability of these mixed finite element methods depends on the well known inf-sup condition which preclude many desirable combinations of velocity and pressure interpolations. Stabilized mixed formulations have been successfully introduced to overcome these limitations. See references [12, 13, 14, 16] on stabilized mixed formulations for Darcy flow.

2.2 Stabilized dual mixed formulation

In [14] an unconditionally stable mixed formulation is proposed for Darcy flow with optimal rates of convergence in $H^1(\Omega)$ and $L^2(\Omega)$ norms for same order C^0 velocity and pressure approximations. Applied to the Helmholtz problem this formulation leads to the following weak form:

Find $[\mathbf{u}, p] \in W \cap [H^1(\Omega)]^2 \times V \cap H^1(\Omega)$ such that, for all $[\mathbf{v}, q] \in W \times V$,

$$(\mathbf{u}, \mathbf{v}) - (p, \text{div}\mathbf{v}) - (\text{div}\mathbf{u}, q) - k^2(p, q) - (f, q) + \int_{\partial\Omega} g\mathbf{v} \cdot \mathbf{n} ds + \tag{11}$$

$$\frac{\delta_1}{k^2}(\text{div}\mathbf{u} - k^2 p - f, \text{div}\mathbf{v} - k^2 q) + \delta_2(\mathbf{u} + \nabla p, \mathbf{v} + \nabla q) + \frac{\delta_3}{k^2}(\text{rot}\mathbf{u}, \text{rot}\mathbf{v}) = 0\,. \tag{12}$$

Of course, we should not expect unconditional stability and optimal rates of convergence for same order C^0 finite element approximations of the above dual mixed formulation of the Helmholtz problem, especially for heterogeneous media. Its stability will certainly be dependent on the choice of the stabilization parameters $\delta_i,\ i = 1, 2, 3$. An other limitation of this formulation is the required C^0 approximations for both velocity and pressure which is not applicable to heterogeneous media with discontinuous material properties. In the next section we introduce hybrid formulations more appropriate to construct finite element approximations of our model problem on heterogeneous media.

3 HYBRIDIZATION

Let

$$\mathcal{T}_h = \{\mathcal{K}\} := \text{union of all elements } \mathcal{K}$$

be a regular finite element mesh on the two dimension domain Ω. To introduce the hybrid formulation we first consider equation $\mathbf{u} + \nabla p = 0$, in the local weak form

$$(\mathbf{u} + \nabla p, \mathbf{v})_{\mathcal{K}} = \int_{\mathcal{K}} \mathbf{u} \cdot \mathbf{v} d\Omega - \int_{\mathcal{K}} p \text{div}\mathbf{v} d\Omega + \int_{\partial\mathcal{K}} p\mathbf{v} \cdot \mathbf{n} ds = 0,$$

defined on each element $\mathcal{K}$ using integration by parts. Considering the spaces

$$Q_{\mathcal{K}} = \{q \in L^2(\mathcal{K})\ \forall \mathcal{K} \in \mathcal{T}_h\},$$

$$U_{\mathcal{K}} = \{\mathbf{v} \in L^2(\mathcal{K}) \times L^2(\mathcal{K}), \text{div}\mathbf{v} \in L^2(\mathcal{K})\ \forall \mathcal{K} \in \mathcal{T}_h\},$$

of local functions defined on each element $\mathcal{K}$ and defining the forms

$$a_{\mathcal{K}}\left([\mathbf{u}, p], [\mathbf{v}, q]\right) = (\mathbf{u}, \mathbf{v})_{\mathcal{K}} - (p, \text{div}\mathbf{v})_{\mathcal{K}} - (\text{div}\mathbf{u}, q)_{\mathcal{K}} + k^2(p, q)_{\mathcal{K}},$$

$$f_{\mathcal{K}}([\mathbf{v}_h, q_h]) = -(f, q)_{\mathcal{K}} - c_{\mathcal{K}}(\bar{p}, \mathbf{v}),$$

$$c_{\mathcal{K}}(\bar{p}, \mathbf{v}) = \int_{\partial\mathcal{K}} \bar{p}\mathbf{v} \cdot \mathbf{n} ds,$$

for given $p = \bar{p}$ on $\partial\mathcal{K}$ we can solve the local problems:

For each $\mathcal{K} \in \mathcal{T}_h$, find $[\mathbf{u}, p] \in U_{\mathcal{K}} \times Q_{\mathcal{K}}$, such that

$$a_{\mathcal{K}}\left([\mathbf{u}, p], [\mathbf{v}, q]\right) = f_{\mathcal{K}}([\mathbf{v}, q])\ \forall\, [\mathbf{v}, q] \in U_{\mathcal{K}} \times Q_{\mathcal{K}}.$$

Local Raviart-Thomas or BDM finite element approximations can be used to solve these local problems on each element $\mathcal{K}$. Following the classical hybrid formulation, an approximation for the pressure trace $p = \bar{p}$ on $\partial\mathcal{K}$ can be obtained by solving a global problem associated with the dual hybrid mixed formulation, as presented in the next section.

3.1 Dual Hybrid mixed formulation

To present the hybrid formulation some additional notations and definitions are need. Let

$$\mathcal{E}_h = \{e : e \text{ is an edge of } \mathcal{K} \text{ for all } \mathcal{K} \in \mathcal{T}_h\} \tag{13}$$

denote the set of all edges of all elements $\mathcal{K}$ of the mesh $\mathcal{T}_h$,

$$\mathcal{E}_h^0 = \{e \in \mathcal{E}_h \; e \text{ is an interior edge}\} \tag{14}$$

the set of interior edges, and

$$\mathcal{E}_h^\partial = \mathcal{E}_h \cap \partial\Omega, \tag{15}$$

the set of edges of $\mathcal{E}_h$ on the boundary of Ω.

Following the commonly adopted notation on DG (Discontinuous Galerkin) formulations, we consider the unit normal vectors $\mathbf{n}^1$ abd $\mathbf{n}^2$ on e pointing exterior to $\mathcal{K}^1$ and $\mathcal{K}^2$, respectively. For a scalar function φ, piecewise smooth on $\mathcal{T}_h$, with $\varphi = \varphi|_\mathcal{K}$ we define on each interior edge e

$$\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\![\varphi]\!] = \varphi^1\mathbf{n}^1 + \varphi^2\mathbf{n}^2 \quad \text{on } \; e \in \mathcal{E}^0 \tag{16}$$

and for a vector function $\mathbf{v}$

$$\{\mathbf{v}\} = \frac{1}{2}(\mathbf{v}^1 + \mathbf{v}^2), \quad [\![\mathbf{v}]\!] = \mathbf{v}^1 \cdot \mathbf{n}^1 + \mathbf{v}^2 \cdot \mathbf{n}^2 \quad \text{on } \; e \in \mathcal{E}^0. \tag{17}$$

Defining the function spaces $M = \{\mu \in L^2(e) \; \forall e \in \mathcal{E}_h^0\}$, $U = \prod_\mathcal{K} U_\mathcal{K}$, $Q = \prod_\mathcal{K} Q_\mathcal{K}$ and the bilinear and linear forms:

$$a\left([\mathbf{u}, p], [\mathbf{v}, q]\right) = \sum_\mathcal{K} a_\mathcal{K}\left([\mathbf{u}, p], [\mathbf{v}, q]\right) \;\; \forall \;\; [\mathbf{u}, p], [\mathbf{v}, q] \;\in U \times Q$$

$$c(\mu, \mathbf{v}) = \sum_\mathcal{K} c_\mathcal{K}(\mu, \mathbf{v}) = \int_{\mathcal{E}^0} \mu[\![\mathbf{v}]\!]ds \;\; \forall \; \mu \in M, \; \forall \mathbf{v} \in U$$

$$f([\mathbf{v}, q]) = \sum_\mathcal{K} (f, q) - \int_{\mathcal{E}_h^\partial} g\mathbf{v} \cdot \mathbf{n} ds \; \forall \mathbf{v} \in U, \; \forall q \in Q$$

the dual hybrid formulation consists in:

Find $[\mathbf{u}, p] \in U \times Q$ and the Lagrange multiplier $\lambda \in M$, such that

$$a\left([\mathbf{u}, p], [\mathbf{v}, q]\right) + c(\lambda, \mathbf{v}) = f([\mathbf{v}, q]), \; \forall [\mathbf{v}, q] \in U \times Q,$$

$$c(\mu, \mathbf{u}) = 0 \; \forall \mu \in M,$$

with $\lambda = \bar{p}$, the trace of the pressure p on $\mathcal{E}_h^0$ and $\lambda = g$ on $\mathcal{E}_k^\partial$.

3.2 Recovering the dual mixed formulation

Considering that $c(\mu, \mathbf{u}) = 0 \; \forall \; \mu \in M$ implies that $[\![\mathbf{u}]\!] = 0$ (continuity of the normal component of the velocity field $\mathbf{u}$), we note that the pair $[\mathbf{u}, p]$, solution of the above defined dual hybrid mixed method, satisfies the dual mixed formulation

Find $[\mathbf{u}, p] \in U \cap H(\mathrm{div}) \times Q$, such that

$$a\left([\mathbf{u}, p], [\mathbf{v}, q]\right) = f([\mathbf{v}, q]) \; \forall [\mathbf{v}, q] \in U \cap H(\mathrm{div}) \times Q$$

or, more explicitly

Find $\mathbf{u} \in W = U \cap H(\mathrm{div})$ and $p \in V = Q$ such that

$$(\mathbf{u}, \mathbf{v}) - (p, \mathrm{div}\mathbf{v}) = \int_{\partial\Omega} g\mathbf{v} \cdot \mathbf{n} ds \;\forall\, \mathbf{v}_h \in W$$

$$-(\mathrm{div}\mathbf{u}, q) = -(f, q) \;\forall\, q \in V$$

which is the classical definition of the dual mixed formulation (Ravirat-Thomas) presented in Section 2.1.

4 STABILIZATION

Adding to the dual mixed formulation least squares stabilization terms defined in each element $\mathcal{K}$ and on the element boundaries we obtained the following residual form

$$a\left([\mathbf{u}, p], [\mathbf{v}, q]\right) + c(\lambda, \mathbf{v}) + c(\mu, \mathbf{u}) - f([\mathbf{v}, q]) + \sum_{\mathcal{K}} \int_{\partial\mathcal{K}} \beta(\lambda - p)(\mu - q) ds +$$

$$\sum_{\mathcal{K}} \left(\frac{\delta_1}{k^2}(\mathrm{div}\mathbf{u} - k^2 p - f, \mathrm{div}\mathbf{v} - k^2 q)_{\mathcal{K}} + \delta_2(\mathbf{u} + \nabla p, \mathbf{v} + \nabla q)_{\mathcal{K}} + \frac{\delta_3}{k^2}(\mathrm{rot}\mathbf{u}, \mathrm{rot}\mathbf{v})_{\mathcal{K}} \right) = 0 \tag{18}$$

in which the added residual forms in the interior of the element, weighted by the stabilization parameters $\delta_i,\ i = 1, 2, 3$, are the same considered in the stabilized dual mixed formulation presented Section 2.2 aiming at stabilizing the pair $[\mathbf{u}, p]$, while the residual form on the element boundaries multiplied by β is introduced to stabilize the multiplier λ.

4.1 Stabilized dual hybrid mixed formulation

Collecting appropriately the residual terms in (18) corresponding to the pair $[\mathbf{u}, p]$ and the multiplier λ, the stabilized dual hybrid mixed method can be presented as:

SDHM: Find $[\mathbf{u}, p] \in U \times Q$ and the Lagrange multiplier $\lambda \in M$ such that

$$a_\delta\left([\mathbf{u}, p], [\mathbf{v}, q]\right) + c(\lambda, \mathbf{v}) + \sum_{\mathcal{K}} \int_{\partial\mathcal{K}} \beta(p - \lambda) q ds = f_\delta([\mathbf{v}, q]) \;\;\forall\, [\mathbf{v}, q] \in U \times Q \tag{19}$$

$$c(\mu, \mathbf{u}) + \sum_{\mathcal{K}} \int_{\partial\mathcal{K}} \beta(\lambda - p)\mu ds = 0 \;\forall\, \mu \in M \tag{20}$$

where

$$a_\delta\left([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]\right) = \sum_{\mathcal{K}} a^\delta_{\mathcal{K}}\left([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]\right); \quad f_\delta([\mathbf{v}, q]) = \sum_{\mathcal{K}} f^\delta_{\mathcal{K}}([\mathbf{v}, q]) \tag{21}$$

with

$$a^\delta_{\mathcal{K}}\left([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]\right) = a_{\mathcal{K}}\left([\mathbf{u}, p], [\mathbf{v}, q]\right) +$$
$$\frac{\delta_1}{k^2}\left(\mathrm{div}\mathbf{u} - k^2 p, \mathrm{div}\mathbf{v} - k^2 q\right)_{\mathcal{K}} - \frac{\delta_2}{k^2}\left(\mathbf{u} + \nabla p, \mathbf{v} + \nabla q\right)_{\mathcal{K}} + \frac{\delta_1}{k^2}\left(\mathrm{rot}\mathbf{u}, \mathrm{rot}\mathbf{v}\right)_{\mathcal{K}} \tag{22}$$

$$f^\delta_{\mathcal{K}}([\mathbf{v}, q]) = f_{\mathcal{K}}([\mathbf{v}, q]) + \frac{\delta_1}{k^2}(f, \mathrm{div}\mathbf{v} - k^2 q)_{\mathcal{K}}$$

Considering the definitions (16) and (17) we restate problem **SDHM** as:

SDHM: Find $[\mathbf{u}, p] \in U \times Q$ and the Lagrange multiplier $\lambda \in M$ such that $\forall\, [\mathbf{v}, q] \in U \times Q$

$$a_\delta\left([\mathbf{u}, p], [\mathbf{v}, q]\right) + \int_{\mathcal{E}_h^0} \lambda [\![\mathbf{v}]\!] ds + \int_{\mathcal{E}_h^0} 2\beta\left(\{p\} - \lambda\right)\{q\} ds + \int_{\mathcal{E}_h^0} \frac{\beta}{2} [\![p]\!] \cdot [\![q]\!] ds = f_\delta([\mathbf{v}, q]) \quad (23)$$

$$\int_{\mathcal{E}_h^0} 2\beta\left((\lambda - \{p\}) + [\![\mathbf{u}]\!]\right)\mu ds = 0 \;\forall\, \mu \in M. \quad (24)$$

In Section 5 finite element approximations will be constructed for this stabilized dual hybrid formulation considering finite dimension polynomial spaces for all fields.

4.2 Hybridizable mixed DG method

Solving equation (24) for the multiplier λ we get

$$\lambda = \{p\} - \frac{1}{2\beta} [\![\mathbf{u}]\!]. \quad (25)$$

Replacing (25) in (23) yields the following stabilized mixed discontinuous Galerkin method

SMDG: Find $[\mathbf{u}, p]$ such that

$$\begin{aligned} &a_\delta\left([\mathbf{u}, p], [\mathbf{v}, q]\right) + \int_{\mathcal{E}_h^0} \left(\{p\}[\![\mathbf{v}]\!] + [\![\mathbf{u}]\!]\{q\}\right) ds + \\ &+ \int_{\mathcal{E}_h^0} \frac{\beta}{2} [\![p]\!] \cdot [\![q]\!] ds - \int_{\mathcal{E}_h^0} \frac{1}{2\beta} [\![\mathbf{u}]\!][\![\mathbf{v}]\!] ds = f_\delta([\mathbf{v}, q]) \;\; \forall\, [\mathbf{v}, q] \in U \times Q \end{aligned} \quad (26)$$

Setting $\delta_i = 0$, for $i = 1, 2, 3$, the above **SMDG** formulation recovers the LDG-H (Local Discontinuous Galerkin - Hybridizable) method analyzed in [15] in the context of elliptic problems.

5 FINITE ELEMENT APPROXIMATIONS

Let us consider the finite dimension spaces

$$Q_h^l = \{q \in Q : \quad q|_{\mathcal{K}} \in \mathfrak{R}^l \;\; \forall \mathcal{K} \in \mathcal{T}_h\},$$

$$U_h^m = \{\mathbf{v} \in U : \quad \mathbf{v}|_{\mathcal{K}} \in \mathfrak{R}^m \times \mathfrak{R}^m \;\; \forall \mathcal{K} \in \mathcal{T}_h\},$$

$$M_h^n = \{\mu \in M : \quad \mu|_e \in \mathfrak{P}^n \;\; \forall e \in \mathcal{E}^0\},$$

where $\mathfrak{R}^l$ is the set of polynomial of degree less then or equal to l when $\mathcal{K}$ is a triangle or less then or equal to l in each coordinate when $\mathcal{K}$ is a quadrilateral, and $\mathfrak{P}^n$ is the set of polynomials of degree less then or equal to n on each edge e.

We can now present a finite element approximation for the stabilized dual hybrid mixed formulation introduced in Section 4.1 as:

SDHM$_h$: Find $[\mathbf{u}_h, p_h] \in U_h^m \times Q_h^l$ and the Lagrange multiplier $\lambda_h \in M_h^n$ such that

$$\begin{aligned} &a_\delta\left([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]\right) + \int_{\mathcal{E}_h^0} \frac{\beta}{2} [\![p_h]\!] \cdot [\![q_h]\!] ds \\ &+ \int_{\mathcal{E}_h^0} \lambda_h [\![\mathbf{v}_h]\!] ds + \int_{\mathcal{E}_h^0} 2\beta\left(\{p_h\} - \lambda_h\right)\{q_h\} ds = f_\delta([\mathbf{v}_h, q_h]) \;\; \forall\, [\mathbf{v}_h, q_h] \in U_h^m \times Q_h^l \end{aligned} \quad (27)$$

$$\int_{\mathcal{E}_h^0} 2\beta\left((\lambda_h - \{p_h\}) + [\![\mathbf{u}_h]\!]\right)\mu_h ds = 0 \;\; \forall\, \mu_h \in M_h. \tag{28}$$

For $n \geq l$ and $n \geq m$ we can solve explicitly equation (28), for the multiplier λ_h, obtaining

$$\lambda_h = \{p_h\} - \frac{1}{2\beta}[\![\mathbf{u}_h]\!], \tag{29}$$

which when replaced in (23) yields the following hybridizable discontinuous Galerkin method

$\mathbf{SMDG}_h$: Find $[\mathbf{u}_h, p_h] \in U_h^m \times Q_h^l$ such that

$$a_\delta\left([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]\right) + \int_{\mathcal{E}_h^0} \left(\{p_h\}[\![\mathbf{v}_h]\!] + [\![\mathbf{u}_h]\!]\{q_h\}\right) ds +$$

$$+ \int_{\mathcal{E}_h^0} \frac{\beta}{2}[\![p_h]\!] \cdot [\![q_h]\!] ds - \int_{\mathcal{E}_h^0} \frac{1}{2\beta}[\![\mathbf{u}_h]\!][\![\mathbf{v}_h]\!] ds = f_\delta([\mathbf{v}_h, q_h]) \;\; \forall\, [\mathbf{v}_h, q_h] \in U_h \times Q_h. \tag{30}$$

Computationally, the best option to solve the systems of linear equations (28) and (29) is to solve first (28) at element level to obtain $[\mathbf{u}_h, p_h]$ in terms of λ_h, replace $[\mathbf{u}_h, p_h]$ in (29) and assemble a global system in λ_h only. After solving the global system in λ_h, the pair $[\mathbf{u}_h, p_h]$ is computed by solving the following set of local problems defined at element level.

$\mathbf{LP}_h$: For given λ_h, find $[\mathbf{u}_h, p_h] \in U_{\mathcal{K}}^m \times Q_{\mathcal{K}}^l$, at each element $\mathcal{K}$, such that

$$a_{\mathcal{K}}^\delta\left([\mathbf{u}_h, p_h], [\mathbf{v}_h, q_h]\right) + \int_{\partial\mathcal{K}} \beta p_h q_h ds =$$

$$f_{\mathcal{K}}^\delta([\mathbf{v}_h, q_h]) + \int_{\partial\mathcal{K}} \beta\lambda_h q_h ds - \int_{\partial\mathcal{K}} \lambda_h \mathbf{v}_h \cdot \mathbf{n} ds \;\; \forall\, [\mathbf{v}_h, q_h] \in U_{\mathcal{K}}^m \times Q_{\mathcal{K}}^l, \tag{31}$$

where $U_{\mathcal{K}}^m = \mathfrak{R}^m \times \mathfrak{R}^m$ and $Q_{\mathcal{K}}^l = \mathfrak{R}^l$.

This strategy, typical of hybridized mixed finite element methods, will be adopted here in the numerical experiments presented next.

6 NUMERICAL RESULTS

The parameters δ_i and β play an important role in the stability and accuracy of the proposed dual hybrid mixed formulation. However, here we will not invest in finding their optimal values. In all numerical results presented next we adopted the following choice:

$$\delta_1 = \delta_3 = -1; \quad \delta_2 = 1; \quad \beta = \frac{1}{h_e};$$

with h_e denoting the length of the edge e of the element $\mathcal{K}$. We have also adopted equal order approximations for all fields, that is, $l = m = n$.

6.1 Homogeneous media

Initially we consider Helmholtz equation, with k^2 = constant and $f(x, y) = 0$, subject to Dirichlet boundary conditions such that the exact solution is a plane wave (real part only) propagating in θ-direction:

$$w(x, y) = \cos[k(x\cos\theta + y\sin\theta)].$$

The objective is check the convergence of the proposed formulation. In this study we consider $\theta = \pi/6$, $k = 30$ and $\Omega = [0,1] \times [0,1]$. First we check the h-convergence of the stabilized formulation for $l = m = n = 2$ and a sequence of 10×10, 20×20, 40×40 and 80×80 uniform meshes. Figure 1 shows convergence results in $L^2(\Omega)$ norm (left) and $H^1(\Omega)$ seminorm (right) for the pressure approximation p_h of the stabilized dual hybrid formulation (p_{DH}) compared to the its interpolant (p_I) and the local projection (p_{LP}), obtained by solving the local problems $\mathbf{LP}_h$ replacing λ_h (approximation of the lagrange multiplier) by λ (exact value). In Figure 1 we observe pollution effects on both DH and LP solutions for coarse meshes, but these pollution effects disappear with refinement. Figure 2 show results of convergence study using a fixed 10×10 uniform mesh and varying the degree of the polynomial approximations by setting $l = m = n = 2, 3, 4, 5, 6, 7$ sequentially. We can see that highly accurate solutions are obtained by increasing the order of the polynomial approximations (l-refinement). In this study, neq is the the number of equation in the global system associated with the approximation of multiplier λ_h.

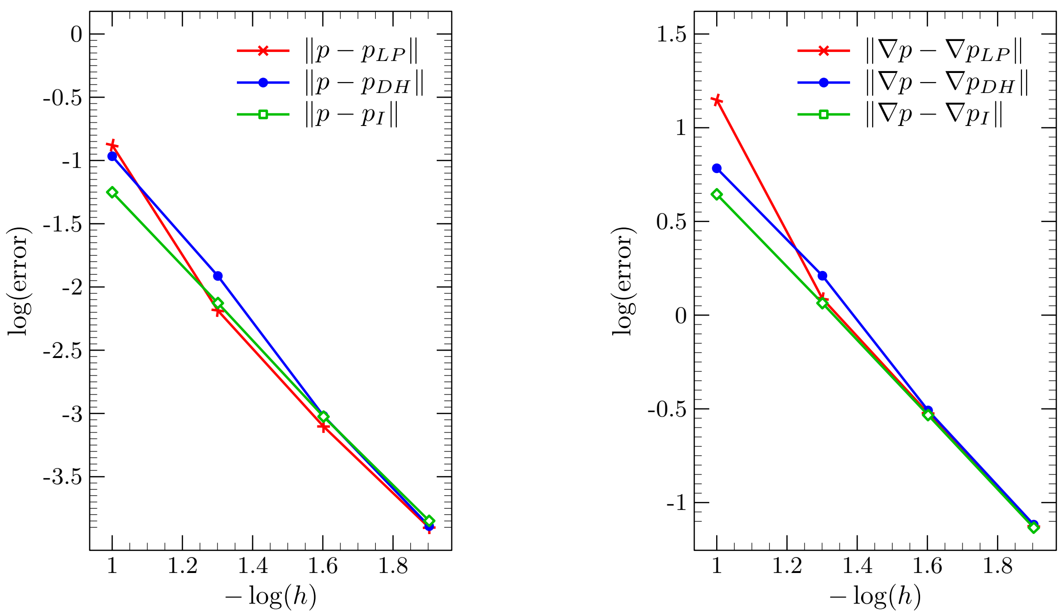

Figure 1: Convergence of the finite element approximations for the homogeneous problem with h-refinement. Error in the L^2-norm (left) and H^1-seminorm (right). Stabilized dual hybrid mixed (DH) solution compared to the local projection (LP) and the interpolant (I).

6.2 Heterogeneous media. Interface problem

We now solve Helmholtz equation in unite domain $\Omega = (-0.5, 0.5) \times (-0.5, 0.5)$ with a discontinuity in the *wavenumber*. We consider $k = k_1$ for $x < 0$, $k = k_2$ for $x > 0$ and impose Dirichlet boundary conditions, weakly as presented in the stabilized dual mixed formulation,

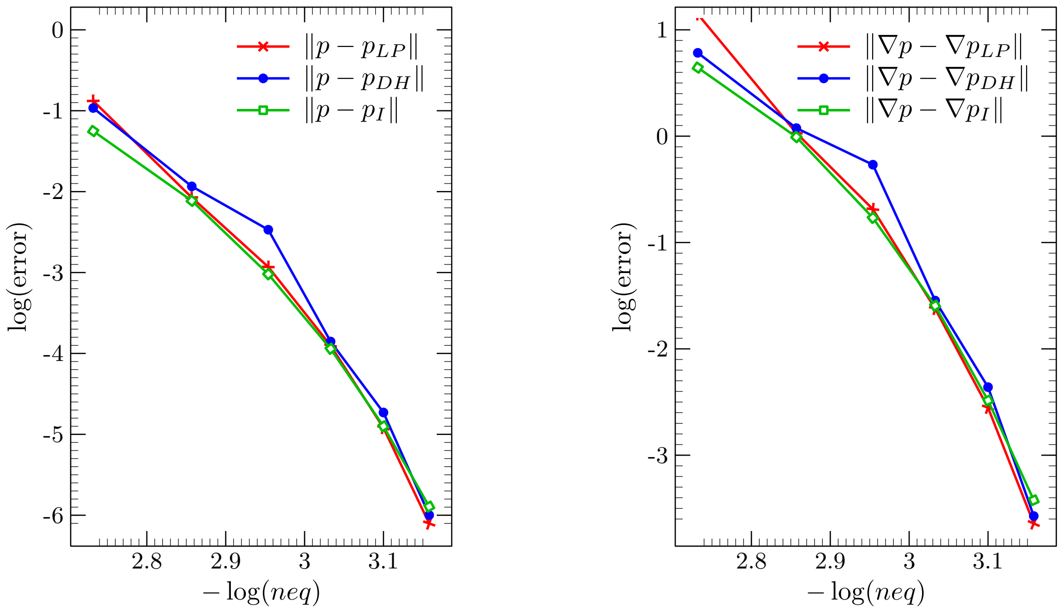

Figure 2: Convergence of the finite element approximations, with the polynomial degree l, for the homogeneous Helmholtz problem. Error in the L^2-norm (left) and H^1-seminorm (right). Stabilized dual hybrid mixed (DH) solution compared to the local projection (LP) and the interpolant (I).

such that the exact solution is given by

for $x < 0$:

$$u(x, y) = \exp[-ik_1(x\cos\theta_1 + y\sin\theta_1)] + R\exp[-ik_1(-x\cos\theta_1 + y\sin\theta_1)]\,, \tag{32}$$

for $x > 0$:

$$u(x, y) = T\exp[-ik_2(x\cos\theta_2 + y\sin\theta_2)]\,, \tag{33}$$

with :

$$R = (k_1\cos\theta_1 - k_2\cos\theta_2)\,/\,(k_1\cos\theta_1 + k_2\cos\theta_2) \tag{34}$$

$$T = 2k_1\cos\theta_1/\,(k_1\cos\theta_1 + k_2\cos\theta_2)\,, \tag{35}$$

$$k_2\sin\theta_2 = k_1\sin\theta_1. \tag{36}$$

In this study we set $k_1 = 30$, $k_2 = 20$ and $\theta_1 = \pi/6$. Again, the same sequence of uniform meshes used in the previous example is adopted in the h-convergence study for $l = m = n = 2$. From Figure 3 we can see pollution effects even higher than those observe in previous example, corresponding to the homogeneous Helmholtz problem, probably due to the an inappropriate choice of the stabilization parameters. Much more accurate solutions are obtained by increasing the degree of the polynomial approximations as shown in Figure 4 where l-convergence results are presented for this heterogeneous problem using a fixed 10×10 uniform mesh and varying the degree of the polynomial approximations by setting $l = m = n = 2, 3, 4, 5, 6, 7$ sequentially, as in the previous study of the homogeneous Helmholtz problem. Finally, we present in Figure 5 a comparison of h-refinement and l-refinement convergence results. From 5 we clearly observe that higher order are much more accurate than lower lower order polynomial approximations for the the same number of degree of freedom (neq).

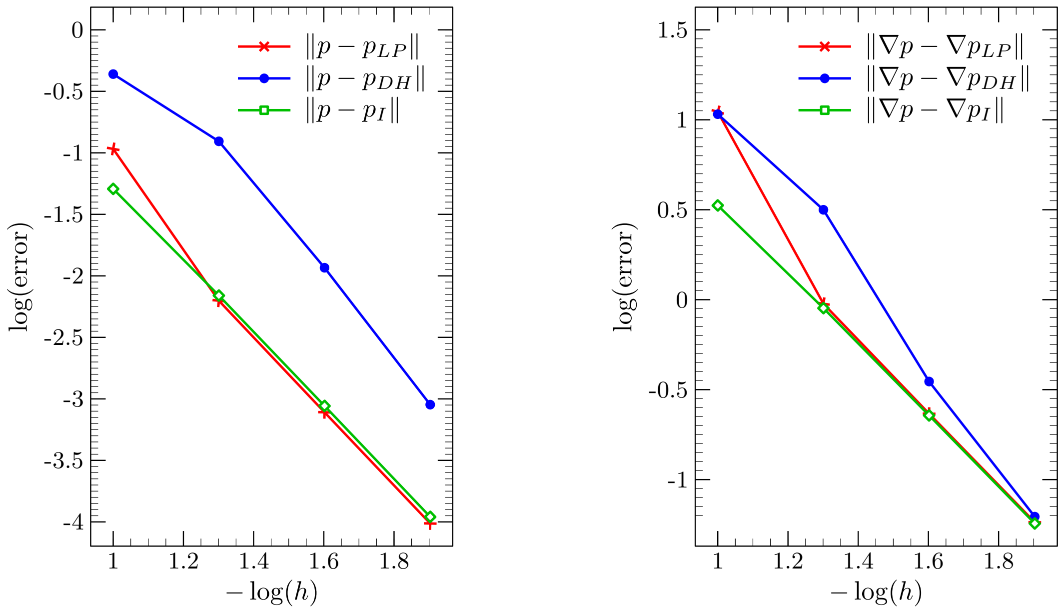

Figure 3: Convergence of the finite element approximations for the heterogeneous problem with h-refinement. Error in the L^2-norm (left) and H^1-seminorm (right). Stabilized dual hybrid mixed (DH) solution compared to the local projection (LP) and the interpolant (I).

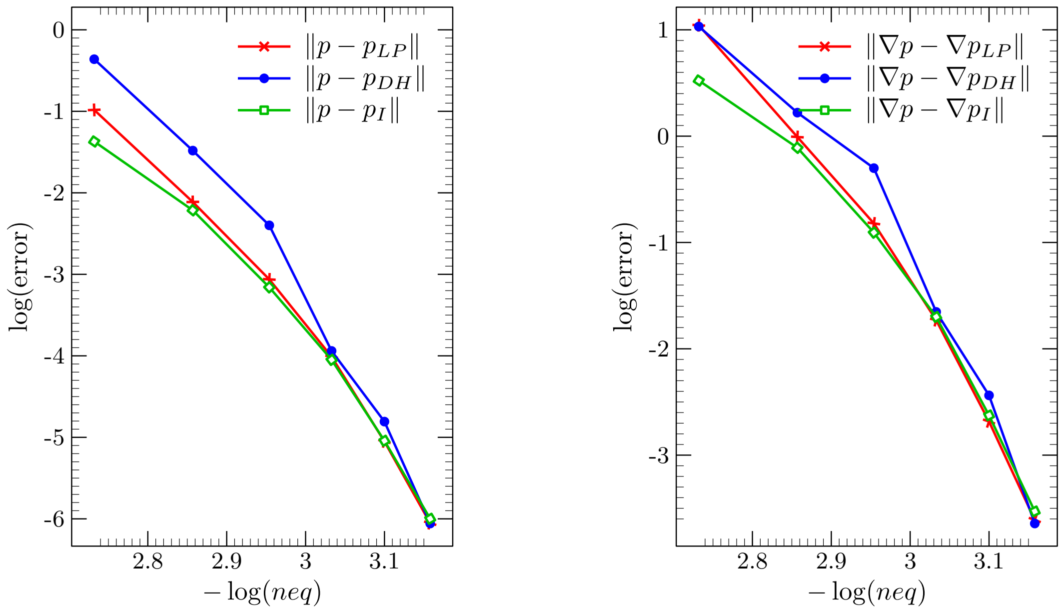

Figure 4: Convergence of the finite element approximations, with the polynomial degree l, for the heterogeneous Helmholtz problem. Error in the L^2-norm (left) and H^1-seminorm (right). Stabilized dual hybrid mixed (DH) solution compared to the local projection (LP) and the interpolant (I).

7 CONCLUSIONS

Stabilized dual hybrid and mixed finite element methods are proposed for solving Helmholtz problems in homogeneous or heterogeneous media using Galerkin and least square residual of the governing equation. Local problems, in the velocity and pressure fields, are solved at element level and these variables are eliminated in favor of the Lagrange multipliers, identified

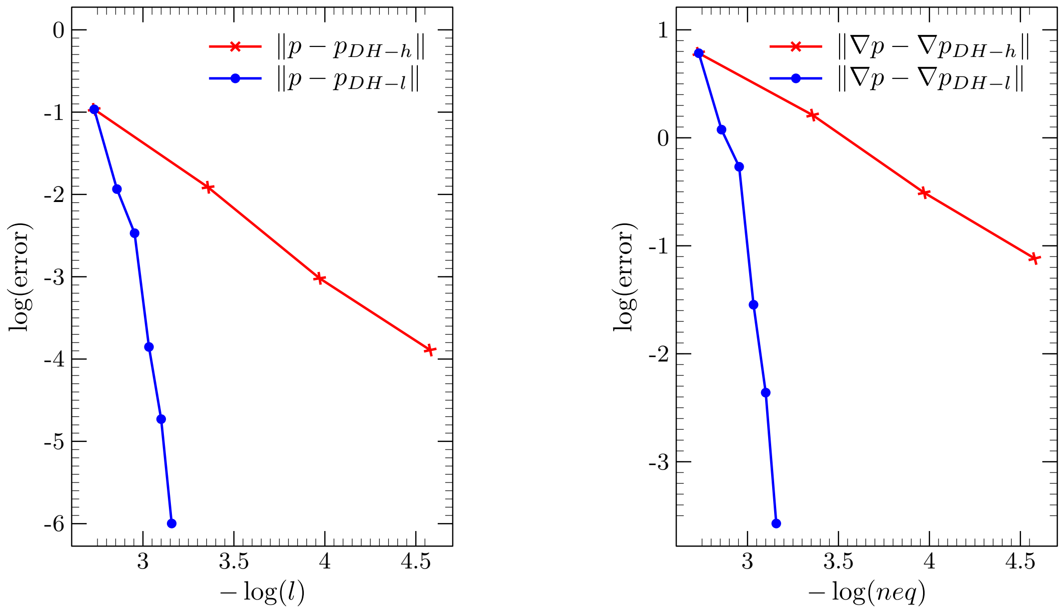

Figure 5: Convergence of the finite element approximations, with the polynomial degree l compared to h-refinement, for the heterogeneous Helmholtz problem. Error in the L^2-norm (left) and H^1-seminorm (right). Stabilized dual hybrid mixed (DH) solution and local projection (LP) corresponding to uniform meshes.

as the trace of the pressure on the element edges of the finite element mesh. A global system is assembled involving only the degrees of freedom associated with the Lagrange multipliers. Polynomial bases are adopted to approximate all fields. Numerical results are presented using equal order approximations for all fields to illustrate the potential of the proposed formulation to efficiently solve Helmholtz problems in homogeneous or heterogeneous media at medium and high frequency regimes. Higher order polynomial approximations are shown to be much more accurate than lower lower order polynomial approximations for the the same number of global degree of freedom.

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