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A NEW POST-PROCESSING PROCEDURE FOR THE INCREASE IN THE ORDER OF ACCURACY OF THE TRAPEZOIDAL RULE AT TIME INTEGRATION OF LINEAR ELASTODYNAMICS PROBLEMS

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Abstract. *In the current presentation, we suggest a very simple and effective post-processing procedure to increase the order of accuracy in time for numerical results obtained at time integration of linear elastodynamics problems by the trapezoidal rule. This technique is based on a new exact, closed-form, a-priori error estimator for time integration of linear elastodynamics equations by the trapezoidal rule with non-uniform time increments. Based on this error estimator, we suggest a new post-processing procedure (containing additional time integration of elastodynamics equations by the trapezoidal rule with few time increments) that systematically improves the order of accuracy of numerical results, with the increase in the number of additional time increments used for post-processing. For example, the use of just one additional time increment for post-processing after time integration with any number of uniform time increments, renders the order of accuracy of numerical results equal to 10/3 = 3.3333. Numerical examples of the application of the new techniques to a system with a single degree of freedom and to a multi-degree system confirm the corresponding increase in the order of convergence of numerical results after post-processing. Because the same trapezoidal rule is used for basic computations and post-processing, the new technique retains all of the properties of the trapezoidal rule, requires no writing of a new computer program for its implementation, and can be easily used with any current commercial and research codes for elastodynamics.*

1 INTRODUCTION

The application of finite elements in space to linear elastodynamics problems leads to a system of ordinary differential equations in time

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}, \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are the mass, damping, and stiffness matrices, respectively, $\boldsymbol{U}$ is the vector of the nodal displacement, $\boldsymbol{R}$ is the vector of the nodal load. Eq. (1) can be also obtained by the application of other discretization methods in space such as the spectral element method, the boundary element method, the smoothed particle hydrodynamics (SPH) method and others. For long-term time integration of semi-discrete elastodynamics equations (1), higher-order accurate methods in time are more computationally effective compared with second order methods. For example, high accuracy in time can be obtained from the unified set of a single step method [1] by the use of higher-order interpolation polynomials in time. However, these methods are not unconditionally stable for elastodynamics. Recently, new high-order accurate methods with a step-by-step time integration scheme have been developed for elastodynamics (see [2, 3, 4, 5, 6, 7, 8, 9, 10] and many others). Most of them are based on semi-discrete equations (1) with the polynomial time approximations of unknown functions. The polynomial coefficients are derived with the use of different approaches such as time-continuous Galerkin (TCG) and time-discontinuous Galerkin (TDG) methods, weighted residual methods, collocation methods and others. The ultimate goal in the development of high-order accurate implicit methods is to construct an unconditionally stable method with controllable numerical dissipation that is much more computationally effective than known second-order methods. Because many high-order accurate methods require the solution of a large system of equations (much larger than a system of equations for second-order methods), the development of effective iterative predictor/multi-corrector solvers is an important component of a high-order method. For example, the predictor/multi-corrector solver suggested in [10] for modified Nørsett methods requires one additional iteration at each time step in order to improve the order of accuracy by one. Many different iterative solvers were developed for the TDG method with linear time approximations of displacements and velocities that correspond to the third order of accuracy (see [2, 3, 8, 9] and others). By the use of three different time increments for each time step, a very effective formulation of fourth-order time-integration methods is obtained from second-order methods in [11]. However, the authors of [11] could not extend their approach to higher orders of accuracy.

To summarize, the development of a computationally effective high-order accurate time-integration method for elastodynamics is still a challenging problem in computational mechanics.

In this paper, we will use a new post-processing procedure (containing additional time integration of elastodynamics equations by the trapezoidal rule with few time increments) that systematically improves the order of accuracy of numerical results, with the increase in the number of additional time increments used for post-processing (see our paper [12]). We should mention that the trapezoidal rule does not include numerical dissipation. However, as was shown in our papers [13, 14, 15, 16], numerical dissipation is not required for long-term integration with the new solution strategy suggested in [13, 14, 16]. It was also shown in [13, 14, 16] that the trapezoidal rule is the best time-integration method (the fastest and most accurate method) for long-term integration of elastodynamics problems (including wave propagation and impact problems) among all implicit second-order time-integration methods.

2 SUMMARY OF THE NUMERICAL TECHNIQUE

In our paper [12], we have developed a new numerical time-integration technique for the increase in the order of accuracy at the integration by the trapezoidal rule. This technique is based on a new exact, closed-form, a-priori error estimator for time integration of linear elastodynamics equations by the trapezoidal rule with non-uniform time increments. Based on this error estimator, we developed a new post-processing procedure that systematically improves the order of accuracy of numerical results, with the increase in the number of additional time increments used for post-processing. The suggested procedure includes time integration at basic computations by the trapezoidal rule with uniform time increments and time integration at post-processing by the trapezoidal rule with few time increments. The sizes of time increments at post-processing (positive and negative time increments are used) are calculated from the new a-priori error estimator and depend on the size and the total number of time increments used at basic computations. For example, the use of just one, three or five additional time increments for post-processing after time integration with any number of uniform time increments at basic computations, renders the order of accuracy of numerical results equal to 10/3, 14/3 and 6, respectively. The sizes of time increments for post-processing should be calculated as follows: $\Delta t = -\sqrt[3]{m}\Delta\bar{t}$ for one time increment; $\Delta t = \alpha_1 \Delta\bar{t}$, $\Delta t = \alpha_1 \Delta\bar{t}$ and $\Delta t = \alpha_2 \Delta\bar{t}$ for three time increments (α_1 and α_2 should be taken from Table 1); $\Delta t = \alpha_1 \Delta\bar{t}$, $\Delta t = \alpha_1 \Delta\bar{t}$, $\Delta t = \alpha_2 \Delta\bar{t}$, $\Delta t = \alpha_2 \Delta\bar{t}$ and $\Delta t = \alpha_3 \Delta\bar{t}$ for five time increments (α_1, α_2 and α_3 should be taken from Table 2) where m and $\Delta\bar{t}$ are the number of uniform time increments and the size of time increments at basic computations by the trapezoidal rule.

Table 1. Coefficients α_1 and α_2 for post-processing with three time increments for different numbers m.

m	2	10	50	100	500	1000	10000	100000	1000000
α_1	-1.48091	-2.68405	-4.65819	-5.8846	-10.092	-12.7221	-27.4292	-59.1037	-127.339
α_2	1.65042	3.0607	5.3386	6.75003	11.5871	14.6095	31.5058	67.8914	146.274

Table 2. Coefficients α_1, α_2 and α_3 for post-processing with five time increments for different numbers m.

m	2	10	50	100	500	1000	10000	100000	1000000
α_1	1.93940	3.68743	6.46987	8.18896	14.0732	17.748	38.2851	82.505	177.762
α_2	-1.57145	-2.89162	-5.03795	-6.3688	-10.9307	-13.7815	-29.7189	-64.0403	-137.977
α_3	-2.06674	-3.9562	-6.95145	-8.80069	-15.1286	-19.0799	-41.161	-88.704	-191.119

3 NUMERICAL EXAMPLES

3.1 A single degree of freedom system

First let's consider the increase in the order of accuracy by post-processing for time integration of the following elastodynamics equation for a single degree of freedom system:

$$\ddot{u}(t) + 2\xi\dot{u}(t) + \omega^2 u(t) = f(t)\,, \tag{2}$$

with the natural frequency $\omega = \pi = 3.1416$ and the following initial conditions $u(0) = 1$ and $v(0)/\pi = \dot{u}(0)/\pi = 1$. Zero damping ($\xi = 0$) for the observation time $T = 50$, nonzero damping ($\xi = 0.1, 0.2$) for the observation time $T = 5$, zero ($f(t) = 0$) and non-zero load

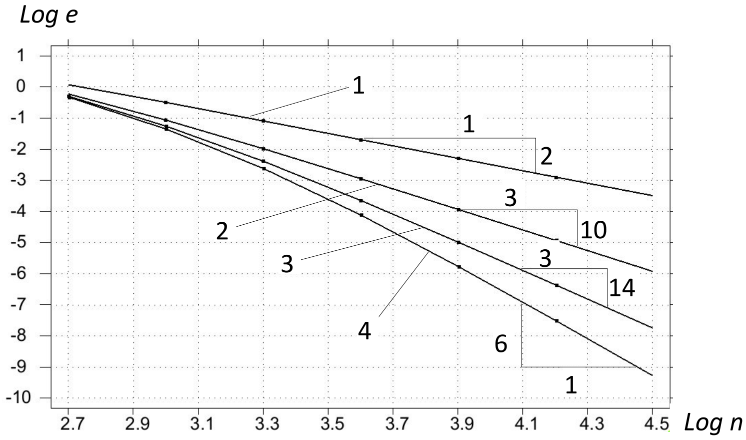

Figure 1: The relative error in displacements and velocities e at the observation time $T = 50$ versus the total number of time increments n in the logarithmic scale at integration of a single degree of freedom system (no damping $\xi = 0$ and zero load $f(t) = 0$) without post-processing (curve 1) and the subsequent post-processing with one (curve 2), three (curve 3) and five (curve 4) time increments.

($f(t) = 2 + 2t$), and post-processing with one, three and five time increments as described in Section 2 are considered. Different total numbers n ($n = m$ plus the number of time increments used at post-processing) of time increments are used in calculations (the corresponding time increments in basic computations can be calculated as $\Delta\bar{t} = \frac{T}{n}$ for the case of basic computations without post-processing, $\Delta\bar{t} = \frac{T}{((n-1)-\sqrt[3]{n-1})}$ for the case of basic computations and the subsequent post-processing with one time increment, $\Delta\bar{t} = \frac{T}{(n-3+2\alpha_1+\alpha_2)}$ for the case of basic computations and the subsequent post-processing with three time increments, and $\Delta\bar{t} = \frac{T}{(n-5+2\alpha_1+2\alpha_2+\alpha_3)}$ for the case of basic computations and the subsequent post-processing with five time increments.

Figs. 1 - 2 show the relative numerical error $e = \frac{e_{uv}}{\sqrt{u^2(0)+(\frac{v(0)}{w})^2}}$ (with $e_{uv} = \sqrt{[u(t_n) - u_{num}(t_n)]^2 + [\frac{v(t_n)-v_{num}(t_n)}{w}]^2}$ where $u(t_n)$, $v(t_n)$ and $u_{num}(t_n)$, $v_{num}(t_n)$ are the exact and numerical solutions of Eq. (2) for the displacement and velocity at time t_n) versus the number of time increments in the logarithmic scale. At the fixed observation time T and a large number of time increments, the number of time increments n is inversely proportional to a time increment Δt ($n \approx m \approx \frac{T}{\Delta t}$ or $Log\, n \approx Log\, T - Log\, \Delta t$. Therefore, the slope of the curves in Figs. 1 - 2 (which are plotted in the logarithmic scale) describes the order of convergence (order of accuracy) of numerical results at large numbers n of time increments.

As can been seen from Fig. 1, at a large number of time increments n, the order of convergence (order of accuracy) of numerical results after post-processing with one, three and five time increments is in agreement with the analytical estimations reported in Section 2 (see the slope of curves 2, 3 and 4). For example, it also follows from Fig. 1 that at the error of 1% (or $Log\, e = -2$) for the observation time $T = 50$, post-processing with one, three and five time increments reduces the total number of time increments by factors of $2.8, 3.8, 4$, respectively (compared with the results with no post-processing, curve 1). Post-processing is even more effective if we are interested in more accurate results (for multi-dimensional problems a general solution is the superposition of the solutions for separate modes and requires more accurate results for individual modes). For example, at the error of 0.01% (or $Log\, e = -4$) for the ob-

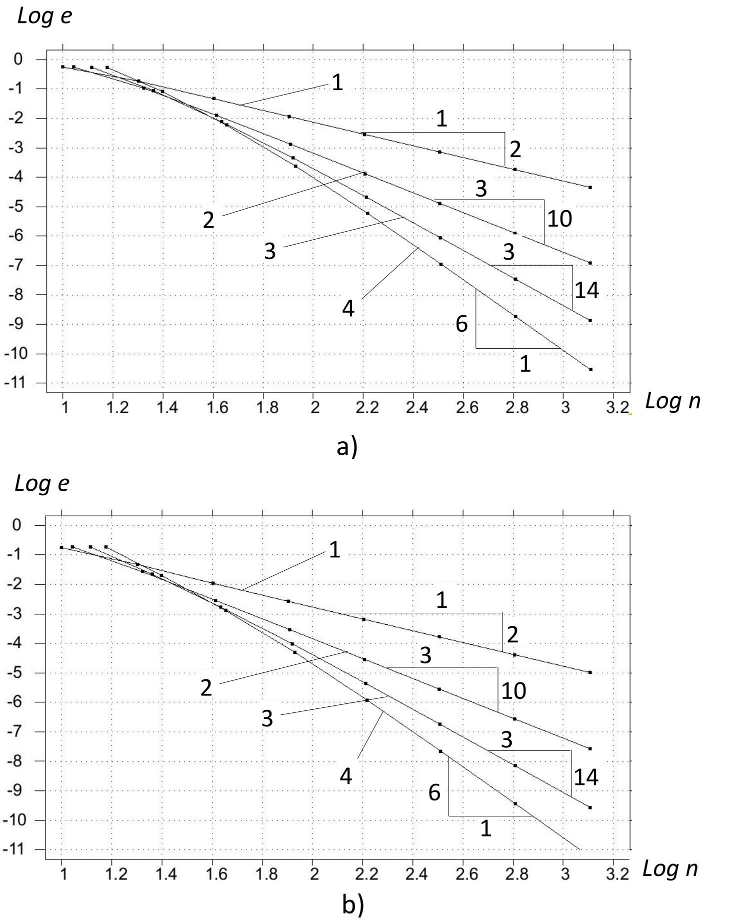

Figure 2: The relative error in displacements and velocities e at the observation time $T = 5$ versus the total number of time increments n in the logarithmic scale at integration of a single degree of freedom system (with zero load $f(t) = 0$ and damping $\xi = 0.1$ (a) and $\xi = 0.2$ (b)) without post-processing (curve 1) and the subsequent post-processing with one (curve 2), three (curve 3) and five (curve 4) time increments.

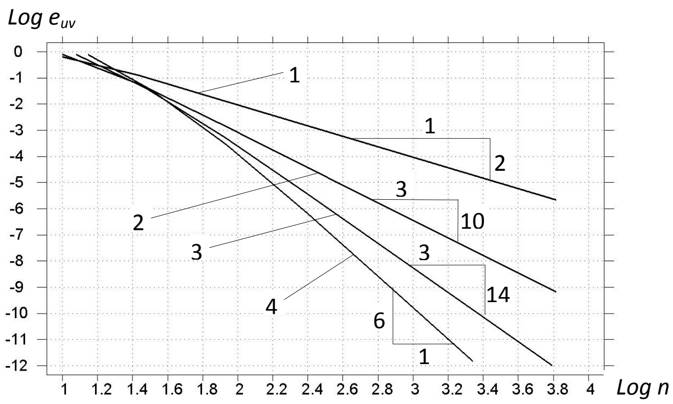

Figure 3: The numerical error in displacements and velocities $\tilde{e}_{uv}$ at the observation time $T = 5$ versus the total number of time increments n in the logarithmic scale at integration of a single degree of freedom system with damping $\xi = 0.1$ and non-zero loading $f(t) = 2 + 2t$ without post-processing (curve 1) and the subsequent post-processing with one (curve 2), three (curve 3) and five (curve 4) time increments.

servation time $T = 50$, post-processing with one, three and five time increments reduces the total number of time increments by factors of $6, 10, 12$, respectively. However, we should also remember that post-processing with one, three and five time increments requires the additional factorization of one, two and three stiffness matrices, respectively. Therefore, the reduction in computation time at the application of the post-processing procedure can be easily calculated if the computation times for the factorization of a stiffness matrix and the back-substitution stage of the solution of a system of linear equations are known for the selected computer code at the given numbers of degrees of freedom.

As can been seen from Figs. 2 and 3, for linear elastic problems with non-zero damping, the order of convergence (order of accuracy) of numerical results after post-processing with one, three and five time increments is in agreement with the analytical estimations described in Section 2. It also follows from Fig. 2 that if after basic computations with the trapezoidal rule the error is smaller than 10% for $\xi = 0.1$ or 5% for $\xi = 0.2$ (for the frequency and observation time used), the suggested post-processing procedure improves the numerical results. For multi-degree problems, numerically solving single degree of freedom problems for the maximum and leading (i.e., those with large amplitudes) modes is recommended in order to roughly estimate the range of time increments at which the suggested post-processing procedure improves the results.

3.2 Harmonic response of an elastic rod

Next we show the application of the post-processing technique described in Section 2 to time integration of the 1-D elastodynamics problem related to harmonic response of an elastic rod. An elastic rod of the length $L = 1$ is considered. Both ends of the rod are fixed, no external loads are applied, the initial velocity is zero, and the initial displacement is proportional to the first harmonic $u_0(x, 0) = sin(\pi x)$; see Fig. 4a. The observation time is assumed to be $T = 50$, Young's modulus to be $E = 1$, and the density to be $\rho = 1$. A uniform mesh with 100 linear finite elements along the bar is used. The elastodynamics problem was integrated in time by the trapezoidal rule with different numbers m of uniform time increments and the subsequent post-processing with one and three time increments as described in Section 2 (a time increment Δt is calculated similar to that described above for a single degree of freedom system). Numerical results show that a numerical solution can be approximated as $u_n(x, t) = sin(\pi x)g_1(t)$, where $g_1(t)$ is a function of time only. This means that only frequencies close to π are excited in the numerical solution. For comparison of the accuracy of the numerical results, the following errors in displacements ($\tilde{e}_u$), velocities ($\tilde{e}_v$) and the combined error in displacements and velocities ($\tilde{e}_{uv}$) at time t were calculated: $\tilde{e}_u(t) = \max_{0\le x\le L}[u_a(x, t) - u_n(x, t)] = u_a(L/2, t) - u_n(L/2, t)$, $\tilde{e}_v(t) = \max_{0\le x\le L}\frac{[v_a(x,t)-v_n(x,t)]}{\pi} = \frac{[v_a(L/2,t)-v_n(L/2,t)]}{\pi}$ and $\tilde{e}_{uv}(t) = \frac{\sqrt{\tilde{e}_u^2 + \tilde{e}_v^2}}{u^0_{max}}$, where $u_n(x, t)$ and $v_n(x, t)$ are numerical solutions for displacements and velocities at current time t, and $u_a(x, t) = sin(\pi x)cos(\pi t)$, $v_a(x, t) = -\pi sin(\pi x)sin(\pi t)$ are the analytical solutions for displacements and velocities at current time t, and the maximum numerical errors in displacements and velocities occur at $x = L/2$, $u^0_{max} = 1$ is the maximum initial displacement. For this problem, we selected a relatively fine mesh which yields a very small error in space. Therefore, for relatively large time increments, the combined error in displacements and velocities $\tilde{e}_{uv}$ should relate to the global error in time with a scaling factor. Fig. 4b shows the numerical error in displacements and velocities $\tilde{e}_{uv}$ versus the number of time increments n in the logarithmic scale. As can be seen from Fig. 4b, at a large number of time increments, the order of convergence (order of accuracy) of numerical results after post-

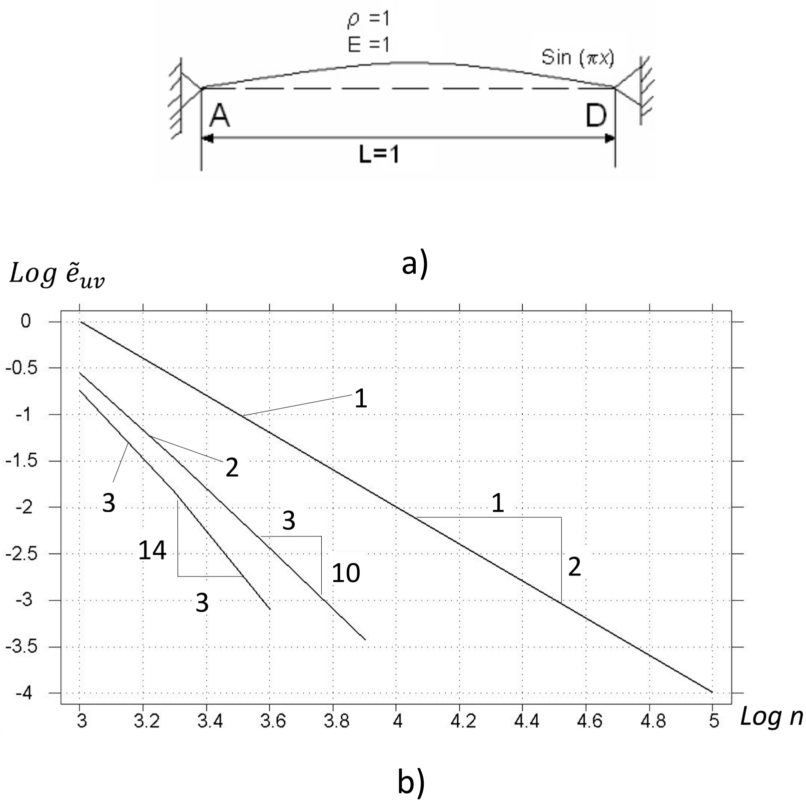

Figure 4: Modeling of harmonic response of an elastic rod (a). The numerical error in displacements and velocities $\tilde{e}_{uv}$ at the observation time $T = 50$ versus the total number of time increments n in the logarithmic scale (b) at time integration by the trapezoidal rule (curve 1) and the subsequent post-processing with one (curve 2) and three (curve 3) time increments.

processing with one (curve 2) and three (curve 3) time increments is in agreement with the analytical estimations reported in Section 2. It also follows from Fig. 4b that at the error of 1% (or $Log\,\tilde{e}_{uv} = -2$) for the observation time $T = 50$, post-processing with one and three time increments reduces the total number of time increments n by factors of 3.5 and 4.6, respectively (compared with the results with no post-processing, curve 1). For more accurate results with the error of 0.1% (or $Log\,\tilde{e}_{uv} = -3$) at the observation time $T = 50$, post-processing with one and three time increments reduces the total number of time increments n by factors of 5.5 and 8.5, respectively. However, we should also remember that post-processing with one and three time increments requires the additional factorization of one and two stiffness matrices, respectively.

3.3 Impact of an elastic bar against a rigid wall

Here we show the application of the post-processing technique described in Section 2 to time integration of a 1-D impact elastodynamics problem for which all frequencies of the semi-discrete system, Eq. (1), are excited. An elastic rod of the length $L = 4$ is considered. The following boundary conditions are applied: the displacement $u(0,t) = t$ (it corresponds to the velocity $v(0,t) = v_0 = 1$) and $u(4,t) = 0$ (it corresponds to $v(4,t) = 0$). Initial displacements and velocities are zero; i.e., $u(x,0) = v(x,0) = 0$. The observation time is assumed to be $T = 2$, Young's modulus to be $E = 1$, and the density to be $\rho = 1$. Zero damping $\boldsymbol{C} = \boldsymbol{0}$ and two cases of non-zero Rayleigh damping $\boldsymbol{C} = b_1\boldsymbol{M} + b_2\boldsymbol{K}$ with the coefficients $b_1 = 0.005$, $b_2 = 0.01$ and with the coefficients $b_1 = 0.05$, $b_2 = 0.1$ are considered. A uniform mesh with

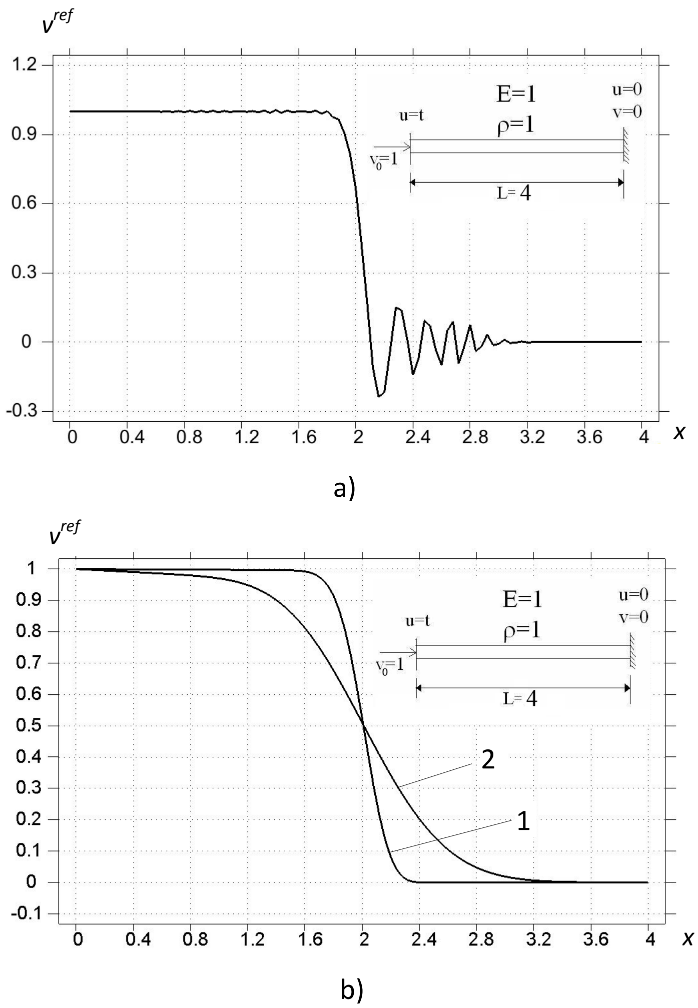

Figure 5: Velocity distribution v^{ref} along the bar at time $T = 2$ for the impact problem with zero (a) and non-zero Rayleigh damping (b). Curves 1 and 2 in b) correspond to the damping coefficients $b_1 = 0.005$, $b_2 = 0.01$ and $b_1 = 0.05$, $b_2 = 0.1$, respectively. The solutions are obtained by the trapezoidal rule with 400000 uniform time increments.

100 linear finite elements along the bar is used. In this paper we will consider the convergence in time of numerical results to a solution of the semidiscrete problem, Eq. (1), which differs from the analytical solution of the continuous impact problem. Because the semidiscrete problem includes 100 degrees of freedom, the simplest way to find the solution of the semidiscrete system is to use accurate time integration of Eq. (1) with very small time increments. This solution, called the reference solution, is obtained by the trapezoidal rule with 400000 time increments and is shown for the velocity v^{ref} in Fig. 5. An accurate numerical solution of the original continuous impact problem with non-zero damping is considered in our papers [14, 16]. To study convergence of the trapezoidal rule after post-processing, the following measure for the velocity error of the semi-discrete system at time $T = 2$ is introduced:

$$e_v(T = 2) = \sqrt{\sum_{i=1}^{100} (v_i^{ref}(T = 2) - v_i^{num}(T = 2))^2}\,, \tag{3}$$

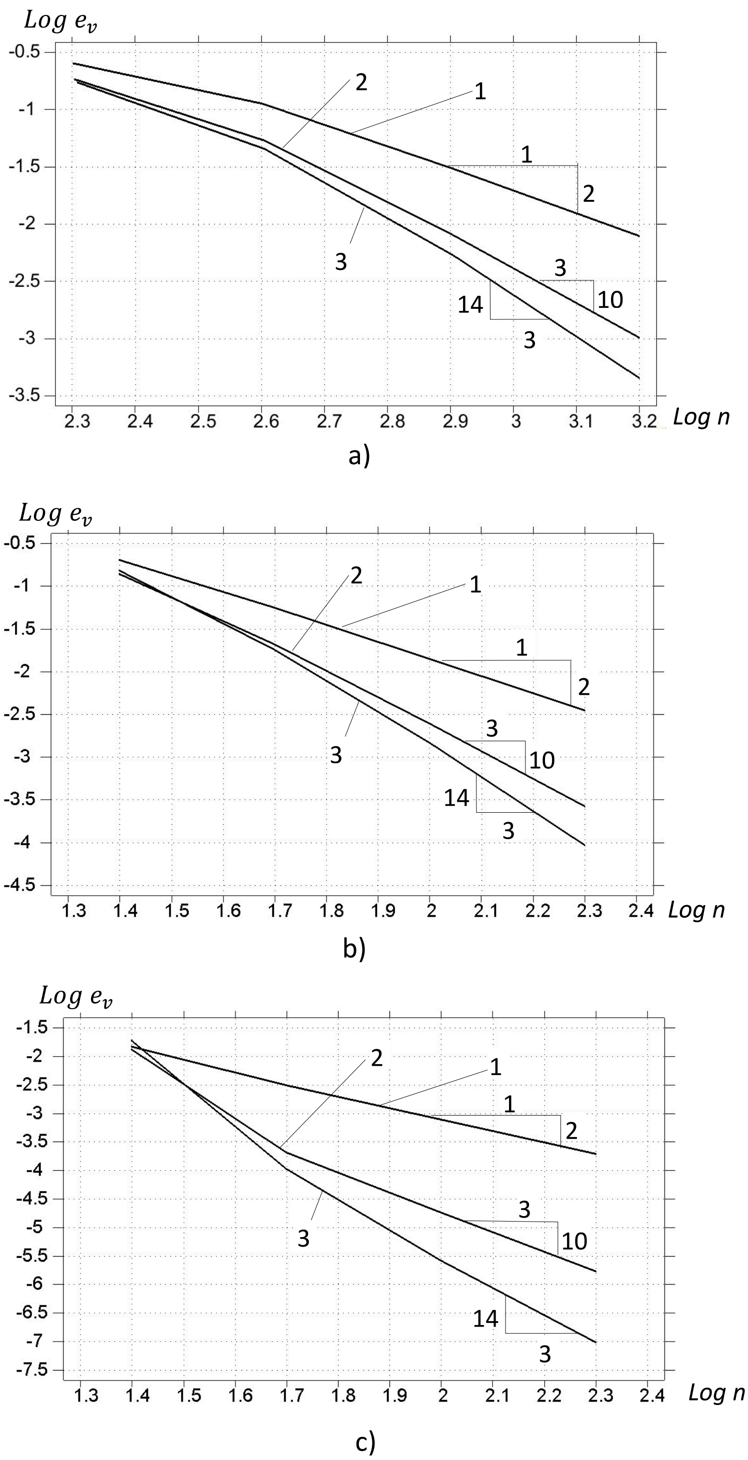

Figure 6: The numerical error in velocity e_v at the observation time $T = 2$ versus the total number of time increments n for the impact problem in the logarithmic scale at time integration by the trapezoidal rule (curve 1) and the subsequent post-processing with one (curve 2) and three (curve 3) time increments. a), b) and c) correspond to non-zero damping and to Rayleigh damping with the damping coefficients $b_1 = 0.005$, $b_2 = 0.01$ and with the damping coefficients $b_1 = 0.05$, $b_2 = 0.1$, respectively.

where $v_i^{ref}(T=2)$ and $v_i^{num}(T=2)$ $(i=1,2,...,100)$ are the nodal velocities of the reference and numerical solutions at time $T=2$, respectively.

Fig. 6 shows the numerical error e_v at time $T=2$ versus the number of time increments in the logarithmic scale. At a large number of time increments, the number of time increments n is inversely proportional to a time increment Δt. Therefore, the slope of the curves at large numbers n of time increments in Fig. 6 (which are plotted in the logarithmic scale) describes the order of convergence (order of accuracy) of numerical results. As can been seen from Fig. 6, the order of convergence (order of accuracy) of numerical results after post-processing with one and three time increments is in agreement with the analytical estimations reported in Section 2 (see the slopes of curves 2 and 3).

4 CONCLUSIONS

In our previous papers, we suggested a new solution strategy for elastodynamics problems and showed that for long-term time integration, a time-integration method at basic computations does not need numerical dissipation even for wave propagation and impact problems. In the current paper, we suggest a very simple and effective post-processing procedure to increase the order of accuracy in time for numerical results obtained by the trapezoidal rule. Because the same trapezoidal rule is used for basic computations and for post-processing, the new technique retains all of the properties of the trapezoidal rule. For example, at zero physical damping C = 0, the trapezoidal rule conserves the total energy and the linear and angular momentum of a mechanical system during time integration. We should also mention that the new technique requires no writing of a new computer program for its implementation and can be easily used with any current commercial and research codes for elastodynamics. Of course, post-processing with a time increment, the size of which differs from that used in basic computations, requires the factorization of a tangent matrix and leads to additional computational costs. However, these additional costs are small compared with those for long-term integration by the trapezoidal rule in basic computations.

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