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DYNAMIC ANALYSIS OF PLATES AND BEAMS BY GFDM

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Abstract. *This paper shows the application of Generalized Finite Difference Method (GFDM) to dynamic analysis of beams and plates. The use of a meshless method with the possibility of using an irregular grid-point distribution can be of interest for modeling this problem.*

1 INTRODUCTION

The Generalized finite difference method (GFDM) is evolved from classical finite difference method (FDM). GFDM can be applied over general or irregular clouds of points [7]. The basic idea is to use moving least squares (MLS) approximation to obtain explicit difference formulae which can be included in the partial differential equations [9]. Benito, Ureña and Gavete have made interesting contributions to the development of this method [1, 2, 6, 4, 5, 12]. The paper [3] shows the application of the GFDM in solving parabolic and hyperbolic equations.
This paper decribes how the GFDM can be applied for solving dynamic analysis problems of plates [10, 11, 13].
The paper is organized as follows. Section 1 is the introduction. Section 2 describes the explicit generalized finite difference schemes. In section 3 is studied the von Neumann stability. In Section 4 is analyzed the relation between stability and irregularity of a cloud of nodes. In Section 5 some applications of the GFDM for solving problems of dynamic analysis are included. Finally, in Section 6 some conclusions are given.

2 EXPLICIT GENERALIZED FINITE DIFFERENCE SCHEMES

2.1 Vibrations of simple beam

Let us consider the problem governed by the following partial differential equation (pde)

$$\frac{\partial^2 U(x,t)}{\partial t^2} + A_1^2 \frac{\partial^4 U(x,t)}{\partial x^4} = F_1(x,t) \quad x \in (0,L), \quad t > 0 \tag{1}$$

with boundary conditions at the ends of the beam of length L for each particular case and initial conditions

$$U(x,0) = 0; \quad \frac{\partial U(x,t)}{\partial t}|_{(x,0)} = F_2(x) \tag{2}$$

where F_1 and F_2 are two known smooth functions, the constant A_1 depends of the material and geometry of the beam.
Firstly, we use the explicit difference formulae for the values of partial derivatives in the space variable. The intention is to obtain explicit linear expressions for the approximation of partial derivatives in the points of the domain.
First of all, an irregular grid or cloud of points is generated in the domain. On defining the composition central node with a set of N points surrounding it (henceforth referred as nodes), the star then refers to the group of established nodes in relation to a central node. Each node in the domain have an associated star assigned [1, 5, 7, 9].
If u_0 is an approximation of fourth-order for the value of the function at the central node (U_0) of the star, with coordinate x_0 and u_j is an approximation of fourth-order for the value of the function at the rest of nodes, of coordinates x_j with $j = 1, \cdots, N$, then, according to the Taylor series expansion

$$U_j = U_0 + h_j \frac{\partial U_0}{\partial x} + \frac{h_j^2}{2} \frac{\partial^2 U_0}{\partial x^2} + \frac{h_j^3}{6} \frac{\partial^3 U_0}{\partial x^3} + \frac{h_j^4}{24} \frac{\partial^4 U_0}{\partial x^4} + \cdots \tag{3}$$

where $h_j = x_j - x_0$.
If in equation 3 the terms over fourth order are ignored. It is then possible to define the function $B_4(u)$ as in [1, 3, 4, 5, 7, 9]

$$B_4(u) = \sum_{j=1}^{N} [(u_0 - u_j + h_j \frac{\partial u_0}{\partial x} + \frac{h_j^2}{2} \frac{\partial^2 u_0}{\partial x^2} + \frac{h_j^3}{6} \frac{\partial^3 u_0}{\partial x^3} + \frac{h_j^4}{24} \frac{\partial^4 u_0}{\partial x^4}) w(h_j)]^2 \tag{4}$$

where $w(h_j)$ is the denominated weighting function.
If the norm 4 is minimized with respect to the partial derivatives the linear equations system is obtained

$$\boldsymbol{A_4}\boldsymbol{D_{u_4}} = \boldsymbol{b_4} \tag{5}$$

where

$$\boldsymbol{A_4} = \begin{pmatrix} \sum_{j=1}^{N} h_j^2 w^2 & \sum_{j=1}^{N} \frac{h_j^3}{2} w^2 & \sum_{j=1}^{N} \frac{h_i^4}{6} w^2 & \sum_{j=1}^{N} \frac{h_j^5}{24} w^2 \\ & \sum_{j=1}^{N} \frac{h_j^4}{4} w^2 & \sum_{j=1}^{N} \frac{h_j^5}{12} w^2 & \sum_{j=1}^{N} \frac{h_j^6}{48} w^2 \\ & & \sum_{j=1}^{N} \frac{h_j^6}{36} w^2 & \sum_{j=1}^{N} \frac{h_j^7}{144} w^2 \\ & SYM & & \sum_{j=1}^{N} \frac{h_j^8}{576} w^2 \end{pmatrix} \tag{6}$$

and

$$\boldsymbol{D_{u_4}} = \left\{ \frac{\partial u_0}{\partial x} \quad \frac{\partial^2 u_0}{\partial x^2} \quad \frac{\partial^3 u_0}{\partial x^3} \quad \frac{\partial^4 u_0}{\partial x^4} \right\}^T \tag{7}$$

$$\boldsymbol{b_4} = \left\{ \begin{array}{c} \sum_{j=1}^{N} (-u_0 + u_j) h_j w^2 \\ \sum_{j=1}^{N} (-u_0 + u_j) \frac{h_j^2}{2} w^2 \\ \sum_{j=1}^{N} (-u_0 + u_j) \frac{h_j^3}{6} w^2 \\ \sum_{j=1}^{N} (-u_0 + u_j) \frac{h_j^4}{24} w^2 \end{array} \right\} \tag{8}$$

and solving system 5 the explicit difference formulae are obtained as in [2]. On including the explicit expressions for the values of the partial derivatives the star equation is obtained

$$\frac{\partial^4 U(x,t)}{\partial x^4}|_{(x_0,n)} = \eta_0 u_0 + \sum_{j=1}^{N} \eta_j u_j \tag{9}$$

with

$$\eta_0 + \sum_{j=1}^{N} \eta_j = 0 \tag{10}$$

Secondly, we shall use an explicit formula for the part of the equation 1 that depends on time. This explicit formula can be used to solve the Cauchy initial value problem. This method involves only one grid point at the advanced time level. The second derivative with respect to time is approached by

$$\frac{\partial^2 U}{\partial t^2}|_{(x_0,y_0,n)} = \frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{(\triangle t)^2} \tag{11}$$

If the equations 9 and 11 are substituted in equation 1 the following recursive relationship is obtained

$$u_0^{n+1} = 2u_0^n - u_0^{n-1} - A_1^2(\triangle t)^2[\eta_0 u_0^n + \sum_{j=1}^{N} \eta_j u_j^n] + F_1(x_0, n) \quad (12)$$

The first derivative with respect to the time is approached by the central difference formula

$$\frac{\partial U}{\partial t}|_{x,0} = \frac{u_0^1 - u_0^{-1}}{2\triangle t} = F_2(x_0) \Rightarrow u_0^{-1} = u_0^1 - 2\triangle t F_2(x_0) \quad (13)$$

If equation 13 is substituted in equation 12 and taking into account initials conditions (2), the following equation is obtained

$$u_0^1 = \triangle t F_2(x_0) + \frac{F_1(x_0, 0)}{2} \quad (14)$$

The equation 14 relates the value of the function at the central node of the star, at time $n = 1$, with the values $F_1(x_0, 0)$ and the initial conditions $F_(x_0)$.

2.2 Vibrations of plates

Let us to consider the problem governed by

$$\frac{\partial^2 U(x,y,t)}{\partial t^2} + A_2^2[\frac{\partial^4 U(x,y,t)}{\partial x^4} + 2\frac{\partial^4 U(x,y,t)}{\partial x^2 \partial^2} + \frac{\partial^4 U(x,y,t)}{\partial y^4}] = G_1(x,y,t)$$
$$(x,y) \in (0,L) \times (0,L), \quad t > 0 \quad (15)$$

with boundary conditions at the edges of the plate $[0,L] \times [0,L]$ for each particular case and initial conditions

$$U(x,y,0) = 0; \quad \frac{\partial U(x,y,t)}{\partial t}|_{(x,y,0)} = G_2(x,y) \quad (16)$$

where G_1 and G_2 are two known smooth functions, the constant A_2 depends of the material and geometry of the plate.
In a similar way that the one used in the subsection previous. If u_0 is an approximation of fourth-order for the value of the function at the central node (U_0) of the star, with coordinates (x_0, y_0) and u_j is an approximation of fourth-order for the value of the function at the rest of nodes, of coordinates (x_j, y_j) with $j = 1, \cdots, N$, then, according to the Taylor series expansion

$$U_j = U_0 + h_j\frac{\partial U_0}{\partial x} + k_j\frac{\partial U_0}{\partial y} + \frac{h_j^2}{2}\frac{\partial^2 U_0}{\partial x^2} + \frac{k_j^2}{2}\frac{\partial^2 U_0}{\partial y^2} + h_j k_j\frac{\partial^2 U_0}{\partial x \partial y} +$$
$$+ \frac{h_j^3}{6}\frac{\partial^3 U_0}{\partial x^3} + \frac{k_j^3}{6}\frac{\partial^3 U_0}{\partial y^3} + \frac{h_j^2 k_j}{2}\frac{\partial^3 U_0}{\partial x^2 \partial y} + \frac{h_j k_j^2}{2}\frac{\partial^3 U_0}{\partial x \partial y^2} + \frac{h_j^4}{24}\frac{\partial^4 U_0}{\partial x^4} + \frac{k_j^4}{24}\frac{\partial^4 U_0}{\partial y^4} +$$
$$+ \frac{h_j^3 k_j}{6}\frac{\partial^4 U_0}{\partial x^3 \partial y} + \frac{h_j^2 k_j^2}{4}\frac{\partial^4 U_0}{\partial x^2 \partial y^2} + \frac{h_j k_j^3}{6}\frac{\partial^4 U_0}{\partial x \partial y^3} + \cdots \quad (17)$$

where $h_j = x_j - x_0; k_j = y_j - y_0$.
If in equation 17 the terms over fourth order are ignored. It is then possible to define the function

$$B_{14}(u) = \sum_{j=1}^{N}[(u_0 - u_j + h_j\frac{\partial u_0}{\partial x} + k_j\frac{\partial u_0}{\partial y} + \frac{h_j^2}{2}\frac{\partial^2 u_0}{\partial x^2} + \frac{k_j^2}{2}\frac{\partial^2 u_0}{\partial y^2} + h_jk_j\frac{\partial^2 u_0}{\partial x\partial y} + \\ + \frac{h_j^3}{6}\frac{\partial^3 u_0}{\partial x^3} + \frac{k_j^3}{6}\frac{\partial^3 u_0}{\partial y^3} + \frac{h_j^2k_j}{2}\frac{\partial^3 u_0}{\partial x^2\partial y} + \frac{h_jk_j^2}{2}\frac{\partial^3 u_0}{\partial x\partial y^2} + \frac{h_j^4}{24}\frac{\partial^4 u_0}{\partial x^4} + \frac{k_j^4}{24}\frac{\partial^4 u_0}{\partial y^4} + \\ + \frac{h_j^3k_j}{6}\frac{\partial^4 u_0}{\partial x^3\partial y} + \frac{h_j^2k_j^2}{4}\frac{\partial^4 u_0}{\partial x^2\partial y^2} + \frac{h_jk_j^3}{6}\frac{\partial^4 u_0}{\partial x\partial y^3})w(h_j,k_j)]^2 \quad (18)$$

where $w(h_j, k_j)$ is the denominated weighting function.
If the norm 18 is minimized with respect to the partial derivatives the linear equation system is obtained

$$\boldsymbol{A_{14}D_{u_{14}}} = \boldsymbol{b_{14}} \quad (19)$$

where $\boldsymbol{A_{14}}$, $\boldsymbol{D_{u_{14}}}$ and $\boldsymbol{b_{14}}$ can be obtained in a similar way that the one used in the expressions 6, 7 and 8, and solving system the explicit difference formulae are obtained. On including the explicit expressions for the values of the partial derivatives the star equation is obtained

$$[\frac{\partial^4 U(x,y,t)}{\partial x^4} + 2\frac{\partial^4 U(x,y,t)}{\partial x^2\partial^2} + \frac{\partial^4 U(x,y,t)}{\partial y^4}]_{(x_0,y_0,n)} = \mu_0 u_0 + \sum_{j=1}^{N}\mu_j u_j \quad (20)$$

with

$$\mu_0 + \sum_{j=1}^{N}\mu_j = 0 \quad (21)$$

If the equations 9 and 20 are substituted in equation 15 the following recursive relationship is obtained

$$u_0^{n+1} = 2u_0^n - u_0^{n-1} - A_2^2(\triangle t)^2[\mu_0 u_0^n + \sum_{j=1}^{N}\mu_j u_j^n] + G_1(x_0, y_0, n) \quad (22)$$

Using a similar process for to obtain the equation 14 the following equation is obtained

$$u_0^1 = \triangle t G_2(x_0, y_0) + \frac{G_1(x_0, y_0, 0)}{2} \quad (23)$$

The expressions 12 and 22 relates the value of the function at the central node of star, at time step $n+1$, with the values of the functions in the nodes of the star at time step n.

3 CONVERGENCE

According to Lax's equivalence theorem, if the consistency condition is satisfied, stability is the necessary and sufficient condition for convergence. In this section we study firstly the truncation error of the equations 1 and 14, and secondly consistency and stability.

3.1 Truncation error

As it is well known, the truncation errors for second order time derivative (TEt) is given as follows:

$$\frac{\partial^2 U(\boldsymbol{x},t)}{\partial t^2} = \frac{u_0^{t+\triangle t} - 2u_0^t + u_0^{t-\triangle t}}{(\triangle t)^2} - \frac{(\triangle t)^2}{12}\frac{\partial^4 U(\boldsymbol{x},t_1)}{\partial t^4} + \Theta((\triangle t)^4), t < t_1 < t + \triangle t \quad (24)$$

$$(TE_t) = -\frac{(\triangle t)^2}{12}\frac{\partial^4 U(\boldsymbol{x},t_1)}{\partial t^4} + \Theta((\triangle t)^4), t < t_1 < t + \triangle t \quad (25)$$

In order to obtain the truncation error for space derivatives, Taylor's series expansion including higher order derivatives is used and then higher order functions $B_p^*[u]$, $p = 4, 14$ are obtained. The expressions of $B_p[u]$, $p = 4, 14$ are similar to the ones given in 4 and 18, but incorporating now higher order derivatives. If the new norms $B_p^*[u]$, $p = 4, 14$ are minimized with respect to the partial derivatives until the fourth order, the following linear equation systems are obtained:

$$\boldsymbol{A_p D_{u_p}} = \boldsymbol{b_p^*} \quad (26)$$

where $\boldsymbol{A_p}$, $\boldsymbol{D_{u_p}}$ and $\boldsymbol{b_p}$ with $(p = 4, 14)$ are as previously calculated in 6, 7, 8 for $p = 4$ and similarly for $p = 14$, and $\boldsymbol{b_p^*}$ can be split in two parts as follows

$$\boldsymbol{b_p^*} = \boldsymbol{b_p} + \boldsymbol{b_p^{**}} \quad (27)$$

where the news terms $\boldsymbol{b_p^{**}}$ correspond to the new higher order derivatives incorporated in the Taylor's series expansion to extend the functions from $B_p[u]$, $p = 4, 14$ to $B_p^*[u]$, $p = 4, 14$.
Then a better approximation of the partial derivatives can be obtained using the inverse matrix $\boldsymbol{A_p^{-1}}$

$$\boldsymbol{D_{u_p}} = \boldsymbol{A_p^{-1} b_p} + \boldsymbol{A_p^{-1} b_p^{**}} \quad (28)$$

In the equation 28 the expression $\boldsymbol{A_p^{-1} b_p}$ is the approximation used in the GFDM (see [3] and [12]) and then the truncation errors for spatial derivatives are given by

$$TE_{x_p} = \boldsymbol{A_p^{-1} b_p^{**}} \quad (29)$$

We develop only the truncation error corresponding to $p = 4$ case. The other truncation error for $p = 14$ case can be obtained in a similar way that the one used in $p = 4$ case.

$$B_4^*(u) = \sum_{j=1}^{N} [(U_0 - U_j + h_j \frac{\partial U_0}{\partial x} + \frac{h_j^2}{2!}\frac{\partial^2 U_0}{\partial x^2} + \frac{h_j^3}{3!}\frac{\partial^3 U_0}{\partial x^3} + \frac{h_j^4}{4!}\frac{\partial^4 U_0}{\partial x^4} + + \frac{h_j^5}{5!}\frac{\partial^5 U_0}{\partial x^5} + \frac{h_j^6}{6!}\frac{\partial^6 U_0}{\partial x^6} + \cdots) w(h_j)]^2 \quad (30)$$

If the function 29 is minimized with respect partial derivatives up to the fourth order, the following linear equations system is defined

$$\boldsymbol{A_4 D_{u_4}} = \left(\sum_{j=1}^{N} \Xi h_j \quad \sum_{j=1}^{N} \Xi \frac{h_j^2}{2!} \quad \sum_{j=1}^{N} \Xi \frac{h_j^3}{3!} \quad \sum_{j=1}^{N} \Xi \frac{h_j^4}{4!} \right)^T \quad (31)$$

where

$$\Xi = [-U_0 + U_j - \frac{h_j^5}{5!}\frac{\partial^5 U_0}{\partial x^5} - \frac{h_j^6}{6!}\frac{\partial^6 U_0}{\partial x^6} - \cdots)]w(h_j)^2 \tag{32}$$

with $N \geq 4$, and then

$$TE_{x_4} = -\frac{1}{A_1^2}\boldsymbol{A_4^{-1}} \times \left(\sum_{j=1}^{N} \Upsilon h_j \quad \sum_{j=1}^{N} \Upsilon \frac{h_j^2}{2!} \quad \sum_{j=1}^{N} \Upsilon \frac{h_j^3}{3!} \quad \sum_{j=1}^{N} \Upsilon \frac{h_j^4}{4!} \right)^T \tag{33}$$

where

$$\Upsilon = -[\frac{h_j^5}{5!}\frac{\partial^5 U_0}{\partial x^5} + \frac{h_j^6}{6!}\frac{\partial^6 U_0}{\partial x^6} + \cdots)]w(h_j)^2 \tag{34}$$

and operating

$$TE_{x_4} = \frac{1}{A_1^2}[\sum_{j=1}^{N} \Psi_{1,j}\frac{\partial^5 U}{\partial x^5} + \Psi_{2,j}\frac{\partial^6 U}{\partial x^6} + ...] + \Theta(h_j) \tag{35}$$

where $\Psi_{1,j}(h_j)$ and $\Psi_{2,j}(h_j)$ are homogeneous rational functions of order two and $\Theta(h_j)$ is a series of third- and higher-order functions.
The expression 35 is the truncation error for spatial derivatives.
Taking into account that the total truncation errors (TTE) is given by

$$TTE = TE_t + TE_{x_4} \tag{36}$$

where TE_t and TE_{x_4} are given by 25 and 35 respectively.

3.2 Consistency

By considering bounded derivatives in 36

$$\lim_{(\triangle t, h_j)\to(0,0)} TTE \to 0 \tag{37}$$

Then, the truncation error condition given in 37 shows the consistency of the approximation.

3.3 Stability criterion

For the difference schemes, the von Neumann condition is sufficient as well as necessary for stability [8]. "Boundary conditions are neglected by the von Neumann method which applies in theory only to pure initial value problems with periodic initial data. It does however provide necessary conditions for stability of constant coefficient problems regardless of the type of boundary condition".
For the stability analysis the first idea is to make a harmonic decomposition of the approximated solution at grid points and at a given time level n. Then we can write the finite difference approximation in the nodes of the star at time n, as

$$u_0^n = \xi^n e^{i\boldsymbol{\nu}^T \boldsymbol{x_0}}; \quad u_j^n = \xi^n e^{i\boldsymbol{\nu}^T \boldsymbol{x_j}} \tag{38}$$

where ξ is the amplification factor,

$$\boldsymbol{x_j} = \boldsymbol{x_0} + \boldsymbol{h_j}; \qquad \xi = e^{-iw\triangle t}$$

$\boldsymbol{\nu}$ is the column vector of the wave numbers

$$\boldsymbol{\nu} = \left\{ \begin{array}{c} \nu_x \\ \nu_y \end{array} \right\}$$

then we can write the stability condition as: $\|\xi\| \leq 1$.
Including the equation 38 into the equation 12 or 22, cancelation of $\xi^n e^{i\boldsymbol{\nu}^T \boldsymbol{x_0}}$, leads to

$$\xi = 2 + \frac{1}{\xi} - (\triangle t)^2 A^2 (m_0 + \sum_1^N m_j e^{i\boldsymbol{\nu}^T \boldsymbol{h_j}}) \tag{39}$$

where A is the constant A_1 or A_2 respectively and m_0, m_j are the coefficients η_0, η_j or μ_0, μ_j.
Using the equations 10 or 21 and after some calculus we obtain the quadratic equation

$$\xi^2 - \xi[2 + A^2(\triangle t)^2(\sum_1^N m_j(1 - \cos \boldsymbol{\nu}^T \boldsymbol{h_j}) - i \sum_1^N m_j \sin \boldsymbol{\nu}^T \boldsymbol{h_j})] + 1 = 0 \tag{40}$$

Hence the values of ξ are

$$\xi = b \pm \sqrt{b^2 - 1} \tag{41}$$

where

$$b = 1 + \frac{A^2(\triangle t)^2}{2} \sum_1^N m_j(1 - \cos \boldsymbol{\nu}^T \boldsymbol{h_j}) - i \frac{A^2(\triangle t)^2}{2} \sum_1^N m_j \sin \boldsymbol{\nu}^T \boldsymbol{h_j} \tag{42}$$

If we consider now the condition for stability, we obtain

$$\|b \pm \sqrt{b^2 - 1}\| \leq 1 \tag{43}$$

Operating with the equations 42 and 43, canceling with conservative criteria, the condition for stability of star is obtained as

$$\triangle t \leq \frac{1}{4A\sqrt{|m_0|}} \tag{44}$$

4 IRREGULARITY OF THE STAR (IIS) AND STABILITY

In this section we are going to define the index of irregularity of a star (IIS) and also the index of irregularity of a cloud of nodes (IIC).
The coefficient m_0 is function of:

- The number of nodes in the star
- The coordinates of each star node referred to the central node of the star
- The weighting function (see references $[1, 4]$)

If the number of nodes by star and the weighting function are fixed, then the equation 44 is function of the coordinates of each node of star referred to its central node.
Denoting τ_0 as the average of the distances between of the nodes of the star and its central node with coordinates (x_0, y_0) and denoting τ the average of the τ_0 values in the stars of the cloud of nodes, then

$$\overline{m_0} = m_0 \tau^4 \tag{45}$$

The stability criterion can be rewritten as

$$\triangle t < \frac{\tau^2}{4A\sqrt{|\overline{m_0}|}} \tag{46}$$

For the regular mesh case, the inequality 44 is for the cases of one and two dimensions as follows

$$\begin{cases} \triangle t < \dfrac{\sqrt{2}\tau^2}{18A\sqrt{3}} & if \quad N=4 \\ \triangle t < \dfrac{9\tau^2}{A\sqrt{13}[3(1+\sqrt{2})+2\sqrt{5}]^2} & if \quad N=24 \end{cases} \tag{47}$$

Multiplying the right-hand side of inequalities 47, respectively, by the factors

$$\begin{cases} \dfrac{9\sqrt{3}}{2\sqrt{2|\overline{m_0}|}} & if \quad N=4 \\ \dfrac{\sqrt{13}[3(1+\sqrt{2})+2\sqrt{5}]^2}{36\sqrt{|\overline{m_0}|}} & if \quad N=24 \end{cases} \tag{48}$$

the inequality 46 is obtained.
For each one of the stars of the cloud of nodes, we define the IIS for a star with central node in (x_0, y_0) as Eq. 48

$$\begin{cases} IIS_{x_0} = \dfrac{9\sqrt{3}}{2\sqrt{2|\overline{m_0}|}} & if \quad N=4 \\ IIS_{(x_0,y_0)} = \dfrac{\sqrt{13}[3(1+\sqrt{2})+2\sqrt{5}]^2}{36\sqrt{|\overline{m_0}|}} & if \quad N=24 \end{cases} \tag{49}$$

that takes the value of one in the case of a regular mesh and $0 < IIS \leq 1$
If the index IIS decreases, then absolute values of $\overline{m_0}$ increases and then according with 43, $\triangle t$ decreases.
The irregularity index of a cloud of nodes (IIC) is defined as the minimum of all the IIS of the stars of a cloud of nodes.

5 NUMERICAL RESULTS

In this section we present different numerical results.

5.1 Transverse vibrations of a simply supported beam

In this section, the weighting function used is

$$\Omega(h_j) = \frac{1}{(\sqrt{h_j^2})^3} \tag{50}$$

The global exact error can be calculated as

$$Global \quad exact \quad error = \sqrt{\frac{\sum_{i=1}^{NT} e_i^2}{NT}} \tag{51}$$

Figure 1: regular and irregular mesh

Figure 2: irregular meshes

where NT is the number of nodes in the domain and e_i is the exact error in the node i.
Let us solve the pde

$$\frac{\partial^2 U(x,t)}{\partial t^2} + \frac{1}{\pi^4}\frac{\partial^4 U(x,t)}{\partial x^4} = 0 \quad x \in (0,1), \quad t > 0 \tag{52}$$

with boundary conditions

$$\begin{cases} U(0,t) = U(1,t) = 0 \\ \frac{\partial^2 U(x,t)}{\partial x^2}|_{(0,t)} = \frac{\partial^2 U(x,t)}{\partial x^2}|_{(1,t)} = 0, \end{cases} \tag{53}$$

and initial conditions

$$U(x,0) = 0; \quad \frac{\partial U(x,t)}{\partial t}|_{(x,0)} = \sin(\pi x) \tag{54}$$

The exact solution is

$$U(x,t) = \sin(\pi x)\sin t \tag{55}$$

Table 1 shows the results of the global error, using a regular mesh of 21 nodes (figure 1), for several values of $\triangle t$.

$\triangle t$	Global error
0.005	0.00276
0.002	0.00109
0.001	0.00017
0.0005	0.00002

Table 1: Influence of $\triangle t$ in the global error.

IIC	Global error
0.96	0.00107
0.78	0.00295
0.62	0.00534
0.46	0.00903

Table 2: Influence of irregularity of mesh in the global error

Table 2 shows the results of global error with $\triangle t = 0.001$ for several irregular meshes of 21 nodes (figures 1 and 2). We have established a measure of the irregularity of the nodes distribution in the domain. For this purpose we have assigned to every node in the domain a value that corresponds with the average of the distances from it to the rest of its star nodes. Then, the index of irregularity (IIC) is defined as the standard deviation of these values.
Figure 3 shows the approximated solution of the equations 52, 53 and 54 in the last time step $(n = 1000)$ with $\triangle t = 0.005$.
As new initial conditions let us assume that due to impact an initial velocity is given to a point of the beam at the distance $x = 0.5$ from the left-hand support, which give the initial conditions

$$U(x,0) = 0; \quad \begin{cases} \frac{\partial U(x,t)}{\partial t}|_{(x,0)} = 1 \quad if \quad x = 0.5 \\ \frac{\partial U(x,t)}{\partial t}|_{(x,0)} = 0 \quad if \quad x \neq 0.5 \end{cases} \tag{56}$$

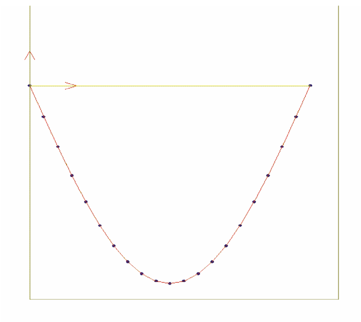

Figure 3: Approximated solution in the last time step ($\triangle t = 0.005$)

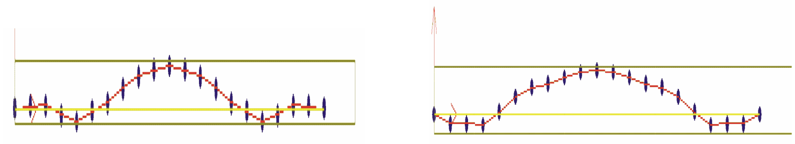

Figure 4: Approximated solution with n=100 Approximated solution with n=200

The exact solution in this case is given by

$$U(x,t) = 2(\sin(\pi x)\sin(t) - \frac{1}{9}\sin(3\pi x)\sin(9t) + \frac{1}{25}\sin(5\pi x)\sin(25t) - \cdots) \tag{57}$$

Table 3 shows the results of the global error, using a regular mesh of 21 nodes (figure 1) and $\triangle t = 0.001$, when we increases the number of time steps (n).
Figures 4 and 5 shows the approximated solution of the equation 52 with the initial conditions

n	Global error
100	0.001628
200	0.001700
500	0.001816
1000	0.002252

Table 3: Variation of global error versus the number of time steps.

56 in the last time step for $n = 100$, $n = 200$, $n = 500$ and $n = 1000$ respectively.

5.2 Forced vibrations of a simply supported beam

In this section, the weighting function used is 50 and the global error is calculated by 51
The pde is given by

$$\frac{\partial^2 U(x,t)}{\partial t^2} + \frac{1}{\pi^4}\frac{\partial^4 U(x,t)}{\partial x^4} = 15\sin(2\pi x)\sin t \quad x \in (0,1), \quad t > 0 \tag{58}$$

with boundary conditions

$$\begin{cases} U(0,t) = U(1,t) = 0 \\ \frac{\partial^2 U(x,t)}{\partial x^2}|_{(0,t)} = \frac{\partial^2 U(x,t)}{\partial x^2}|_{(1,t)} = 0, \end{cases} \tag{59}$$

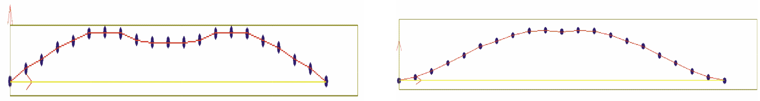

Figure 5: Approximated solution withn=500   Approximated solution with n=1000

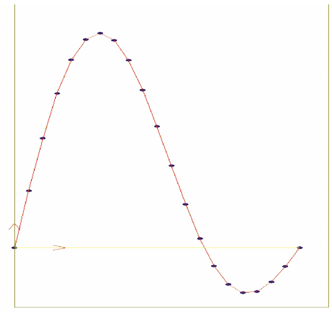

Figure 6: Approximated solution in the last step ($\triangle t = 0.005$)

and initial conditions

$$U(x,0) = 0; \qquad \frac{\partial U(x,t)}{\partial t}|_{(x,0)} = \sin(\pi x) + \sin(2\pi x) \tag{60}$$

The exact solution for this case is given by

$$U(x,t) = (\sin(\pi x) + \sin(2\pi x)) \sin t \tag{61}$$

Table 4 shows the results of the global error, using a regular mesh of 21 nodes (figure 1) for several values of $\triangle t$.
Table 5 shows the results of global error with $\triangle t = 0.001$ for several irregular meshes of 21 nodes (figures 1 and 2).
Figure 6 shows the approximated solution of the equation 57 in the last time step $(n = 1000)$

$\triangle t$	Global error
0.005	0.01025
0.002	0.00987
0.001	0.00625
0.0005	0.00126

Table 4: Influence of $\triangle t$ in the global error.

IIC	Global error
0.96	0.00744
0.78	0.00751
0.62	0.01960
0.46	0.04496

Table 5: Influence of irregularity of mesh in the global error.

with $\triangle t = 0.005$.

5.3 Natural vibrations of a simply supported plate

In this section, the weighting function used is

$$\Omega(h_j, k_j) = \frac{1}{(\sqrt{h_j^2 + k_j^2})^3} \tag{62}$$

and the global exact error can be calculated by 51
The pde is

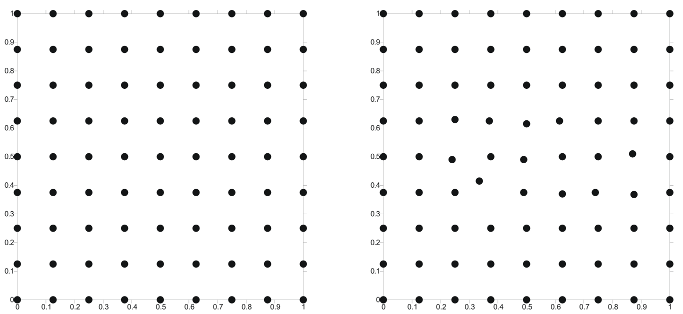

Figure 7: Regular and irregular mesh

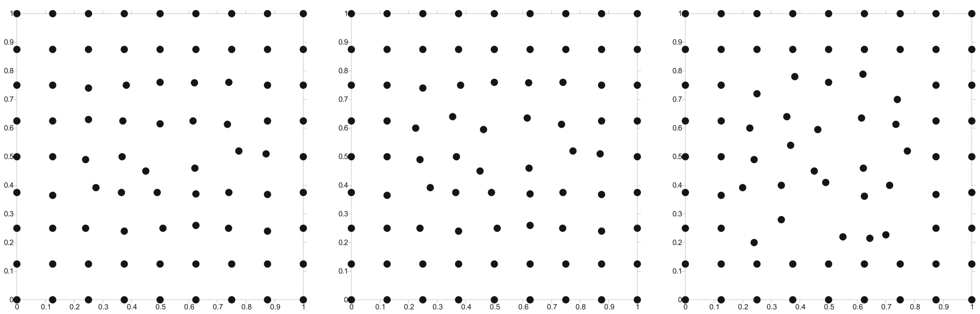

Figure 8: Three irregular meshes

$$\frac{\partial^2 U(x,y,t)}{\partial t^2}+\frac{1}{4\pi^4}[\frac{\partial^4 U(x,y,t)}{\partial x^4}+2\frac{\partial^4 U(x,y,t)}{\partial x^2\partial^2}+\frac{\partial^4 U(x,y,t)}{\partial y^4}]=$$
$$15\sin t\sin(2\pi x)\sin(2\pi y)\quad (x,y)\in(0,1)\times(0,1),\quad t>0 \quad (63)$$

with boundary conditions

$$\begin{cases} U(x,y,t)|_{\Gamma}=0 \\ \frac{\partial^2 U(x,y,t)}{\partial y^2}|_{(0,y,t)}=\frac{\partial^2 U(x,y,t)}{\partial y^2}|_{(1,y,t)}=0,\forall y\in[0,1] \\ \frac{\partial^2 U(x,y,t)}{\partial x^2}|_{(x,0,t)}=\frac{\partial^2 U(x,y,t)}{\partial x^2}|_{(x,1,t)}=0,\forall x\in[0,1] \end{cases} \quad (64)$$

where Γ is the boundary of the domain $[0,1]\times[0,1]$, and initial conditions

$$U(x,y,0)=0;\quad \frac{\partial U(x,y,t)}{\partial t}|_{(x,y,0)}=\sin(\pi x)\sin(\pi y) \quad (65)$$

The exact solution is given by

$$U(x,y,t)=\sin(\pi x)\sin(\pi y)\sin t \quad (66)$$

Table 6 shows the results of the global error, using a regular mesh of 81 nodes (figure 7), for several values of $\triangle t$.

Table 7 shows the results of global error with $\triangle t=0.001$ for several irregular meshes of 81 nodes (figures 7 and 8).

Figure 9 shows the approximated solution of the equation 63 in the last time step $(n=1000)$.

As new initial conditions let us assume that due to impact an initial velocity is given to a point $(x=y=0.5)$ of the plate, which give the conditions

$$U(x,y,0)=0;\quad \begin{cases} \frac{\partial U(x,y,t)}{\partial t}|_{(x,y,0)}=1 \quad if \qquad x=y=0.5 \\ \frac{\partial U(x,y,t)}{\partial t}|_{(x,y,0)}=0 \quad if \quad (x,y)\neq(0.5,0.5) \end{cases} \quad (67)$$

$\triangle t$	Global error
0.01	0.08254
0.005	0.03513
0.002	0.01339
0.001	0.00212

Table 6: Influence of $\triangle t$ in the global error.

IIC	Global error
0.92	0.00224
0.83	0.00224
0.76	0.00231
0.58	0.00251

Table 7: Influence of irregularity of mesh in the global error.

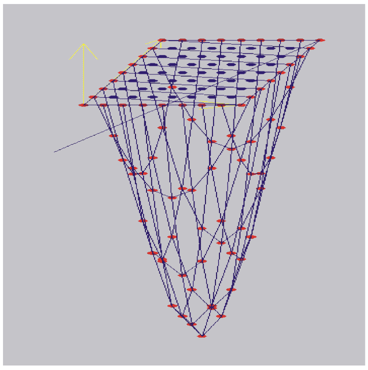

Figure 9: Approximated solution in the last time step

The exact solution is given by

$$U(x,y,t) = 2[\sin(\pi x)\sin(\pi y)\sin(t) - \frac{1}{9}\sin(3\pi x)\sin(3\pi y)\sin(9t) + \frac{1}{25}\sin(5\pi x)\sin(5\pi y)\sin(25t) - \cdots] \quad (68)$$

Table 8 shows the results of the global error, using a regular mesh of 81 nodes (figure 7) and

n	Global error
100	0.01122
200	0.01858
600	0.02690
1200	0.03363

Table 8: Variation of global error versus the number of time steps

$\triangle t = 0.001$, versus the number of time steps (n).

Figures 10 and 11 show the approximated solution of the equation 63 with the initial conditions 67 in the last time steps for the cases $n = 100$, $n = 200$, $n = 600$ and $n = 1200$ time steps respectively.

5.4 Forced vibrations of a simply supported plate

In this section, the weighting function used is 62 and the global error is calculated by 51
The pde is

$$\frac{\partial^2 U(x,y,t)}{\partial t^2} + \frac{1}{4\pi^4}[\frac{\partial^4 U(x,y,t)}{\partial x^4} + 2\frac{\partial^4 U(x,y,t)}{\partial x^2 \partial^2} + \frac{\partial^4 U(x,y,t)}{\partial y^4}] = 15\sin t \sin(2\pi x)\sin(2\pi y)) \quad (x,y) \in (0,1)\times(0,1), \quad t > 0 \quad (69)$$

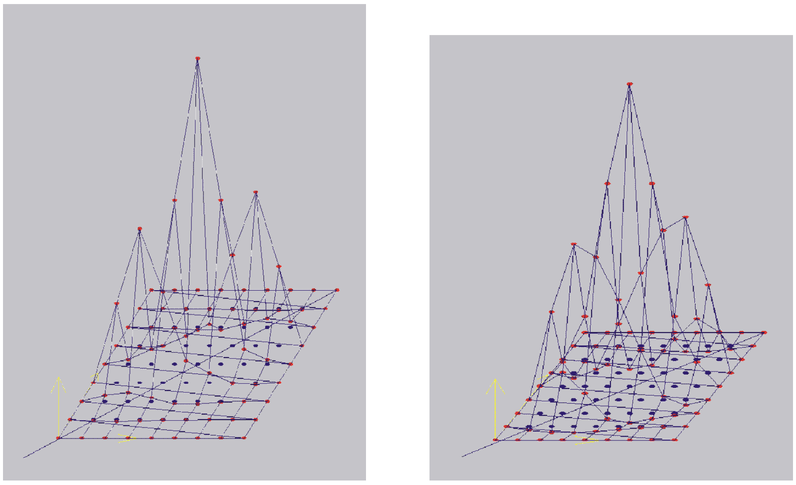

Figure 10: Approximated solution with n=100 Approximated solution with n=200

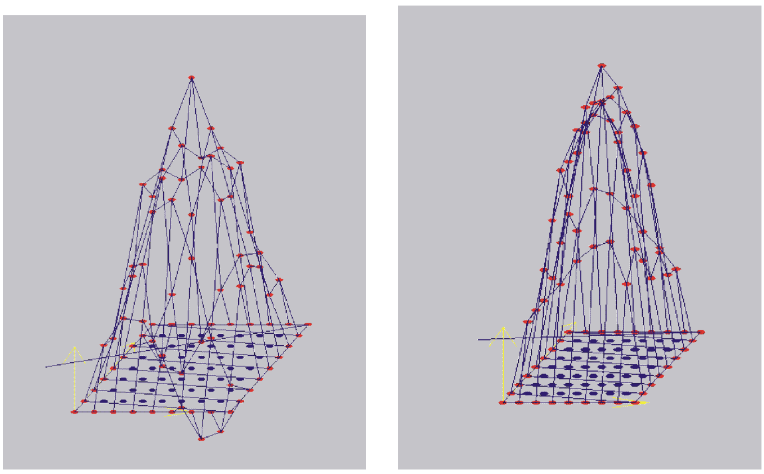

Figure 11: Approximated solution with n=600 Approximated solution with n=1200

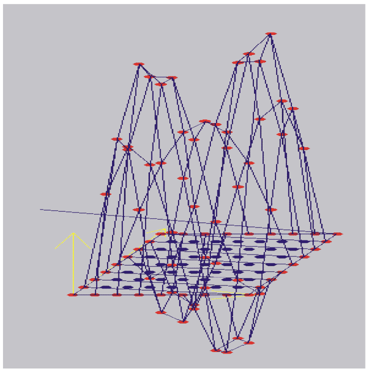

Figure 12: Approximated solution in the last time step

with boundary conditions

$$\begin{cases} U(x,y,t)|_{\Gamma} = 0 \\ \frac{\partial^2 U(x,y,t)}{\partial y^2}|_{(0,y,t)} = \frac{\partial^2 U(x,y,t)}{\partial y^2}|_{(1,y,t)} = 0, \forall y \in [0,1] \\ \frac{\partial^2 U(x,y,t)}{\partial x^2}|_{(x,0,t)} = \frac{\partial^2 U(x,y,t)}{\partial x^2}|_{(x,1,t)} = 0, \forall x \in [0,1] \end{cases} \tag{70}$$

and initial conditions

$$U(x,y,0) = 0; \quad \frac{\partial U(x,y,t)}{\partial t}|_{(x,y,0)} = \sin(\pi x)\sin(\pi y) + \sin(2\pi x)\sin(2\pi y) \tag{71}$$

The exact solution is given by

$$U(x,y,t) = (\sin(\pi x)\sin(\pi y) + \sin(2\pi x)\sin(2\pi y))\sin t \tag{72}$$

Table 9 shows the results of the global error, using regular mesh of 81 nodes (figure 7), for several values of $\triangle t$. Table 10 shows the results of global error with $\triangle t = 0.001$ for several irregular meshes of 81 nodes (figures 7 and 8).

$\triangle t$	Global error
0.01	0.53070
0.005	0.14640
0.002	0.07837
0.001	0.01444

Table 9: Influence of $\triangle t$ in the global error.

IIC	Global error
0.92	0.01412
0.83	0.01437
0.76	0.01442
0.58	0.01447

Table 10: Influence of irregularity of mesh in the global error.

Figure 12 shows the approximated solution of the equation 68 in the last time step $(n = 1000)$.

5.5 Transverse vibrations of a beam with fixed ends

In this section, the weighting function used is 50 and the global exact error can be calculated as 51.
The pde is

$$\frac{\partial^2 U(x,t)}{\partial t^2} + \frac{1}{4.73^4}\frac{\partial^4 U(x,t)}{\partial x^4} = 0 \quad x \in (0,1), \quad t > 0 \tag{73}$$

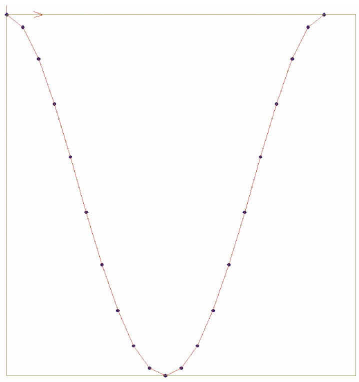

Figure 13: Approximated solution in last time step

with boundary conditions

$$\begin{cases} U(0,t) = U(1,t) = 0 \\ \frac{\partial U(x,t)}{\partial x}|_{(0,t)} = \frac{\partial U(x,t)}{\partial x}|_{(1,t)} = 0, \end{cases} \quad (74)$$

and initial conditions

$$U(x,0) = 0; \quad \frac{\partial U(x,t)}{\partial t}|_{(x,0)} = \cos(4.73x) - \cosh(4.73x) - 0.982501[\sin(4.73x) - \sinh(4.73x)] \quad (75)$$

The exact solution is given by

$$U(x,t) = (\cos(4.73x) - \cosh(4.73x) - 0.982501[\sin(4.73x) - \sinh(4.73x)]) \sin t \quad (76)$$

Table 11 shows the results of the global error, using a regular mesh of 21 nodes (figure 1), for several values of $\triangle t$.

$\triangle t$	Global error
0.01	0.04649
0.005	0.01960
0.002	0.00798
0.001	0.00216

Table 11: Influence of $\triangle t$ in the global error.

IIC	Global error
0.96	0.00419
0.78	0.00423
0.62	0.00763
0.46	0.00781

Table 12: Influence of irregularity of mesh in the global error.

Table 12 shows the results of global error with $\triangle t = 0.001$ for several irregular meshes of 21 nodes (figures 1 and 2).
Figure 13 shows the approximated solution of the equation 73, 74 and 75 in the last time step $(n = 1000)$ with $\triangle t = 0.001$.

5.6 Natural vibrations of a fixed plate

In this section, the weighting function used is 62 and the global exact error can be calculated by 51.
The pde is

$$\frac{\partial^2 U(x,y,t)}{\partial t^2} + \frac{1}{4(4.73)^4}[\frac{\partial^4 U(x,y,t)}{\partial x^4} + 2\frac{\partial^4 U(x,y,t)}{\partial x^2 \partial^2} + \frac{\partial^4 U(x,y,t)}{\partial y^4}] =$$
$$15 \sin t \sin(2\pi x) \sin(2\pi y) \quad (x,y) \in (0,1) \times (0,1), \quad t > 0 \quad (77)$$

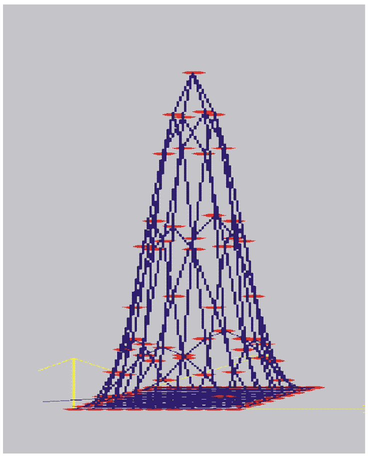

Figure 14: Approximated solution in the last time step

with boundary conditions 64, and initial conditions

$$\begin{cases} U(x,y,0)=0 \\ \frac{\partial U(x,y,t)}{\partial t}|_{(x,y,0)} = (\cos(4.73x)-\cosh(4.73x)-0.982501[\sin(4.73x)- \\ \sinh(4.73x)])(\cos(4.73y)-\cosh(4.73y)-0.982501[\sin(4.73y)-\sinh(4.73y)]) \end{cases} \quad (78)$$

The exact solution is given by

$$U(x,y,t) = (\cos(4.73x)-\cosh(4.73x)-0.982501[\sin(4.73x)-\sinh(4.73x)])(\cos(4.73y) \\ -\cosh(4.73y)-0.982501[\sin(4.73y)-\sinh(4.73y)])\sin t \quad (79)$$

Table 13 shows the results of the global error, using a regular mesh of 81 nodes (figure 7), for several values of $\triangle t$.

Table 14 shows the results of global error with $\triangle t = 0.001$ for several irregular meshes of 81 nodes (figures 7 and 8).

$\triangle t$	Global error
0.005	0.36490
0.002	0.03519
0.001	0.00492
0.0005	0.00064

Table 13: Influence of $\triangle t$ in the global error.

IIC	Global error
0.92	0.00492
0.83	0.00494
0.76	0.00496
0.58	0.00504

Table 14: Influence of irregularity of mesh in the global error.

Figure 14 shows the approximated solution of the equation 77, 78 and 79 in the last time step $(n = 500)$.

6 CONCLUSIONS

The use of the generalized finite difference method using irregular clouds of points is an interesting way of solving partial differential equations. The extension of the generalized finite difference to the explicit solution of some dynamic analysis problems has been developed.

The von Neumann stability criterion has been expressed in function of the coefficients of the

star equation for irregular cloud of nodes.
The index of irregularity of a clouds of nodes (IIC) is given and, also, its relation with the stability. As it is shown in the numerical results, a decrease in the value of the time step, always below the stability limits, leads to a decrease of the global error.

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