COMPDYN 2011
III ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis, V. Plevris (eds.)
Corfu, Greece, 25–28 May 2011

# THEORETICAL ANALYSIS AND NUMERICAL MODELING OF ELASTIC WAVE PROPAGATION IN HONEYCOMB-TYPE THIN LAYER

**B. –Y. Tian, B. Tie, D. Aubry**

LMSSMat (CNRS UMR8579), Ecole Centrale Paris
Grande Voie des Vignes, 92295 Châtenay-Malabry Cedex, France
(biyu.tian, bing.tie, denis.aubry)@ecp.fr

**Keywords:** Honeycomb sandwich, Elastic wave propagation, periodic media, Bloch wave transformation.

**Abstract.** *The present work is devoted to a theoretical analysis and numerical modeling of the phenomena of high-frequency wave propagation in honeycomb-type periodic media. A one-dimensional periodic elastic rod model and a two-dimensional periodic elastic beam hexagonal model are considered. The Bloch wave direct and inverse transformations are applied to find general or particular solutions. By identifying eigenfrequencies and the corresponding eigen-modes of the periodic systems, important information, such as the frequency bandgaps and the diffracted waves caused by the periodic cells, is obtained. The aims are to improve our under-standing of how HF waves are transmitted and attenuated in the studied periodic media and to link the theoretical analyses to our numerical assessments.*

# 1 INTRODUCTION

Sandwich shells using honeycomb-type core are very useful composite materials to design lightweight but high-strength structures, especially in aerospace and aeronautics industries, for the honeycomb core provides an efficient solution to increase bending stiffness without significant increase in structural weight. When high-frequency (HF) flexural waves propagating in such sandwich shells, the involved wavelengths are of the same order of the characteristic lengths of the honeycomb cells, or even shorter, the honeycomb cellular microstructure can interact with the waves and highly perturb the propagation phenomena, as they are governed by the geometry and the material properties of the cells. Indeed, due to the periodic geometric and material discontinuities within the cellular structure, waves can propagate completely or be attenuated even stopped, depending upon the involved frequencies and upon the spatial direction, so we observe the existence of passing and stop bands in frequency domain ([1]). As the classical homogenized models cannot take into account this kind of interaction, they fail to correctly describe the transient flexural behaviors of the sandwich shells in HF ranges ([2]). Therefore, relevant analytical and numerical models should be established by understanding how HF waves propagate in a honeycomb thin layer and interact with its cellular structure.

To simplify and optimize numerical models but still taking into account the detail characteristics of the cellular microstructure, the Bloch wave theory is more and more widely used now. The basic idea is to transform a non-periodic function defined in a periodic structure to a set of periodic functions having the same periodicity as the structure. Therefore the study of the original function in the whole structure can be replaced by considering the periodic functions in a unique cell. The theory was firstly applied in the quantum mechanics to solve the Schrödinger equation in periodic lattice of particles ([3]). Now it has been introduced to the structural mechanics. In the literature, the dispersion equations relating the frequency to the Bloch wave vector have been given for several types of honeycomb cells composed by beams and the effects of the cellular characteristics, for example the slenderness ratio and the internal angle, have been discussed in [4] and [5].

The honeycomb thin layer that we are interested is composed by thin-walled hexagonal regular cells, which give rise to an in-plane two-dimension periodic hexagonal cellular microstructure. Each hexagonal cell has two parallel sides whose thickness is doubled due to the manufacturing process, which forms ribbons with a semi-hexagonal profile and bonds them together in pairs to obtain hexagonal cells. These doubled thickness walls result in plane anisotropic properties of the honeycomb thin layer. Our first research works presented herein consist in the development of theoretical and numerical modeling tools based on the Bloch wave theory and then in their validation by application to a one-dimensional (1D) periodic elastic rod model and a two-dimensional (2D) elastic beam hexagonal model, for which the previously mentioned double thickness aspect is considered. The Bloch wave modes inside one cell are considered and the dispersion equations are obtained.

The paper is organized as follows: The section 2 introduces the direct and inverse Bloch wave transformations. The section 3 is devoted to the theoretical and numerical analyses of the wave propagation in a 1D periodic structure composed by elastic rods. The modeling of a more complex 2D periodic structure composed by elastic beams is then presented in the section 4, before some concluding remarks in the section 5.

## 2 BLOCH WAVE THEORY

Let us consider a periodic structure $\Omega$ of space dimension *N*, a *primitive cell* $Q_0$ and a set of basis vector $\mathbf{e}_i$ (i=1,...,N), called *direct cell basis*, are defined so that the entire structure $\Omega$ can be obtained by repeating the primitive cell along the direct cell basis. Dual to the primitive cell $Q_0$, a *reciprocal cell* $Q_0^*$ and a *reciprocal cell basis*, $\mathbf{e}_j^*$, are defined, which satisfies the following relation:

$$\mathbf{e}_i . \mathbf{e}_j^* = \delta_{ij} \tag{1}$$

where $\delta_{ij}$ is the Kronecker delta. The reciprocal cell is also called the *first Brillouin zone* [6]. The areas of $Q_0$ and $Q_0^*$verify the following equation:

$$\text{volume}(Q_0)\text{volume}(Q_0^*) = 1 \tag{2}$$

For any non periodic function $\mathbf{V}(\mathbf{x})$ defined on $\Omega$, the Bloch wave theory states that, for each wave vector $\mathbf{k}$ restricted in the first Brillouin zone $Q_0^*$, the Bloch transformation $\mathbf{V}^B(\mathbf{x},\mathbf{k})$ of $\mathbf{V}(\mathbf{x})$ is a periodic function having the same periodicity as the periodic structure, which is also called *Bloch wave function*:

$$\mathbf{V}^B(\mathbf{x},\mathbf{k}) = \sum_n \mathbf{V}(\mathbf{x} + n_i\lambda_i\mathbf{e}_i)e^{i\mathbf{k}.(\mathbf{x}+n_i\lambda_i\mathbf{e}_i)} \tag{3}$$

Then, $\mathbf{V}(\mathbf{x})$ can be recovered by the following inverse Bloch transformation:

$$\mathbf{V}(\mathbf{x}) = \frac{1}{\text{volume}(Q_0^*)}\int_{Q_0^*} \mathbf{V}^B(\mathbf{x},\mathbf{k})e^{-i\mathbf{k}.\mathbf{x}}d\mathbf{k} \tag{4}$$

By virtue of the Bloch wave transformation, the propagation phenomena of HF elastic waves, transmission, reflection, conversion and attenuation, through the periodic microstructure can be understood by investigating the Bloch wave modes within the primitive cell, which allows to save lots of the efforts when doing analysis and simulation.

## 3 WAVE PROPAGATION IN 1D ELASTIC ROD MODEL

The first periodic structure considered herein is a 1D elastic rod model, which is a topology of a primitive cell composed by two rigidly jointed elastic rods respectively of lengths $l_1$ and $l_2$, therefore the period of the 1D medium is $\lambda = l_1 + l_2$ (Figure 1).

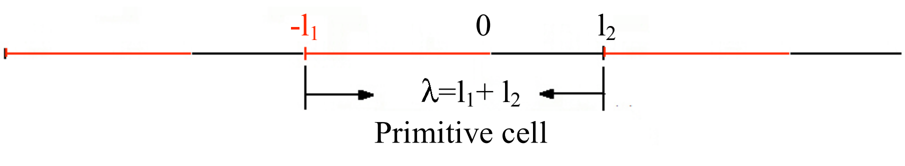


Figure 1: 1D elastic rod model

The equilibrium eigen equation of the i-th rod (i=1, 2) reads as:

$$\frac{d}{dx}\left[E_i\frac{dU(x)}{dx}\right] = -\rho_i\omega^2 U(x) \tag{5}$$

where ($E_i$, $\rho_i$) denote the Young's Modulus and the density, $\omega$ the eigenfrequency and U(x) the corresponding eigenmode. By applying Bloch wave transformation to (5), the following Bloch eigen equation is obtained:

$$E_i \frac{\partial^2 U^B(x,k)}{\partial x^2} - 2ikE_i \frac{\partial U^B(x,k)}{\partial x} - E_i k^2 U^B(x,k) = -\rho_i \omega^2 U^B(x,k) \tag{6}$$

with $k \in Q_0^* = [0, 2\pi/\lambda[$ the Bloch wave vector. Given k, the goal is to find the eigenvalue $\omega$ and the corresponding Bloch eigenmode $U^B(x, k)$ of the eigen problem (6). It is straightforward that the general solution of (6) has the following analytical form:

$$U_i^B(x,k) = a_i e^{i\left(k+\frac{\omega}{c_i}\right)x} + b_i e^{i\left(k-\frac{\omega}{c_i}\right)x} \tag{7}$$

where, for the i-th rod, $c_i = \sqrt{E_i/\rho_i}$ is the wave velocity and ($a_i$, $b_i$) are four constants to be determined.

### 3.1 Dispersion equation

To calculate the four constants ($a_i$, $b_i$), (i=1, 2), we consider the following the interface conditions between the two rods inside the primitive cell and the periodic conditions on its extremities:

$$\begin{aligned} &U^B(0^-) = U^B(0^+),\ N^B(0^-) = N^B(0^+) \\ &U^B(-l_1) = U^B(l_2),\ N^B(-l_1) = N^B(l_2) \end{aligned} \tag{8}$$

where $N^B = dU^B/dx - ikU^B$ denotes the 1D Bloch axial force vector. Substituting the general solution form (7) to these conditions (8), a system of four linear equations is obtained. To ensure that system admits nontrivial solutions, its determinant must vanish, which finally gives a relation between k and $\omega$, called *dispersion equation*.

For our 1D rod model, the following analytical dispersion equation can be obtained:

$$\cos(\lambda k) = \cos(\omega T_1)\cos(\omega T_2) - (\frac{Z_1}{2Z_2} + \frac{Z_2}{2Z_1})\sin(\omega T_1)\sin(\omega T_2) \tag{9}$$

where $Z_i = \rho_i c_i$ denotes the characteristic acoustic impedance and $T_i = l_i / c_i$ the time for wave to propagate through the whole i-th rod. Using (9), for each given frequency $\omega$, a Bloch wave vector $k = k_r + i\, k_{im}$ can be found and result in the following Bloch eigenmode in the i-th rod:

$$U_i^B(x,k) = a_i e^{i\left(k_r+\frac{\omega}{c_i}\right)x} e^{-k_{im}x} + b_i e^{i\left(k_r-\frac{\omega}{c_i}\right)x} e^{-k_{im}x} \tag{10}$$

Therefore, when k is real ($k_{im} = 0$), the Bloch wave mode $U_i^B$ is a propagating mode, which is transmitted to the adjacent cells with the same amplitude and there is no energy losing when propagating through the periodic medium. Otherwise, when k is complex or pure imaginary ($k_{im} \neq 0$), $U_i^B$ is an evanescent Bloch wave mode, it vanishes rapidly when propagating to the adjacent cells and the pure energy exchanging between periodic cells is equal to zero ([1]).

The frequency ranges that give real values of k is called *passing band* and the others *stop band*. The figure 2 gives the frequency bandgap of the 1D model: The red curves plot the two real solutions of k and so indicate the passing bands, while the blue curves present the stop bands, by plotting the imaginary part $k_{im}$ of the two complex solutions of k: $\frac{j\pi}{\lambda} + i\,k_{im}$, (j=0,1). We remark that the location and the width of the stop bands mainly depend on the ratio between the two characteristic acoustic impedances $Z_1/Z_2$ ([7]).

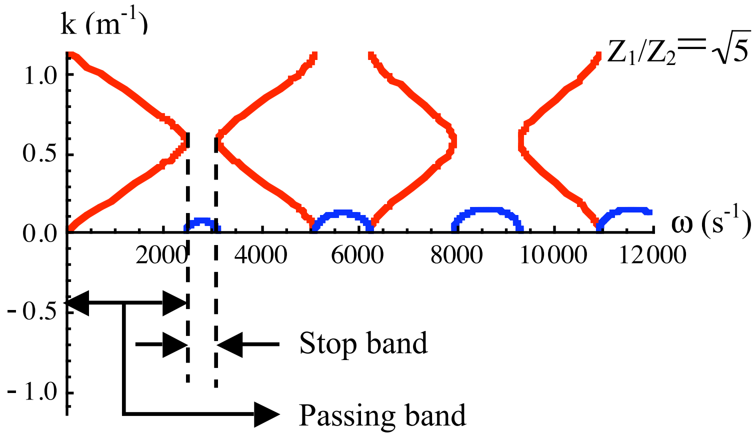


Figure 2: Dispersion relation of eigenfrequency on k, $Z_1/Z_2 = \sqrt{5}$

### 3.2 Diffracted wave analysis

Now, we consider an incident plane wave $u_0(x,t) = e^{i(k_0 x - \omega_0 t)}$ and investigate how its propagation through the 1D periodic structure that is perturbed by the periodic cells. To do this, the wave solution u(x, t) is decomposed into two parts:

$$u(x,t) = \left[e^{ik_0 x} + u_d(x)\right]e^{-i\omega_0 t} \tag{11}$$

where ($k_0$, $\omega_0$) are the wave vector and the angular frequency of the incident wave. We are interested in finding $u_d(x)$, the diffracted wave caused by the periodicity of the cells, which indicate us in fact what's the difference between the wave motion in a periodic medium and in a homogenous medium.

By substituting the equation (11) in the equilibrium eigenequation (5), we get:

$$\frac{d}{dx}\left[E_i \frac{du_d(x)}{dx}\right] + \underbrace{\left\{\frac{d}{dx}\left[E_i \frac{du_0(x)}{dx}\right] + \rho_i \omega_0^2 u_0(x)\right\}}_{f_e} = -\rho_i \omega_0^2 u_d(x) \tag{12}$$

where the second term of the left member is considered as an external loading $f_e$ due to the incident wave. By expanding $u_d(x)$ as a linear combination of eigenmode U(x):

$$u_d(x) = \sum_n \alpha_n U_n(x) \tag{13}$$

the Bloch diffracted wave $u_d^B$ reads as:

$$u_d^B(x,k) = \sum_n \alpha_n(k) U_n^B(x,k) \tag{14}$$

where $U^B(x,k)$ is the Bloch eigenmode and $\alpha_n(k)$ the Bloch coefficient.

Since we have already calculated $U^B(x, k)$, we look for $\alpha_n(k)$ now. Applying the Bloch wave transformation to the equation (12) to get Bloch equilibrium eigenequation and substituting the equation (14) into it, we get:

$$\sum_n \rho_i(\omega^2 - \omega_0^2)\alpha_n(k)U_n^B(x, k) = f_e^B(x, k) \tag{15}$$

where $f_e^B$ is the Bloch external loading that can be expanded using the same Bloch modes $U^B(x, k)$:

$$f_e^B(x, k) = \sum_n F_n(k)U_n^B(x, k) \tag{16}$$

Therefore $\alpha_k$ can be obtained in the following way:

$$\alpha_n(k) = \frac{F_n(k)}{\rho_i(\omega^2 - \omega_0^2)} \tag{17}$$

Subsequently the $u_d^B$ is got using (14) and finally $u_d$ is obtained using the inverse Bloch transformation (4).

The figure 3 illustrates the ratio of amplitude between $u_0$ and $u_d$ with two incident waves with having respectively two different frequencies $f_0$=2.5 kHz and $f_0$=10 kHz, whose corresponding wave lengths in the first rod, $\lambda_0$, are respectively 0.3 and 0.075 times of the period. We observe an important amplification of wave propagation phenomena due to diffracted wave caused by the periodicity of the cells and the amplification level is not affected a lot by the frequency of the incident wave.

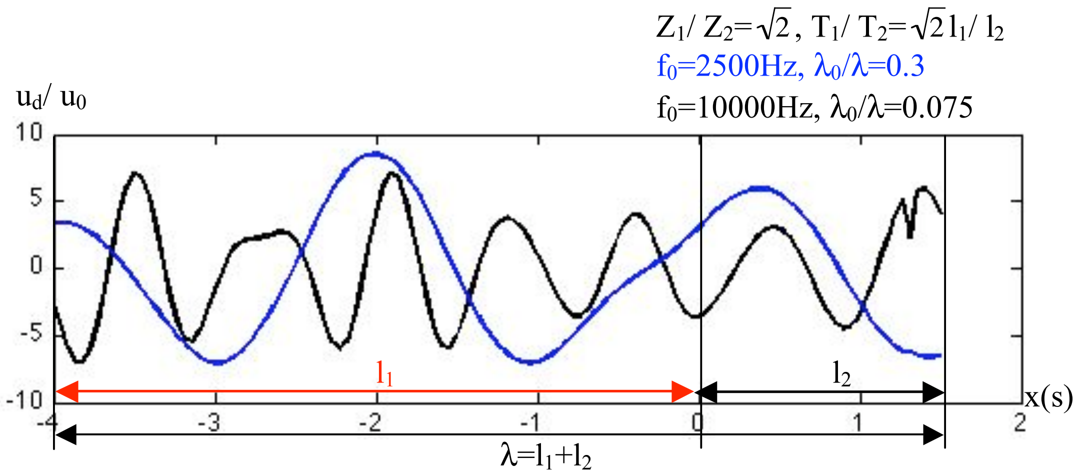


Figure 3: Diffracted wave inside the primitive cell.

## 4 WAVE PROPAGATION IN 2D ELASTIC BEAM MODEL

The second periodic structure considered here is a 2D elastic beam model (Figure 4). It is a topology of a primitive cell composed by three rigidly jointed elastic beams ([8]) with the same length s. The Lamé constants and the density of the beams are ($\lambda$, $\mu$, $\rho$). The thickness of beam (1) is twice of the thickness of the other two beams (2) and (3) (see Figure 4). Each beam is firstly considered in its local basis (**s**, **n**), in which the unit vector **s** is parallel to the beam's axis and the unit vector **n** is perpendicular to the beam's axis. Then the entire model is considered in a global basis (**x**, **y**).

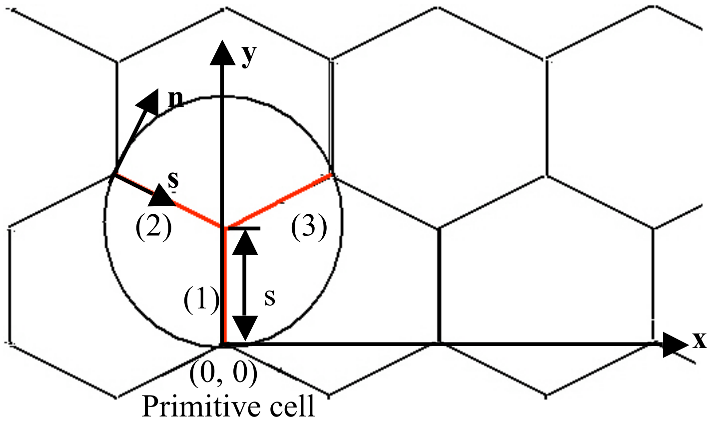


Figure 4: 2D elastic periodic structure composed by beams

### 4.1 Dispersion relation

The well-known Timoshenko kinetics for thick beams is used, so the displacement $\mathbf{u}(s, t)$ in each beam reads as:

$$\mathbf{u}(s,t) = u_{01}(s)\mathbf{s} + u_{02}(s)\mathbf{n} + t\, u_{13}(s)\mathbf{s} \tag{18}$$

where $u_{01}(s)$ and $u_{02}(s)$ are the displacement fields of the middle line in direction **s** and **n** and $u_{13}(s)$ is the rotation of the fibers.

For each beam, the Bloch equilibrium eigenequations of the 2D model are:

$$\begin{aligned}
&\frac{d^2U_{01}^B}{ds^2} - 2i(\mathbf{k}.\mathbf{s})\frac{dU_{01}^B}{ds} - (\mathbf{k}.\mathbf{s})^2 U_{01}^B = -\frac{\rho\omega^2}{\lambda+2\mu}U_{01}^B \\
&\frac{d^2U_{02}^B}{ds^2} - 2i(\mathbf{k}.\mathbf{s})\frac{dU_{02}^B}{ds} - (\mathbf{k}.\mathbf{s})^2 U_{02}^B + \frac{dU_{13}^B}{ds} = -\frac{\rho\omega^2}{\mu}U_{02}^B,\ (m=0,\ 1) \\
&\frac{d^2U_{13}^B}{ds^2} - 2i(\mathbf{k}.\mathbf{s})\frac{dU_{13}^B}{ds} - \left[(\mathbf{k}.\mathbf{s})^2 + \frac{12\mu}{(\lambda+2\mu)H_m^2}\right]U_{13}^B - \frac{12\mu}{(\lambda+2\mu)H_m^2}\frac{dU_{02}^B}{ds} = -\frac{\rho\omega^2}{\lambda+2\mu}U_{13}^B
\end{aligned} \tag{19}$$

with $(U_{01}^B,\ U_{02}^B, U_{13}^B)$ the Bloch eigenmode and $\mathbf{k} \in Q_0^*$ the Bloch wave vector. In this 2D model, the first Brillouin zone $Q_0^*$ is also a hexagonal cell. Similar to the analysis we did for the 1D model, giving **k**, we look for the eigenvalues and the corresponding Bloch eigenmodes of the equation (19).

According to the interface conditions between the beams inside the primitive cell and the periodic conditions on their extremities as following:

$$\begin{aligned}
&U_{01}^{B(1)} + U_{02}^{B(1)} = U_{01}^{B(2)} + U_{02}^{B(2)} = U_{01}^{B(3)} + U_{02}^{B(3)} \\
&U_{13}^{B(1)} = U_{13}^{B(2)} = U_{13}^{B(3)} \\
&N^{B(1)} + Q^{B(1)} + N^{B(2)} + Q^{B(2)} + N^{B(3)} + Q^{B(3)} = 0 \\
&M^{B(1)} = M^{B(2)} = M^{B(3)}
\end{aligned} \tag{20}$$

where $\mathbf{N}^B$ is the Bloch axial force vector, $\mathbf{Q}^B$ is the Bloch transverse shear force vector and $\mathbf{M}^B$ is the Bloch bending moment, we get a system of 18 linear equations. Let the determinant of the system equal to zero, we can get finally the dispersion equations between **k** and ω. Unfortunately, in the 2D model, we are not able to explicit the dispersion equation and we can only plot the dispersion surface numerically (Figure 5). We find that the ratio of $H_0/H_1$ largely affects the width of the stop bands and the bandgap will move to HF range if we shorter the lengths of the beams.

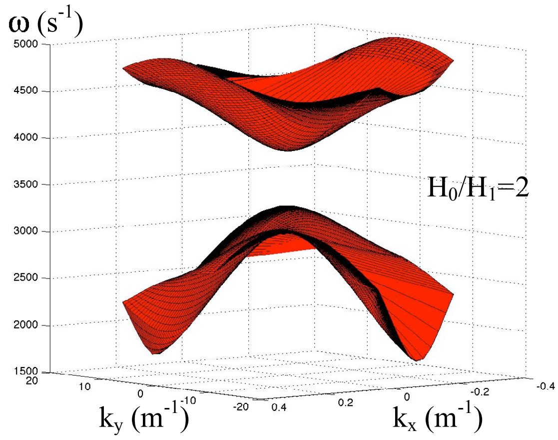


Figure 5: Dispersion surface of eigenfrequency upon Bloch wave vector.

## 5 CONCLUSIONS

The classical homogenized models fail to correctly describe the HF transient flexural behavior of the honeycomb-type sandwich shells. It is necessary to look into the effects coming from the periodic microstructure of the honeycomb thin layer. The Bloch wave theory has been adopted in order to take into account the characteristics of the honeycomb cells while optimizing numerical models and saving calculation costs.

Theoretical analyzing and numerical modeling tools base on the Bloch wave theory and the FE method has been developed. They are at first applied to a 1D periodic structure composed by elastic rods and validated. By considering only the primitive cell, the dispersion equation is obtained, which allows identifying the passing and stopping bands of frequency. With an incident plane wave, the diffracted wave due to the periodicity of the cells is calculated and amplification phenomena are observed independently from the frequency of the incident wave.

Then, a 2D periodic hexagonal regular structure composed by elastic beams is investigated. The first result of eigenfrequency bandgap is obtained.

Our current research work is to apply our theoretical and numerical tools to the honeycomb-type thin layer composed of periodic thin-walled hexagonal regular cells.

## REFERENCES

[1] D. J. Mead, A general theory of harmonic wave propagation in linear periodic systems with multiply coupling. *Journal of Sound and Vibration*, **27**, 235-260, 1973.

[2] A. Grédé, B. Tie, D. Aubry, Elastic wave propagation in hexagonal honeycomb sandwich panels: Physical understanding and numerical modelling. *Journal de Physique*, **134**, 507–514, 2006.

[3] P. Atkins, R. Friedman, *Molecular quantum mechanics*. Oxford university press, New York, 2005.

[4] A. Srikantha, J. Woodhouse, N. A. Fleck, Wave propagation in two-dimensional periodic lattice. *J. Acoust. Soc. Am*, **119**(4), 1995-2005, 2006.

[5] S. Gonella, M. Ruzzene, Analysis of in-plane wave propagation in hexagonal and re-entrant lattices. *Journal of Sound and Vibration*, **312**, 125-139, 2008.

[6] L. Brillouin, *Wave propagation in the periodic structures*. Dover Publications, New York, 1946.

[7] B. –Y. Tian, B. Tie, D. Aubry, Théorie des Ondes de Bloch Appliquée à la Modélisation de la Propagation d'Ondes Elastiques dans des Milieux Périodiques. *$10^{th}$ Colloque National en Calcul des Structures (CSMA 2011),* Giens, France, May 9-13, 2011.

[8] C. Davini, F. Ongaro, A Homogenized Model for Honeycomb Cellular Materials. *Journal of Elasticity*, **Published online**, 2011.