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SH SURFACE WAVES IN A HALF SPACE WITH RANDOM HETEROGENEITIES

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Abstract. *Horizontally polarized shear waves (SH waves) do not exist in a homogeneous half space according to the traditional elastic wave theory. However, in this study, we proved both theoretically and numerically that there will be surface waves in a half space which has small, random density, but the mean value of the density is homogeneous. Historically, this type of half space is often treated as a homogeneous one with deterministic methods. In this investigation, a closed-form dispersion equation was derived stochastically, and the frequency spectrum, dispersion equation, phase/group velocity were plotted numerically to study how the random inhomogeneities will affect the dispersion properties of the half space with random density. This research may find its application in seismology, non-destructive test/evaluation, etc.*

1 Introduction

In this study, the dispersion and attenuation properties of waves propagating in a half space (see figure 1) with random heterogeneities are investigated.

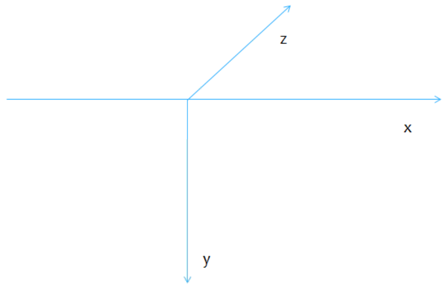

Figure 1: coordinate system of the half space

Shear horizontal surface waves (SHSW) are the most destructive waves in an earth quake and they can propagate through a very long distance without much loss of its energy. But, scientists have proved long ago that there is no SHSW in a homogeneous isotropic linearly elastic half-space [1]. However, in 1911, love predicted mathematically that SHSW could exist if the half-space is covered by a layer of a different material.

Since then, SHSW in a half space was mostly explained theoretically by Love's theory or its variant theories. But we know that the earth's surface is very complex. It is a mixture of many kinds of rocks, sands, soil, water, etc., and more complicatedly, these materials do not often distribute in deterministic ways, but distribute randomly. So do SHSW exist in such a complex, random half space?

Similar problems have been explored by some scientists. B. Collet et al. [3] studied SHSW in a Functionally Graded Material of which some material constants share the same depth-dependent function , and derived some of the depth-dependent functions which could be solved exactly. Using their solutions, they studied the influence of different inhomogeneity functions on the properties of SHSW. J. Achenbach et al. [2] studied SHSW in a purely elastic half-space whose shear modulus and mass density depend arbitrarily on the depth and gave a general solution that is quite exact for high frequencies. T.C.T. Ting [4] recently investigated SHSW in a half space of which C_{44} and ρ have the same function form, and C_{55}, C_{45} are correlated. Ting also got an asymptotic solution of general graded materials for large wave number k. Anti-plane shear waves for anisotropic graded materials have been considered for periodic half-spaces by A. Shuvalov et al. [5] and for a single plate by A. Shuvalov et al. [6]. Shear horizontal waves in functionally graded piezoelectric materials are also greatly studied by Tianjian Lu et al. [7, 8, 9].

But these researches haven't given an explicit solution of dispersion and attenuation of SHSW in a half space with random density in the depth direction by strict stochastic methods. In this study, we get the explicit dispersion equation by the first order smoothing approximation (FOSA) method. And we then analyze the dispersion and attenuation properties using the dispersion equation.

In this study we proved mathematically and numerically that SHSW could exist in a stochastically homogeneous half space. Some interesting properties of dispersion and attenuation found in this study could promote our understanding of waves propagating in a half space with random heterogeneities, e.g. earth's upper crust, alloys or composites. It will also help us to do the inverse problems, for example, to use seismic waves to detect the earth's crust structure, and to use ultrasonic waves to evaluate a structure with randomly distributed micro-cracks or heterogeneities.

2 Modeling and mathematical analysis

The fundamental dynamic equation system for statistically homogenous, isotropic, linearly elastic solid is

$$\tau_{ij,j} + \rho f_i = \rho \ddot{u}_i \tag{1}$$
$$\tau_{ij} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \tag{2}$$
$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

To account for the random heterogeneities, we change the constants ρ, μ, λ in the equation system to functions of space, time and random variables.

Consider SH waves propagating in x direction in a half space (see Fig. 1).

It is known that for anti-plane waves that $u_x = u_y = 0$ and $\partial/\partial z = 0$. And if we assume that there is no body force, the equation system reduces to

$$\tau_{zj,j} = \rho \ddot{u}_z \tag{4}$$
$$\tau_{zj} = \mu u_{z,j} \tag{5}$$

in which, $j = x, y$. So the dynamic equation for SH waves in a random half space is

$$(\mu u_{z,j})_{,j} = \rho \ddot{u}_z \tag{6}$$

And the boundary condition is

$$\tau_{zy}|_{y=0} = 0 \quad i.e. \tag{7}$$
$$\mu u_{z,y}|_{y=0} = 0 \tag{8}$$

Assume here that there is randomness only in the y direction. Consider an harmonic wave motion of the form

$$u_z = f(y)exp[i(k_1 x - \omega t)] \tag{9}$$

in which, $f(y)$ is a random function. To study the surface shear wave, we assume the averaged $f(y)$ to be

$$< f(y) >= Ae^{-by} \tag{10}$$

, in which $b > 0$. Thus the mean wave motion $< u_z >$ could be written as

$$< u_z >= Ae^{-by}e^{i(k_1 x - \omega t)} \tag{11}$$

If there is no random heterogeneities in the solid, a solution of Eq. (6) would be of the form [1]

$$u_z = Ae^{-by}e^{i(kx - \omega t)} \tag{12}$$

Substituting Eq. (12) into Eq. (6), we find

$$\frac{\omega^2}{C_s^2} - k_1^2 + b^2 = 0 \tag{13}$$

For a free surface, the boundary condition at $y = 0$ is

$$\frac{\mathrm{d}u_z}{\mathrm{d}y} = 0 \tag{14}$$

The boundary condition Eq. (14) can be satisfied only if either $A = 0$ or $b = 0$. Therefore, there is no surface SH wave in an homogenous, isotropic, linearly elastic half space.

Firstly, we consider that random heterogeneities are only on the surface (as a practical example, the roughness of the earth surface could be viewed as a half space but with random density on the surface), then the boundary conditions at $y = 0$ can be written as

$$\mu\frac{\partial u_z}{\partial y} = 0 \quad \Rightarrow$$
$$(\mu_0 + \epsilon\mu_1)\frac{\partial(< u_z > +\epsilon u_{z1})}{\partial y} = 0 \tag{15}$$

By averaging both sides of Eq. (15), when $y = 0$, we get

$$\mu_0\frac{\mathrm{d} < u_z >}{\mathrm{d}y} + \epsilon^2 < \mu_1\frac{\partial u_{z1}}{\partial y} >= 0 \tag{16}$$

The randomness of the surface takes effect through the term $\epsilon^2 < \mu_1\frac{\partial u_{z1}}{\partial y} >$. We assume here that

$$\epsilon^2 < \mu_1\frac{\partial u_{z1}}{\partial y} > |_{y=0} = \mu_0 A\beta e^{i(k_1x-\omega t)} \tag{17}$$

Substituting Eqs. (17) and (11) into Eq. (16), we get

$$b = \beta \tag{18}$$

Considering Eq. (13), the dispersion equation for SH waves in a half space with random heterogeneities only on the surface is

$$\frac{\omega^2}{C_s^2} - k_1^2 + \beta^2 = 0 \tag{19}$$

Next, we will investigate the problem of the half space with random heterogeneities in the whole depth direction. Substituting Eq. (9) in Eq. (6) gives

$$(\rho\omega^2 - \mu k_1^2)f + (\mu f_{,y})_{,y} = 0 \tag{20}$$

Assuming that ρ, μ differ slightly from the mean value of them, ρ, μ can be written as

$$\rho(y) = \rho_0 + \epsilon\rho_1(y)$$
$$\mu(y) = \mu_0 + \epsilon\mu_1(y) \tag{21}$$

where, ϵ is a small parameter, and

$$< \rho_1 >=< \mu_1 >= 0 \tag{22}$$

Substituting Eq. (21) in Eq. (20), we have

$$(\rho_0\omega^2 - \mu_0k_1^2)f + \mu_0 f_{,yy} + \epsilon\left((\rho_1\omega^2 - \mu_1k_1^2)f + (\mu_1 f_{,y})_{,y}\right) = 0 \tag{23}$$

According to FOSA theory (see Appendix A for a brief deduction of FOSA), the deterministic operator of Eq. (20) is

$$L_0(y) = \mu_0 \left(k_0^2 + \frac{\partial^2}{\partial y^2} \right) \tag{24}$$

in which,

$$k_0^2 = \frac{\omega^2}{C_s^2} - k_1^2 \tag{25}$$

and, C_s is the shear velocity of the homogeneous material without random heterogeneities,

$$C_s = \sqrt{\frac{\mu_0}{\rho_0}} \tag{26}$$

And the first order random operator of Eq. (20) is

$$L_1(y) = P(y) + \mu_1(y)_{,y} \frac{\partial}{\partial y} + \mu_1(y) \frac{\partial^2}{\partial y^2} \tag{27}$$

in which,

$$P(y) = \rho_1(y)\omega^2 - \mu_1(y)k_1^2 \tag{28}$$

Considering Eq. (22), we can see that $< L_1 >= 0$. For steady waves, G_0 can be taken as

$$G_0(y_1, y_2) = -\frac{1}{2k_0\mu_0} \sin(k_0|y_1 - y_2|) \tag{29}$$

According to the stochastic theory, the FOSA equation is

$$L_0 < f(y_1) > -\epsilon^2 \left\langle L_1(y_1) \int G_0(y_1, y_2) L_1(y_2) < f(y_2) > \, \mathrm{d}y_2 \right\rangle = 0 \tag{30}$$

To solve Eq. (30), let's calculate $L_1(y_1)G_0(y_1, y_2)$ first,

$$L_1(y_1)G_0(y_1, y_2) = -\left(P(y_1) + \mu_1(y_1)_{,y_1} \frac{\partial}{\partial y_1} + \mu_1(y_1) \frac{\partial^2}{\partial y_1^2} \right) * \frac{1}{2k_0\mu_0} \sin(k_0|y_1 - y_2|) \tag{31}$$

When $y_2 < y_1$

$$\begin{aligned} L_1(y_1)G_0(y_1, y_2) &= Q_1 \sin(k_0(y_1 - y_2)) + Q_2 \cos(k_0(y_1 - y_2)) \\ &= M(y_1, y_2) \end{aligned} \tag{32}$$

in which,

$$Q_1 = \left(\frac{\mu_1(y_1)k_0}{2\mu_0} - \frac{P(y_1)}{2k_0\mu_0} \right) \tag{33}$$

$$Q_2 = -\frac{\mu_1(y_1), y_1}{2\mu_0} \tag{34}$$

and, when $y_2 > y_1$

$$L_1(y_1)G_0(y_1, y_2) = -M(y_1, y_2) \tag{35}$$

Then, using Eq. (10), $L_1(y_2) < f(y_2) >$ can be expressed as

$$\begin{aligned} L_1(y_2) < f(y_2) >= & \left(P(y_2) + \mu_1(y_2)_{,y_2}\frac{\partial}{\partial y_2} + \mu_1(y_2)\frac{\partial^2}{\partial y_2^2}\right) Ae^{-by_2} \\ = & A\left(P(y_2) - \mu_1(y_2)_{,y_2}b + \mu_1(y_2)b^2\right)e^{-by_2} \\ = & N(y_2) \end{aligned} \tag{36}$$

If we assume that $\mu_1 = 0$, we could study the influence of the randomness of the density on the dispersion properties of the plate.

The random function $\rho_1(y_1;\gamma)$ is taken as Uhlenbeck-Ornstein process [10]. Although its correlation function is not mean-square differentiable, this process has been used in a number of investigations because it fits experimental data the best [11]. This process is a centered and stationary random function [10] and its correlation function is

$$\begin{aligned} R_{\rho_1(y_1;\gamma)\rho_1(y_2;\gamma)} = & \int \rho_1(y_1;\gamma)\rho_1(y_2;\gamma)\,\mathrm{d}\gamma \\ = & \zeta^2 e^{-\frac{|y_1-y_2|}{R_c}} = R(y_1 - y_2) \end{aligned} \tag{37}$$

In which, $\zeta = \sqrt{<\rho_1{}^2>}$ and it is the standard deviation of the random density function; γ is a random variable. And R_c is the integral radius (the correlation length) of the correlation function, which physically means the scale of heterogeneity [12], and it should be positive.

From Eq. (10), we have

$$L_0 < f(y_1) >= \mu_0\left(k_0^2 + \frac{\partial^2}{\partial y^2}\right)Ae^{-by_1} = \mu_0\left(k_0^2 + b^2\right)Ae^{-by_1} \tag{38}$$

Substituting Eqs. (31), (36) and (38) into Eq. (30), we get the dispersion equation,

$$k_0^2 + b^2 - \frac{\omega^4\zeta^2\epsilon^2 b}{2k_0\mu_0^2}\left(\frac{1}{(b+\frac{1}{R_c})^2 + b^2} + \frac{1}{(b-\frac{1}{R_c})^2 + b^2}\right) = 0 \tag{39}$$

It could be seen from the dispersion equation Eq. (39) that if there is no random fluctuation, i.e. $\epsilon = 0$ or $\zeta = 0$ then $k_0^2 + b^2 = 0$ —the equation becomes the dispersion equation without random heterogeneities;

Considering the surface condition Eq. (18), the dispersion equation could be written as,

$$k_0^2 + \beta^2 - \frac{\omega^4\zeta^2\epsilon^2 \beta}{2k_0\mu_0^2}\left(\frac{1}{(\beta+\frac{1}{R_c})^2 + \beta^2} + \frac{1}{(\beta-\frac{1}{R_c})^2 + \beta^2}\right) = 0 \tag{40}$$

To conveniently evaluate numerically the effect of random heterogeneities, the dispersion equation Eq. (40) is transformed into a dimensionless equation in the following.

Introduce new dimensionless variables as,

$$\begin{aligned} \overline{\omega} &= \frac{2h\omega}{\pi C_s} & \overline{k} &= \frac{2hk_1}{\pi} \\ \overline{R_c} &= \frac{\pi R_c}{2h} & \overline{\zeta} &= \frac{\epsilon\zeta}{\rho_0} \\ \overline{\mu_0} &= \frac{\mu_0}{\rho_0 C_s^2} = 1 & \overline{\beta} &= \frac{2h\beta}{\pi} \end{aligned} \tag{41}$$

From Eqs. (25) and (41), we get

$$k_0^2 = \frac{\omega^2}{C_s^2} - k_1^2 = \left(\frac{\pi}{2h}\right)^2 \left(\overline{\omega}^2 - \overline{k}^2\right) \tag{42}$$

so the dimensionless k_0 is defined as,

$$\overline{k_0}^2 = \overline{\omega}^2 - \overline{k}^2 \tag{43}$$

Using Eqs. (41), we could get the dimensionless dispersion equation from Eq. (19),

$$\overline{\omega}^2 - \overline{k_1}^2 + \overline{\beta}^2 = 0 \tag{44}$$

Using Eqs. (41) and (43), the dimensionless dispersion equation of Eq. (40) is,

$$\overline{k_0}^2 + \overline{\beta}^2 - \Lambda = 0 \tag{45}$$

Λ denote the random term,

$$\Lambda = \frac{\overline{\omega}^4 \overline{\zeta}^2 \overline{\beta}}{2\overline{k_0}} \left(\frac{1}{(\overline{\beta} + \frac{1}{R_c})^2 + \overline{\beta}^2} + \frac{1}{(\overline{\beta} - \frac{1}{R_c})^2 + \overline{\beta}^2} \right) \tag{46}$$

3 Numerical results and analysis

The SH surface waves propagating in a half space with random densities is further studied numerically. The dimensionless dispersion equation Eq. (45) is used to compute the curves. The numerical results are explained and discussed in the following.

3.1 Random heterogeneities only on the surface

The geomorphy of the earth's surface is always very complex. The reason for this complexness can come from both natural and man-made actions. In this study, we model the complex geomorphy by giving a surface parameter β. So in this section, we will study the dispersion properties for half spaces with random heterogeneities only on the surface. The dispersion curves are plotted according to Eq. (19).

From figure 2, it can be seen that the phase velocity will grow to 1 slowly, but for $\overline{k} < 2$, the phase velocity will be 0, i.e. the waves become standing waves in this circumstance.

From figure 3, it can be seen that, given a wave number, the phase velocity will decrease to 0 as the surface parameter $\overline{\beta}$ grows, i.e. the waves propagate more and more slowly when the surface becomes more and more rough, and all the waves will be blocked when $\overline{\beta}$ is large enough.

3.2 Frequency spectrum analysis

In the following, we will study the dispersion properties for half spaces with random heterogeneities not only on the surface but also in the whole half space. The related parameters are set to $\epsilon = 0.1, \overline{\zeta} = 2, \overline{R_c} = 0.4, \overline{\beta} = 2$ respectively.

From figures 4, 5 and 6, we can see that

1. As the wave number grows, the velocity will grow to a value—approximately 0.93 in this case. The reason that it can not reach to 1 could be that the waves are reflected and scattered by the random heterogeneities.

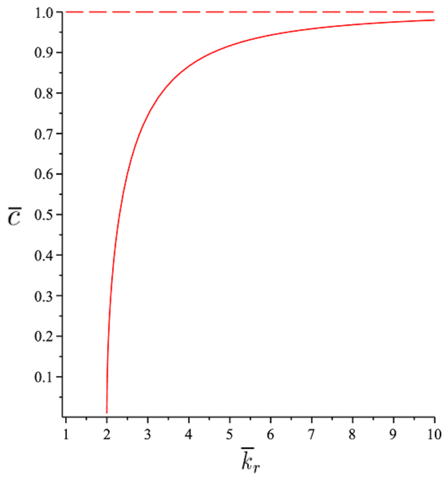

Figure 2: Normalized phase velocity—normalized wave number. $\overline{\beta} = 2$

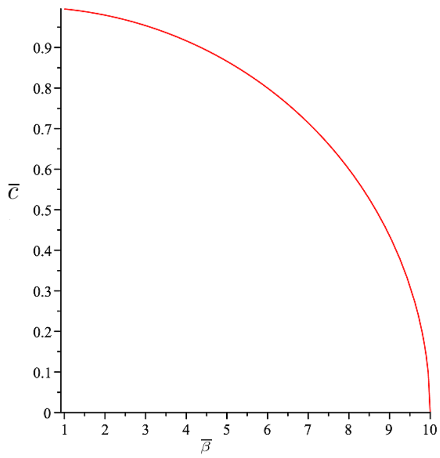

Figure 3: Normalized phase velocity—normalized surface parameter. $\overline{k} = 10$

2. The wave number does not start from 0, but 2. We can call this value the cut-off wave number. 2 is also the value of $\overline{\beta}$. From Eq. (45), we can see that the cut-off wave number equals the surface parameter.

Also, from figure 5, it can be seen that the phase velocity will decrease to 0 when the wave number decreases. This phenomenon agrees with the common knowledge that when the wave number decreases (the wave length increases), the effect of the random heterogeneities will be averaged out gradually, that is, the stochastically homogeneous half space will be more and more like a homogeneous half space, and we know that SHSW could not exist in a homogeneous half space, therefore, the phase velocity will decrease gradually to 0.

The imaginary wave number represents the attenuation rate. Therefore, we know from figure 7 that the bigger the circular frequency is, the faster the wave attenuates. This phenomenon should be caused by reflection and scattering. And from figure 6 we see that the wave length will decrease as the circular frequency grows. It is known that the smaller the wave length is, the easier the waves can be reflected or scattered by the random heterogeneities. Thus the wave attenuates more fast as the frequency grows.

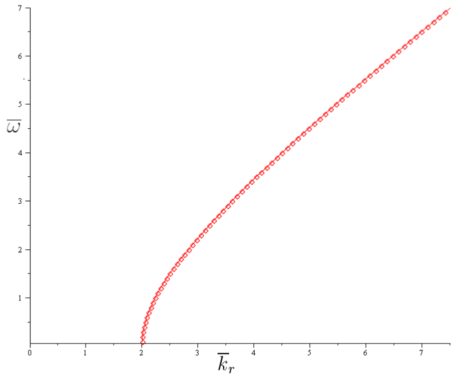

Figure 4: Normalized circular frequency—normalized wave number

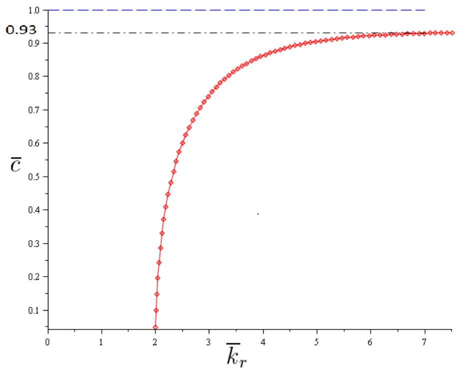

Figure 5: Normalized phase velocity—normalized wave number

4 Conclusion

In this study, we proved that SHSW could exist in a stochastically homogeneous half space. The dispersion properties of SHSW in an half space with random density in the depth direction or only near the surface have been investigated both theoretically and numerically. The first order smoothing approximation method is used to solve the random differential equation. The dimensionless dispersion equation is obtained. And the dispersion properties is further studied numerically. The phase velocity is found increasing to an asymptotic value when the wave number is bigger than a critical value—the cut-off wave number, below which the phase velocity is 0. The interesting properties of dispersion and attenuation found here will help us understanding properties of waves in a half space with random heterogeneities, e.g. the earth's crust. It will also help us to do the inverse problems, for example, to use seismic waves to detect the earth's upper crust structure, and to extract information more exactly from the acoustic testing results.

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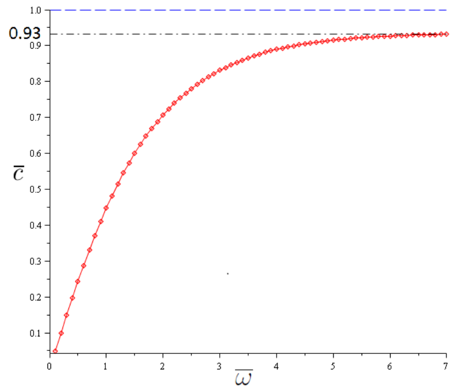

Figure 6: Normalized phase velocity—normalized circular frequency

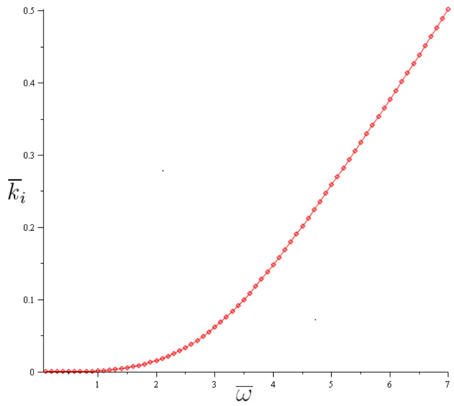

Figure 7: Normalized imaginary wave number—normalized circular frequency

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