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COMPUTATIONAL STOCHASTIC DYNAMICS BASED ON ORTHOGONAL EXPANSION OF RANDOM EXCITATIONS

X. Frank Xu[1], George Stefanou[2]

[1] Department of Civil, Environmental and Ocean Engineering
Stevens Institute of Technology, Hoboken, NJ 07030, USA
e-mail: xxu1@stevens.edu

[2] Institute of Structural Analysis & Antiseismic Research
National Technical University of Athens, 15780 Athens, Greece
e-mail: stegesa@mail.ntua.gr

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Abstract. *A major challenge in stochastic dynamics is to model nonlinear systems subject to general non-Gaussian excitations which are prevalent in realistic engineering problems. In this work, an n-th order convolved orthogonal expansion (COE) method is proposed. For linear vibration systems, the statistics of the output can be directly obtained as the first-order COE about the underlying Gaussian process. The COE method is next verified by its application on a weakly nonlinear oscillator. In dealing with strongly nonlinear dynamics problems, a variational method is presented by formulating a convolution-type Lagrangian and using the COE representation as trial functions.*

1 INTRODUCTION

To evaluate probabilistic response of a structural dynamic system subject to parametric and external excitations, there are generally two approaches. The first approach uses Fokker-Planck-Kolmogorov (FPK) equation to directly find probability density function (pdf) by assuming a white noise excitation. To solve FPK equation especially for nonlinear systems, various techniques have been proposed, including weighted residual, path integral, etc, which however are all limited to systems of low dimension (up to 4). The second approach includes perturbation method, moment closure method, and statistical equivalent techniques. While the perturbation method is limited to weak nonlinearity, the accuracy of moment closure method and statistical equivalent techniques on highly nonlinear problems remains an open question.

A major deficiency of the existing approaches is their incapability in dealing with general non-Gaussian excitations which are prevalent in realistic engineering problems [5]. The marginal pdf and power spectral density of a loading process play a major role in determining the response of systems, e.g. seismic wave in earthquake engineering. Therefore, a new approach to model dynamic systems subject to non-Gaussian excitations is highly desired.

A novel stochastic computation method based on orthogonal expansion of random fields is recently proposed [6]. In this study, the idea of orthogonal expansion is extended to the so-called n-th order convolved orthogonal expansions (COE) especially in dealing with nonlinear dynamics. For linear vibration systems, the statistics of the output can be directly obtained as the first-order COE about the underlying Gaussian process. The COE is next verified by its application on a weakly nonlinear oscillator. In dealing with strongly nonlinear dynamics problems, a variational method is presented by formulating the convolution-type Lagrangian and using the COE representation as trial functions [7]. Theoretically, substitution of the trial response function into the Lagrangian will lead to the optimal solution. The effect of using different trial functions (COE of different orders) on the accuracy and efficiency of the proposed approach will be examined in a forthcoming paper.

2 CONVOLVED ORTHOGONAL EXPANSIONS

2.1 The zero-th order convolved orthogonal expansion

An underlying stationary Gaussian excitation $h_1(t,\vartheta)$ is characterized with the autocorrelation function $\rho(t)$ and unit variance, where $\vartheta \in \Theta$ indicates a sample point in random space. Based on the so-called diagonal class of random processes [1], the zero-th order convolved (or memoryless) orthogonal expansion of $h_1(t,\vartheta)$ is proposed as [6]

$$u(t,\vartheta) = \sum_{i=0} u_i(t) h_i(t,\vartheta) \tag{1}$$

where the random basis function h_i corresponds to the i-th degree Hermite polynomial with $h_0 = 1$. According to the generalized Mehler's formula [4] the correlations among the random basis functions are given as

$$\begin{aligned} R_{s_1 s_2 \cdots s_n}(t_1, t_2, \cdots, t_n) &= \overline{h_{s_1}(t_1,\vartheta) \cdots h_{s_n}(t_n,\vartheta)} \\ &= \sum_{\nu_{12}=0}^{\infty} \cdots \sum_{\nu_{n-1,n}=0}^{\infty} \delta_{s_1 r_1} \cdots \delta_{s_n r_n} \prod_{j<k} \frac{\rho^{\nu_{jk}}(t_j - t_k)}{\nu_{jk}!} s_1! \cdots s_n! \end{aligned} \tag{2}$$

where $r_k = \sum_{j \neq k} \nu_{jk}$, $\nu_{jk} = \nu_{kj}$, $\delta_{s_k r_k} = \begin{cases} 1 & s_k = r_k \\ 0 & s_k \neq r_k \end{cases}$, and the overbar denotes ensemble average. Following Eq. (2), the two-point and three-point correlation functions are specifically obtained as

$$R_{ij}(t_1 - t_2) = \overline{h_i\,(t_1, \vartheta) h_j\,(t_2, \vartheta)} = \delta_{ij} i! \rho^i (t_1 - t_2) \tag{3}$$

$$\begin{aligned} &R_{ijk}(t_1 - t_2, t_1 - t_3, t_2 - t_3) = \overline{h_i\,(t_1, \vartheta) h_j\,(t_2, \vartheta) h_k\,(t_3, \vartheta)} \\ &= \frac{i!\,j!\,k!}{i'!\,j'!\,k'!} \rho^{k'}(t_1 - t_2) \rho^{j'}(t_1 - t_3) \rho^{i'}(t_2 - t_3) \\ &i' = \frac{j + k - i}{2}, j' = \frac{i + k - j}{2}, k' = \frac{i + j - k}{2} \end{aligned} \tag{4}$$

where i', j', k' must be non-negative integers, otherwise $R_{ijk} = 0$.

The correlation relations can be extended to the derivatives of the random basis functions, e.g.

$$\begin{aligned} R_{ij,pq}(t_1 - t_2) &= \overline{h^{(0)}_{i,p}(t_1, \vartheta) h^{(0)}_{j,q}(t_2, \vartheta)} = \delta_{ij} i! \frac{\partial^{p+q}}{\partial t_1^p \partial t_2^q} \rho^i (t_1 - t_2) \\ &= \delta_{ij} (-1)^q i! \frac{\partial^{p+q}}{\partial \tau^{p+q}} \rho^i (\tau) \end{aligned}$$

where $\tau = t_1 - t_2$, and the subscripts p, q denote p-th and q-th derivatives. Similarly, the derivations can be made for the convolution of the random basis functions, e.g.

$$\begin{aligned} C_{ij} &= \overline{h_i\,(t_1, \vartheta) * h_j\,(t_2, \vartheta)} = \delta_{ij} i! \int_{-\infty}^{\infty} \rho^i (t_1 - 2t_2) dt_2 = \delta_{ij} i! \tau_i \\ C_{ij,11} &= \overline{h_{i,1}(t_1, \vartheta) * h_{j,1}(t_2, \vartheta)} = \delta_{ij} i! \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t_1 \partial t_2} \rho^i (t_1 - 2t_2) dt_2 = 0 \end{aligned} \tag{5}$$

where $\tau_i = \int_{-\infty}^{\infty} \rho^i (t) dt$ is the correlation time.

2.2 *n*-th order convolved orthogonal expansion

The idea of the memoryless orthogonal expansion presented above can be generalized to an n-th order convolved orthogonal expansion (COE) for representation of nonlinear output processes

$$u(t, \vartheta) = \sum_{n=0} \sum_{i=0} u_i^{(n)}(t) h_i^{(n)}(t, \vartheta) \tag{6}$$

$$h_i^{(n)}(t, \vartheta) = \overbrace{g * g * \cdots * g}^{n} * h_i = g^{*n} * h_i \tag{7}$$

where g is a given kernel, and the symbol * denotes the convolution operator. For notational simplicity, the superscript (0) of the zero-th order COE is usually dropped throughout the paper. The memoryless orthogonal expansion thus corresponds to the zero-th order COE with n=0 in (6). The correlation functions of the n-th order basis functions are therefore obtained as

$$\begin{aligned} R_{s_1\cdots s_n}^{m_1\cdots m_n}(t_1,t_2,\cdots,t_n) &= \overline{h_{s_1}^{(m_1)}(t_1,\vartheta)\cdots h_{s_n}^{(m_n)}(t_n,\vartheta)} \\ &= \sum_{\nu_{12}=0}^{\infty}\cdots\sum_{\nu_{n-1,n}=0}^{\infty}\delta_{s_1r_1}\cdots\delta_{s_nr_n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g^{*m_1}(t_1,\tau_1)\cdots \\ &\quad g^{*m_n}(t_n,\tau_n)\prod_{j<k}\frac{\rho^{\nu_{jk}}(t_j-t_k)}{\nu_{jk}!}s_1!\cdots s_n!\,d\tau_1\cdots d\tau_n \end{aligned} \tag{8}$$

with the two-point correlations

$$\begin{aligned} R_{ij}^{mn}(t_1-t_2) &= \overline{h_i^{(m)}(t_1,\vartheta)h_j^{(n)}(t_2,\vartheta)} \\ &= \delta_{ij}i!\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^{*m}(t_1,\tau_1)g^{*n}(t_2,\tau_2)\rho^i(\tau_1-\tau_2)d\tau_1 d\tau_2 \end{aligned} \tag{9}$$

The derivatives of the n-th order basis functions can be similarly obtained, e.g.

$$\begin{aligned} R_{ij,pq}^{mn}(t_1-t_2) &= \overline{h_{i,p}^{(m)}(t_1,\vartheta)h_{j,q}^{(n)}(t_2,\vartheta)} \\ &= \delta_{ij}i!\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g^{*m}(t_1,\tau_1)g^{*n}(t_2,\tau_2)\frac{\partial^{p+q}}{\partial\tau_1^p\partial\tau_2^q}\rho^i(\tau_1-\tau_2)d\tau_1 d\tau_2 \end{aligned} \tag{10}$$

By letting $U=\Phi(u)$, $H=\Phi(h)$, $S=\Phi(R)$, $G^n=\Phi(g^{*n})$ and $S^{*i}=\Phi(\rho^i)$, with Φ being the Fourier transform operator, we specially rewrite the two-point correlation functions of the COE basis functions in frequency domain

$$S_{ij}^{mn}(\omega) = \overline{H_i^{(m)}(\omega,\vartheta)\tilde{H}_j^{(n)}(\omega,\vartheta)} = \delta_{ij}i!G^m(\omega)\tilde{G}^n(\omega)S^{*i}(\omega) \tag{11}$$

$$\begin{aligned} S_{ij,pq}^{mn}(\omega) &= \left(\omega\sqrt{-1}\right)^{p+q}\overline{H_i^{(m)}(\omega,\vartheta)\tilde{H}_j^{(n)}(\omega,\vartheta)} \\ &= \delta_{ij}i!\left(\omega\sqrt{-1}\right)^{p+q}G^m(\omega)\tilde{G}^n(\omega)S^{*i}(\omega) \end{aligned} \tag{12}$$

where the tilde denotes complex conjugate. Note that in the cases of stationary correlation functions, it specially follows

$$\begin{aligned} &\overline{H_{i,p}^{(m)}(\omega_k,\vartheta)\tilde{H}_{j,q}^{(n)}(\omega_l,\vartheta)} = \delta_{kl}S_{ij,pq}^{mn}(\omega) \\ &\overline{H_{i,p}^{(m)}(\omega_k,\vartheta)H_{j,q}^{(n)}(\omega_l,\vartheta)} \\ &= \delta_{kl}\delta_{ij}i!\left(\omega\sqrt{-1}\right)^{p+q}G^m(\omega)\tilde{G}^n(\omega)\tau_i \end{aligned} \tag{13}$$

Remark: The advantage of the n-th order COE (6) can be demonstrated by comparing it with the classical Volterra series expansion

$$u(t,\vartheta)=\sum_{n=0}\frac{1}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}k^{(n)}(t_1,t_2\cdots t_n)h(t-t_1,\vartheta)$$
$$h(t-t_2,\vartheta)\cdots h(t-t_n,\vartheta)dt_1dt_2\cdots dt_n \tag{14}$$

The Volterra representation typically suffers from severe difficulties in solving for the unknown kernels $k^{(n)}$. In the COE representation, the kernels are all explicitly given, and the problem is significantly reduced to solving of the unknown coefficients $u_i^{(n)}$.

3 THE COE METHOD ON RANDOM VIBRATION

3.1 Linear oscillators

Suppose the linear oscillator

$$\begin{aligned}&\ddot{u}+2\zeta\omega_n\dot{u}+\omega_n^2u=f\\&u(0)=\dot{u}(0)=0\end{aligned} \tag{15}$$

is subjected to a non-stationary non-Gaussian translation input, i.e.

$$f(t,\vartheta)=\sum_{i=0}f_i(t)h_i(t,\vartheta) \tag{16}$$

By using the Green function

$$\begin{aligned}&g(t)=\frac{1}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)\\&\omega_d=\omega_n\sqrt{1-\zeta^2}\\&G(\omega)=\frac{1}{\omega_n^2-\omega^2+\sqrt{-1}2\zeta\omega\omega_n}\end{aligned} \tag{17}$$

the first three correlations of the non-stationary output u can be directly calculated from

$$\bar{u}(t)=\int_0^t g(t-\tau)f_0(\tau)d\tau \tag{18}$$

$$R_{uu}(t_1,t_2)=\int_0^{t_2}\int_0^{t_1}g(t_1-\tau_1)g(t_2-\tau_2)\sum_{i=0}i!$$
$$\rho^i(\tau_1-\tau_2)f_i(\tau_1)f_i(\tau_2)d\tau_1d\tau_2 \tag{19}$$

$$R_{uuu}(t_1,t_2,t_3)=\int_0^{t_3}\int_0^{t_2}\int_0^{t_1}g(t_1-\tau_1)g(t_2-\tau_2)g(t_3-\tau_3)$$
$$\sum_{i,j,k=0}R_{ijk}(\tau_1-\tau_2,\tau_1-\tau_3,\tau_2-\tau_3)d\tau_1d\tau_2d\tau_3 \tag{20}$$

where R_{ijk} is given in Eq. (4).

When the excitation in Eq. (15) is stationary, the output can be directly given as

$$u(t,\vartheta)=\sum_{i=0} f_i\ h_i^{(1)}(t,\vartheta) \tag{21}$$

which is a special case of the COE representation (6). Note that, with the Green function g and the underlying Gaussian process being given, the stationary probability density function (pdf) of the output in Eq. (21) can be rapidly estimated by using Monte Carlo method in the frequency domain.

A numerical example for application of the COE on linear oscillator is given in [6]. With regard to the multi-degree-of-freedom linear systems, the oscillator equations given above can be directly applied by using the modal decomposition as shown in [8].

3.2 Weakly nonlinear oscillators

A Duffing oscillator subjected to a Gaussian white noise excitation with intensity D is considered

$$\ddot{u}+2\zeta\omega_n\dot{u}+\omega_n^2(u+\alpha u^3)=W \tag{22}$$

The Gaussian response of the linear filter can be given as $u_0=\sigma_0 h_1$, and h_1 is characterized by unit variance and power spectral density (PSD)

$$S=\frac{D}{\sigma_0^2}\left|G(\omega)\right|^2 \tag{23}$$

$$\sigma_0^2=D\int_{-\infty}^{\infty}\left|\frac{1}{\omega_n^2-\omega^2+\sqrt{-1}2\zeta\omega\omega_n}\right|^2 d\omega=\frac{D\pi}{2\zeta\omega_n^3} \tag{24}$$

For small α, the nonlinear output of Eq. (22) can be approximated as

$$u=\sigma_0 h_1-\alpha\omega_n^2\sigma_0^3 g*h_1^3+3\alpha^2\omega_n^4\sigma_0^5 g*\left(h_1^2 g*h_1^3\right)+O(\alpha^3) \tag{25}$$

By noting $h_1^3=h_3+3h_1$ and $h_1^2=h_2+1$, Eq. (25) can be rewritten in terms of the random basis functions

$$\begin{aligned}u=\sigma_0 h_1-\alpha\omega_n^2\sigma_0^3 g*\left(h_3+3h_1\right)+\\ 3\alpha^2\omega_n^4\sigma_0^5 g*\left[(h_2+1)g*(h_3+3h_1)\right]+O(\alpha^3)\end{aligned} \tag{26}$$

$$\begin{aligned}U=\sigma_0 H_1-\alpha\omega_n^2\sigma_0^3 G\left(H_3+3H_1\right)+\\ 3\alpha^2\omega_n^4\sigma_0^5 G\left[(H_2+\delta(0))*G(H_3+3H_1)\right]+O(\alpha^3)\end{aligned} \tag{27}$$

By using the correlations of Eqs. (3)-(4) and (11), it follows that the stationary mean

$$\bar{u}=O(\alpha^3) \tag{28}$$

and the stationary PSD

$$\begin{aligned}S_{UU}=\overline{U\tilde{U}}=\sigma_0^2 S-3\alpha\omega_n^2\sigma_0^4(G+\tilde{G})S+\alpha^2\omega_n^4\sigma_0^6[9S(\left|G\right|^2+G^2\\ +\tilde{G}^2)+6\left|G\right|^2 S^{*3}]+O(\alpha^3)\end{aligned} \tag{29}$$

Since

$$\int_{-\infty}^{\infty} \frac{2(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} \left| \frac{1}{\omega_n^2 - \omega^2 + \sqrt{-1}2\zeta\omega\omega_n} \right|^2 d\omega = \frac{\pi}{2\zeta\omega_n^5} \tag{30}$$

the variance calculated from the first two terms of Eq. (29) is simply obtained as

$$\sigma^2 = \sigma_0^2(1 - 3\alpha\sigma_0^2) \tag{31}$$

which is identical to the result obtained using other approaches, e.g. [2,3]. In addition to serving as verification to the COE method, this example shows simplicity and efficiency of the orthogonal expansions in nonlinear problems.

3.3 Strongly nonlinear oscillators

For strongly nonlinear systems, the perturbation method is inapplicable. In this part, a variational method will be presented following the variational principles formulated for random media elastodynamics [7]. The variational functional, or Lagrangian, of a nonlinear oscillator

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2\big(u + g(u,\dot{u})\big) = f \tag{32}$$

can be formulated by using the convolution form

$$\delta\ell = \delta u * [\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2\big(u + g(u,\dot{u})\big) - f] = 0 \tag{33}$$

For a Duffing oscillator $g(u,\dot{u}) = \alpha u^3$, the Lagrangian is derived from Eq. (33) as

$$\begin{aligned} \ell(u) &= \frac{1}{2}\dot{u} * \dot{u} + \zeta u * \dot{u} + \frac{1}{2}\omega_n^2 u * u \\ &+ \alpha\omega_n^2(u^3 * u - \frac{3}{4}u^2 * u^2) - f * u + \dot{u}(0)u \end{aligned} \tag{34}$$

where any trial function u satisfies the specified initial condition $u(0)$. To the authors' knowledge, the convolution Lagrangian (34) is the first variational form formulated for nonlinear dissipative systems. It is especially noted that the classical point-wise Lagrangian form does not work on the dissipative term.

For nonlinear random vibrations, the stochastic Lagrangian is directly obtained by taking ensemble average of Eq. (34), i.e.

$$\delta\bar{\ell} = 0 \tag{35}$$

with the trial function u based on the COE representation (6).

For stationary solutions, Eq. (35) can be rewritten in frequency domain as

$$\begin{aligned} &\delta\bar{L}(U) = 0 \\ &\bar{L}(U) = \left(-\frac{1}{2}\omega^2 + \sqrt{-1}\omega\zeta + \frac{1}{2}\omega_n^2 \right)\overline{U^2} + \\ &\alpha\omega_n^2\left[\overline{(U * U * U)U - \frac{3}{4}(U * U)^2} \right] - \overline{FU} \end{aligned} \tag{36}$$

Suppose the excitation is stationary

$$f(t,\vartheta) = \sum_{i=0} f_i h_i(t,\vartheta) \tag{37}$$

and choose the zeroth-order COE

$$u(t,\vartheta) = \sum_{i=0} u_i h_i(t,\vartheta) \tag{38}$$

as the trial function for the stationary solution. By substituting Eq. (38) into Eqs. (34)-(35) and taking derivative with respect to u_i, it leads to a series of equations to solve for u_i

$$\frac{\partial \overline{\ell}}{\partial u_i} = 0 \tag{39}$$

Similarly the first- or higher-order COE can be chosen as the trial function. The detail of numerical examples and investigation of accuracy and computational efficiency of the different trial functions will be provided in a forthcoming paper.

4 CONCLUDING REMARK

By developing a diagonal class of random fields/stochastic processes to represent high-dimensional uncertainty, the proposed convolved orthogonal expansion method opens a new direction to deal with nonlinear stochastic dynamics. The advantage is especially noted for its efficiency in computing of large and nonlinear dynamical systems, in comparison with the classical Volterra series representation and the recently developed random variable based polynomial chaos expansions.

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