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APPLICATION OF INTERVAL FIELDS FOR UNCERTAINTY MODELING IN A GEOHYDROLOGICAL CASE

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Abstract. *In situ soil remediation requires a good knowledge about the processes that occur in the subsurface. Groundwater transport models are needed to predict the flow of contaminants. Such a model must contain information on the material layers. This information is obtained from in situ point measurements which are costly and thus limited in number. The overall model is thus characterised by uncertainty. This uncertainty has a spatial character, i.e. the value of an uncertain parameter can vary based on the location in the model itself. In other words the uncertain parameter is non-uniform throughout the model. On the other hand the uncertain parameter does have some spatial dependency, i.e. the particular value of the uncertainty in one location is not totally independent of its value in a location adjacent to it. To deal with such uncertainties the authors have developed the concept of interval fields. The main advantage of the interval field is its ability to represent a field uncertainty in two separate entities: one to represent the uncertainty and one to represent the spatial dependency. The main focus of the paper is on the application of interval fields to a geohydrological problem. The uncertainty taken into account is the material layers' hydraulic conductivity. The results presented are the uncertainties on the contaminant's concentration near a river. The second objective of the paper is to define an input uncertainty elasticity of the output. In other words, identify the locations in the model, whose uncertainties influence the uncertainty on the output the most. Such a quantity will indicate where to perform additional in situ point measurements to reduce the uncertainty on the output the most.*

1 INTRODUCTION

In recent years, the study of uncertainties in numerical modeling has gained a lot of attention. Probabilistic and non-probabilistic methods were developed for dealing with scalar parameter uncertainties. However, scalar parameter uncertainties are not the only kind of uncertainties influencing numerical models. Often scalar parameter uncertainties represent uncertainties that have uncertainty on a smaller scale spatial dimension too. The spatial influence of such uncertainties is often neglected, as it is assumed captured by assumptions of uniformity and homogeneity. This neglect is not without reasons, for a thorough discretisation of an uncertain property over the spatial domain would result in an explosion of independent uncertainties and thus a drastic increase in the computation time for the uncertainty analysis. However, a go-between approach is possible when certain patterns describing the spatial behaviour of an uncertainty are available. Taking into account the patterns reduces the explosion of uncertainties in going from one spatially uniform uncertainty to a thorough discritisation of the spatial domain. The authors have developped an interval field approach [1] to formalize these notions.

The paper first presents the general problem of interval finite element analysis and the interval field approach to it. Secondly, a section details the choice of certain spatial patterns in the interval field approach, based on random field analogies. Next, the concept of input uncertainty elasticity of the output is introduced in the context op spatial uncertainties. In the next section the geohydrological problem is introduced and the obtained results are presented. The paper concludes with some suggestions for further research.

2 INTERVAL AND INTERVAL FIELD ANALYSIS

This section first describes the general concept of Interval Finite Element (IFE) analysis and the method used to deal with it. Next the interval field concept is introduced to deal with dependent uncertain quantities.

2.1 Interval Finite Element analysis

Generally an IFE problem can be represented by [2]:

$$\mathbf{y}^s = \{\mathbf{y} \mid (\mathbf{x} \in \mathbf{x}^I)(\mathbf{y} = f(\mathbf{x}))\} \tag{1}$$

with $\mathbf{x}^I$ the interval vector representing the bounds on the input uncertainties and $f(\mathbf{x})$ the function representing the input-output relationship. The solution is expressed as a set $\mathbf{y}^s$, rather than an interval vector $\mathbf{y}^I$ to stress that certain value combinations of components within a hypercubic approximation of the uncertain vector result $\mathbf{y}$ are not necessarily physically coherent. However in most cases the individual ranges of only some components of $\mathbf{y}$ are really of interest. Several implementation strategies for interval numerical analysis have been proposed. Because global optimisation based strategies yield physically correct results, they are more and more acknowledged as the standard approach for non-intrusive IFE analysis. The core of this analysis (the $f(\mathbf{x})$) is a black-box FE calculation which can roughly be any analysis (for example a static or dynamic structural analysis, but also a heat-conductivity problem, hydrogeological problem or vibro-accoustic problem), limited only by the capabilities of the FE solver. The global optimisation based solution strategies actively search in the non-deterministic input interval space for the combination that results in the minimum or maximum value of an output quantity. In theory, the global optimisation approach results in the exact interval vector.

However, despite the smooth behaviour of typical objective functions, the computational cost of the global optimisation based approach remains high. Hence, most research on this method focuses on fast approximate optimisation techniques. The approximating technique used in this paper starts by building a Kriging response surface based on a number of initial sample points. From this preliminary response surface the optimal additional samples are determined by focusing on the location of the possible extremes of the approximated output quantity in the uncertainty space [3]. The response surface is thus improved by additionally sampling the core FE-model till a pre-specified maximum number of samples are taken. Subsequently, global optimisation and anti-optimisation is performed on this response surface model to yield the bounds on the considered output quantity. For a thorough discussion of this adaptive response surface optimisation method, the interested reader is referred to [4].

For completeness the extension of an interval number to a fuzzy set is presented. A fuzzy set [5] is a set in which every member has a degree of membership, represented by the membership function $\mu_x(x)$, associated with it. If $\mu_x(x) = 1$, x is definitely a member of the fuzzy set. If $\mu_x(x) = 0$, x is definitely not a member of the fuzzy set. Analysis using fuzzy sets is very often done by using so-called α-cuts. An α-cut contains all the x for which $\mu_x(x) > \alpha$ is true. These α-cuts are essentially classical intervals, which means that the interval analysis is the basis of a fuzzy analysis.

2.2 Interval fields

The interval field framework as developped in [1] has an explicit and an implicit implementation. For the application presented here the explicit implementation is needed.

For a spatially dependent uncertainty, the interval vector $\mathbf{x}^I$ containing an independent interval component for every spatial location is not a realistic description. Furthermore, it would result in an infeasibly high dimensional optimisation problem. To describe spatially dependent variation, numerical modelling approaches often use some type of shape functions (e.g. the modes used to represent the dynamic behaviour of a structure using the modal superposition technique). The actual solution is a linear combination of these shape functions.

Accordingly, the explicit interval field $\mathbf{x}^F$ is defined as a superposition of n_b base vectors $\boldsymbol{\psi}_i$ using interval factors α_i^I:

$$\mathbf{x}^F = \sum_{i=1}^{n_b} \alpha_{x,i}^I \boldsymbol{\psi}_{x,i} \tag{2}$$

The base vectors represent a limited set of reference patterns over the spatial domain, each of which is scaled by an interval factor. The components of the interval fields themselves (the local value of the uncertainty) are coupled through the reference patterns. Once the reference patterns are chosen, the definition of the interval field requires the specification of the interval factors that define the field on x, which can be assembled in a classical (hypercubic) interval vector $\boldsymbol{\alpha}_x^I$. In matrix notation, the interval field is denoted as:

$$\mathbf{x}^F = [\boldsymbol{\psi}_x]\, \boldsymbol{\alpha}_x^I \tag{3}$$

The application of an explicit interval field on the input side of an analysis is rather straightforward. Since expert knowledge about the modelled system dominates the definition of the uncertainties, the freedom in choosing the base vectors is ideal to reflect this knowledge (for example: the sinusodal (= base vector) deviation of the thickness of a rolled plate with uncertain amplitude (= interval factor)). The main limitation of the explicit interval field is that its definition only allows a linear relation between the base vectors and the interval factors.

The application of an explicit interval field on the output side of an analysis is less straightforward. The base vectors and interval factors are determined by the analysis itself. Furthermore, in order to obtain an explicit interval field that introduces no conservatism in its derived response variables (i.e. derivatives of the primary response variables), the output interval factors should be completely independent. An analysis of the application of the interval field approach to the output of static FE analysis is presented in [6].

Once the spatially dependent uncertainty on the input side of an analysis is defined by means of an explicit interval field, the dimensions of the uncertainty space are drastically reduced. This allows for the use of the adaptive response surface technique as described in the above subsection.

3 THE CHOICE OF BASE VECTORS

The use of the explicit interval field on the input side of an analysis requires the selection of appropriate base vectors and interval factors. This section first presents the factors influencing the selection of these base vectors and interval factors. The choice for base vectors and interval factors based on random field expansions is explained in the next subsection.

3.1 Factors influencing the choice of base vectors

- The bounds on the uncertainty on a model parameter $\mathbf{x}$ are specified by two functions of the spatial coordinate $\mathbf{r}$, one function for the upper bound $\overline{x}(\mathbf{r})$ and one for the lower bound $\underline{x}(\mathbf{r})$ of the uncertainty. The linear combination of the base vectors with the interval factors that makes up an interval field must remain within these bounds for any value of the interval factors.

- The base vectors must represent the expert's knowledge of the spatial dependency of the model parameter. Most often knowledge about this dependency is limited and the set of base vectors preferably allows for a range of small and large scale dependency.

- The number of base vectors and corresponding interval factors to represent the input uncertainty will influence the calculation time to get the output uncertainty.

3.2 Base vectors derived from random field expansion

In an attempt to construct a base vector set that takes into account the above described factors, the expansion of a random field is studied.

The objective of a random field is to represent a spatial variation of a specific model property by a stochastic variable defined over the region on which the variation occurs [7]. A random field can thus be denoted as $H(\mathbf{r}, \theta)$ with $\mathbf{r}$ the spatial coordinate and θ the outcome of a random phenomenon. A random field is a random variable for a given $\mathbf{r_0}$ and is a realization of the field for a given θ_0. The specification of a random field generally comes down to the specification of the spatial evolution of the first two statistical moments of the field variable and a corresponding covariance function, expressing the spatial dependency of the field variable. In most cases the random field is considered to be weakly stationary, resulting in a constant for the first few statistical moments throughout the spatial domain (i.e. zero mean and unit variance). Furthermore the covariance function for weakly stationary random fields depends only on the distance between observation points, not on their actual location.

The application of the concept of random fields in a numerical modelling framework requires some sort of discretisation of the spatially varying stochastic field over the defined geometry. A good overview of methods can be found in the report by Sudret and Der Kiureghian [8]. The technique studied here is the Karhunen-Loève expansion [9] that has gained particular attention in literature. This approach is based on the spectral decomposition of the autocovariance function $C_{HH}(\mathbf{r_1}, \mathbf{r_2})$. The set of deterministic functions over which any realization of the field $H(\mathbf{r}, \theta_0)$ is expanded is defined by the eigenvalue problem:

$$\int_{\omega} C_{HH}(\mathbf{r_1}, \mathbf{r_2}) \varphi_i(\mathbf{r_2}) d\omega_{\mathbf{r_2}} = \lambda_i \varphi_i(\mathbf{r}) \tag{4}$$

with ω the spatial domain and $i = 1, \ldots$. Once the eigenfunctions are found the random field can be expressed as:

$$H(\mathbf{r}, \theta) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} \xi_i(\theta) \varphi_i(\mathbf{r}) \tag{5}$$

with $\{\xi_i\}$ a set of orthonormal random variables. In stochastic analysis, this expansion is truncated after N terms to reduce the computational costs.

Several features of the random field expansion can be used in the interval field implementation after some adaptations. To begin, an off-set function $f_{mid}(\mathbf{r})$ to describe the mid value of the model parameter throughout the spatial domain is calculated

$$x_{mid}(\mathbf{r}) = \frac{\overline{x}(\mathbf{r}) + \underline{x}(\mathbf{r})}{2} \tag{6}$$

The eigenfunctions $\varphi_i(\mathbf{r})$ of the covariance function are then used as base vectors $\boldsymbol{\psi}_i(\mathbf{r})$ for the interval field with

$$\boldsymbol{\psi}_i(\mathbf{r}) = \lambda_i \varphi_i(\mathbf{r}) |\varphi_i(\mathbf{r})| \tag{7}$$

and replacing the orthonormal random variables $\{\xi_i\}$ by interval factors $\alpha_i^I \in \lfloor -1 \; 1 \rceil$. These adaptations make sure that for $N \to \infty$ the interval field will assign a value from the interval $\lfloor -1 \; 1 \rceil$ to the model parameter throughout the spatial domain. This unit interval is then scaled by the difference function

$$x_{dif}(\mathbf{r}) = \overline{x}(\mathbf{r}) - \underline{x}(\mathbf{r}) \tag{8}$$

describing the actual range of uncertainty on the model paramater for every location in the model. The description of the model parameter by the interval field is thus

$$\mathbf{x}^F = x_{mid}(\mathbf{r}) + \sum_{i=1}^{N} (\lambda_i \; \varphi_i(\mathbf{r}) \; |\varphi_i(\mathbf{r})| \; \alpha_i^I) \; (x_{dif}(\mathbf{r})) \tag{9}$$

With this equation the considerations from the first and last item in the list of influencing factors is accounted for. Next is the issue of uncertainty about the spatial dependency.

The base vectors taken from the expansion of a random field with a given autocovariance function only take into account the given correlation length L. In [10] a method is described to take into account interval correlation lengths with interval fields. Essentially the method relies on building an interval field description for the base vectors themselves in the correlation length space using a limited number of autocovariance expansions. In this way the base vectors are depending on the correlation length and can be calculated by a simple matrix vector product. The resulting interval field for the model parameter can thus be summarised by

$$\mathbf{x}^F = x_{mid}(\mathbf{r}) + \sum_{i=1}^{N} (\lambda_i(L) \; \varphi_i(\mathbf{r}, L) \; |\varphi_i(\mathbf{r}, L)| \; \alpha_i^I) \; (x_{dif}(\mathbf{r})) \quad L \in \lfloor L_{min} \; L_{max} \rceil \tag{10}$$

This approach only introduces one additional interval to represent the uncertainty about the amount of spatial dependency. The solution strategy to find the uncertainty on the output remains the same, for example a response surface based optimisation and anti-optimisation, with only one additional dimension in the uncertainty space.

4 INPUT UNCERTAINTY ELASTICITY OF THE OUTPUT

To assess the relative importance of an input uncertainty on an output uncertainty, the concept of input uncertainty elasticity of the output is introduced in general terms and then applied to the case of spatial uncertainty.

4.1 General concept

As in economics, an elasticity R is defined as the ratio of the relative change (more precisely, the derivative with respect to some quantity) in one parameter Y to the relative change in an other parameter X

$$R_X^Y = \frac{\Delta Y}{\Delta X}\frac{X}{Y} \tag{11}$$

Let Y be the range of the uncertain output and X be the range of the uncertain input. The reduction (i.e. the Δ) on the range of the interval for the input X, will affect the range of the interval for the output Y to a greater or lesser extent. The relative magnitude of this influence is described by the input uncertainty elasticity of the output R_X^Y.

4.2 Spatial uncertainty context

In the context of spatial uncertainty, the influence of an input uncertainty on an output uncertainty has a spatial component. The influence of an uncertain input parameter will depend on the spatial distribution of its uncertainty. Figure 1 shows in a generic way the influence of the spatial uncertainty distribution. For an investigated spatial location the amount of uncertainty $x_{dif}(\mathbf{r})$ is reduced and some sort of coherent distribution of the uncertainty is assumed over the spatial domain (as illustrated at the top left in the figure). The uncertainty analysis is carried out for this spatial uncertainty distribution and a resulting uncertainty (an interval) for the output is found (bottom left). By repeating this for other investigated spatial locations, one finds the combined result which is shown at the right of the figure. It presents the different output uncertainties for several investigated spatial locations. This data is then used to calculate the input uncertainty elasticity of the output over the spatial domain. In this context the R_X^Y is in particular usefull to identify the spatial location where an input uncertainty influences the output uncertainty the most. In allocating resources to reduce the uncertainty, the spatial location with the highest R_X^Y should get priority.

An appropriate selection of the $\overline{x}(\mathbf{r})$ and $\underline{x}(\mathbf{r})$ is needed to make a study over the spatial domain to give a scalar field of input uncertainty elasticities of the output. Important choices to be made in the selection of $\overline{x}(\mathbf{r})$ and $\underline{x}(\mathbf{r})$ to investigate a particular spatial location's uncertainty influence are listed below. Figure 2 shows $\overline{x}(\mathbf{r})$ and $\underline{x}(\mathbf{r})$ for three cases. The first case, at the left on the figure, is the reference case. The two other cases illustrate particular choices for $\overline{x}(\mathbf{r})$ and $\underline{x}(\mathbf{r})$ explained in the list below.

Important choices to be made in the selection of $\overline{x}(\mathbf{r})$ and $\underline{x}(\mathbf{r})$:

- the magnitude of the reduction of $x_{dif}(\mathbf{r})$ for the investigated spatial location. The second case in figure 2 shows a reduction of 50% for $x_{dif}(0.2)$, the third case shows a reduction of 90% for $x_{dif}(0.7)$.

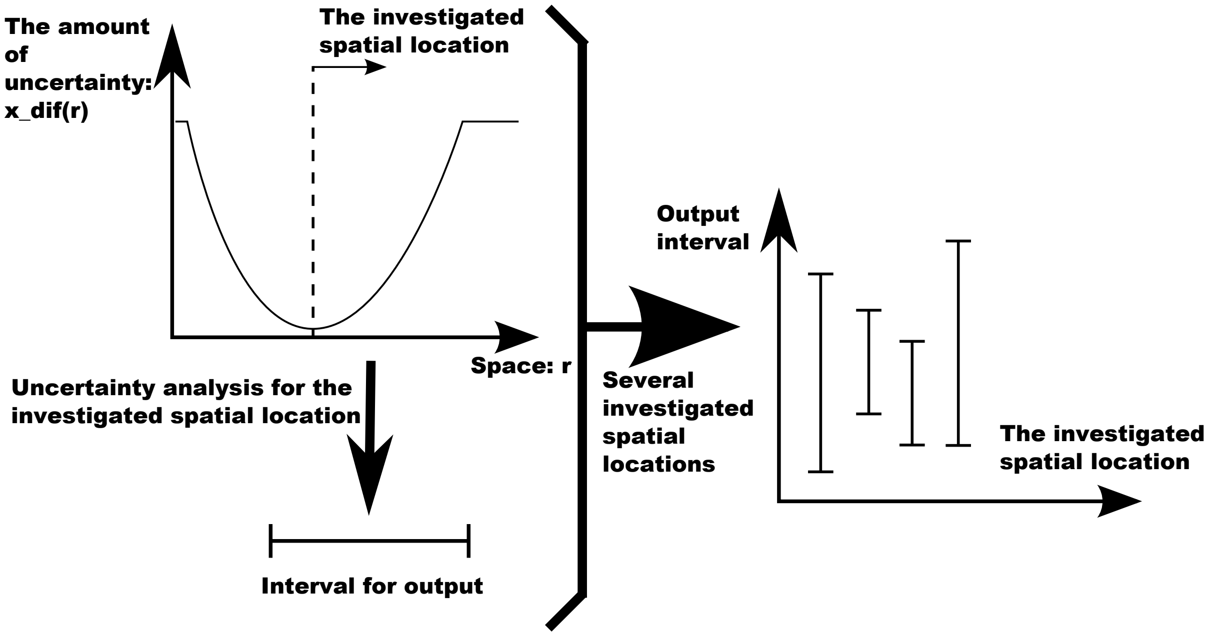

Figure 1: Concept to determine R_X^Y in a spatial context

- the magnitude of the reduction of $x_{dif}(\mathbf{r})$ in the local influence zone of the investigated spatial location. By reducing the amount of uncertainty for the investigated spatial location, the amount of uncertainty for the region around the investigated spatial location is also affected. In this so called local influence zone, a transition from the reduced amount of uncertainty to the reference amount of uncertainty is needed. In this paper a quadratic transition is suggested.

- the magnitude of the local influence zone of the investigated spatial location. The second case in figure 2 shows a zone of influence from -0.1 to $+0.1$ around the investigated spatial location 0.2. The third case shows a zone of influence from -0.3 to $+0.3$ around the investigated spatial location 0.7.

- the change in $x_{mid}(\mathbf{r})$. If $\overline{x}(\mathbf{r})$ and $\underline{x}(\mathbf{r})$ are not changed symmetrically with respect to $x_{mid}(\mathbf{r})$ in the reference case, then $x_{mid}(\mathbf{r})$ is affected as well. For simplicity this influence is not presented here.

These notions are explained further in the case study.

5 GEOHYDROLOGICAL CASE STUDY

A geohydrological case study was chosen to apply the above presented techniques. The case study deals with a groundwater pollution problem where benzene was spilled and is now being transported in groundwater to a river. To characterize the flow and transport of the benzene spill, a groundwater flow and transport model was built in HYDRUS3D. First, the problem together with its uncertainty is described and the results of a fuzzy analysis without taking into account the spatial dependency are presented. Next, the spatial dependency is introduced and an investigation of the input uncertainty elasticity of the output is performed.

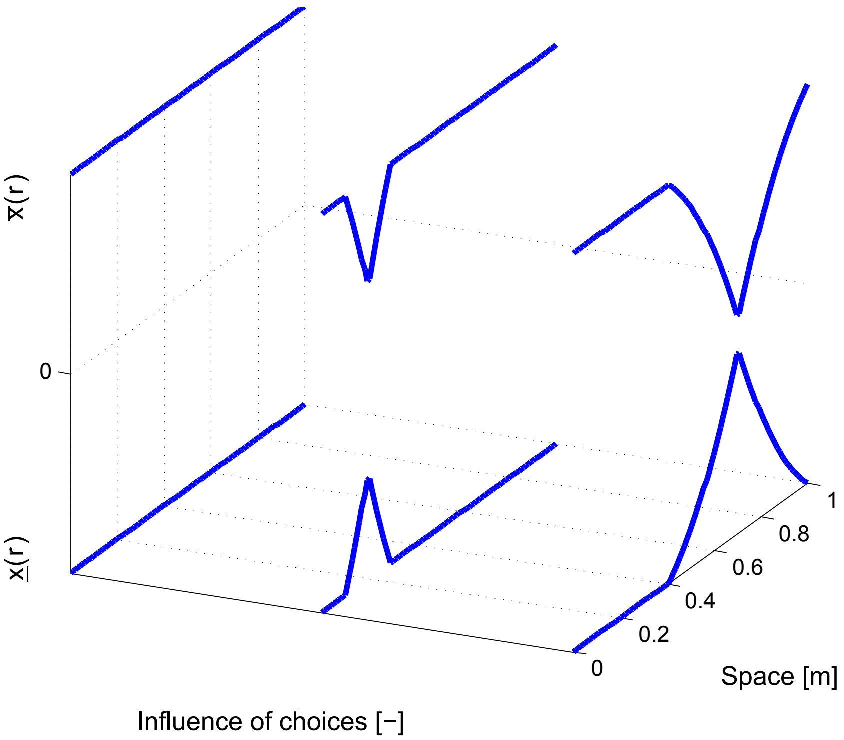

Figure 2: Choices in the selection of $\overline{x}(\mathbf{r})$ and $\underline{x}(\mathbf{r})$

Material Layer	Minimum K	Maximum K
1	1.4	2.1
2	8	12
3	3.6	5.4
4	2.6	3.9
5	4	6

Table 1: Intervals for the hydraulic conductivity K [m/day], ordered from top to bottom.

5.1 Problem description

The governing equation for solute transport in groundwater is a convection-diffusion equation based on conservation of mass. Convection is determined by groundwater flow which is based on the constitutive equation for variably saturated flow in porous media, called the Darcy Buckingham equation. For the solute (the contaminant: Benzene) and ground water flow problem at hand the following input was given:

- FE-model (14661 nodes) for the HYDRUS3D [11] solver (see figure 3). The dimensions of the problem are 1100 m in the length direction and between 32 and 36.5 m in the depth direction. In the time domain a period of 11000 days (approximately 30 years) is calculated. A deterministic run of this model takes 10 minutes.

- Intervals for the material properties, i.e. the saturated hydraulic conductivity K (see table 1) of the five different material layers.

A river is situated at the left side of the domain (see the red ellipse on figure 3) and the two sources of the contaminant are in the middle of the domain (see the red arrows on figure 3).

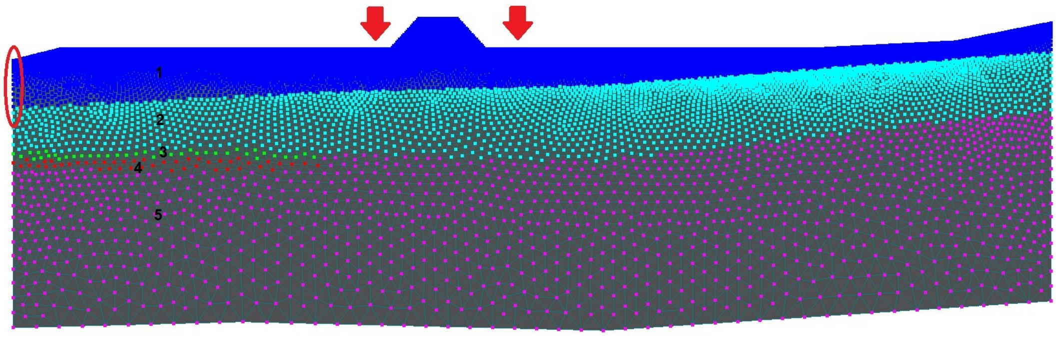

Figure 3: FE-model for solute and ground water flow showing the five different material layers.

The requested output is the concentration of the contaminant over time at the river given the uncertainties on the material properties.

5.2 Fuzzy analysis without spatial uncertainty

In the first fuzzy analysis, the uncertainty on the material properties is represented by fuzzy numbers. The hydraulic conductivity of each layer is considered independent and modelled as a triangular fuzzy number with the given intervals (see table 1) as base and the mid value as the top of the triangle. In each material layer the hydraulic conductivity is considered homogeneous through space. Figure 4 shows for example the spatial fuzzy number in blue and a possible sample of the fuzzy number in red for the hydraulic conductivity of material layer 1. Two types of fuzzy analyses were performed:

- Reduced Transformation Method (TM) [12] with 5 alpha-cuts, resulting in 161 samples.
- An optimisation on a Kriging response surface (ARSM) [4] that was built using 32 initial latinhypercube samples and 32 additional samples.

Additionally, a reference Monte Carlo Simulaion using 200 samples was performed at each alpha-level, based on a uniform distribution within the interval at each alpha-level Figure 5 shows the fuzzy concentration through time for location 11 (at the river, 3 m below the surface). In blue is the result of the reduced transformation method ($5 * 32 + 1$ samples); in green is the result of the optimisation on a Kriging response surface (32 initial + 32 additional samples); in red is the result of the Monte Carlo Simulation ($5 * 200$ samples). From these results it is clear that the TM and ARSM results are close to each other. The MCS result, despite being the computationally most expensive, does not yield good results for the maxima: the value given by the TM is an actual solution of the problem (i.e. a genuine sample) and results in higher maxima. The ARSM has problems identifying the proper minima since it tends to give negative (non-physical) results.

5.3 Fuzzy analysis with spatial uncertainty

In this fuzzy analysis, only the uncertainty on the hydraulic conductivity in material layer 1 is taken into account. The other hydraulic conductivities are set at their minimal value. To model the spatial uncertainty for the hydraulic conductivity of material layer 1, the following assumptions are made:

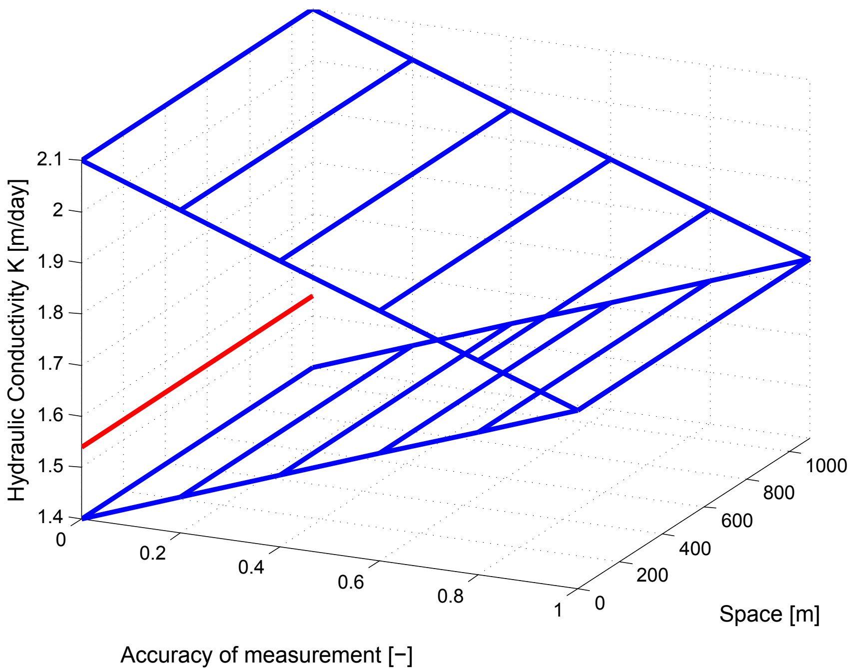

Figure 4: The spatial fuzzy number in blue and a sample of it in red for the hydraulic conductivity of material layer 1, assuming homogeneity.

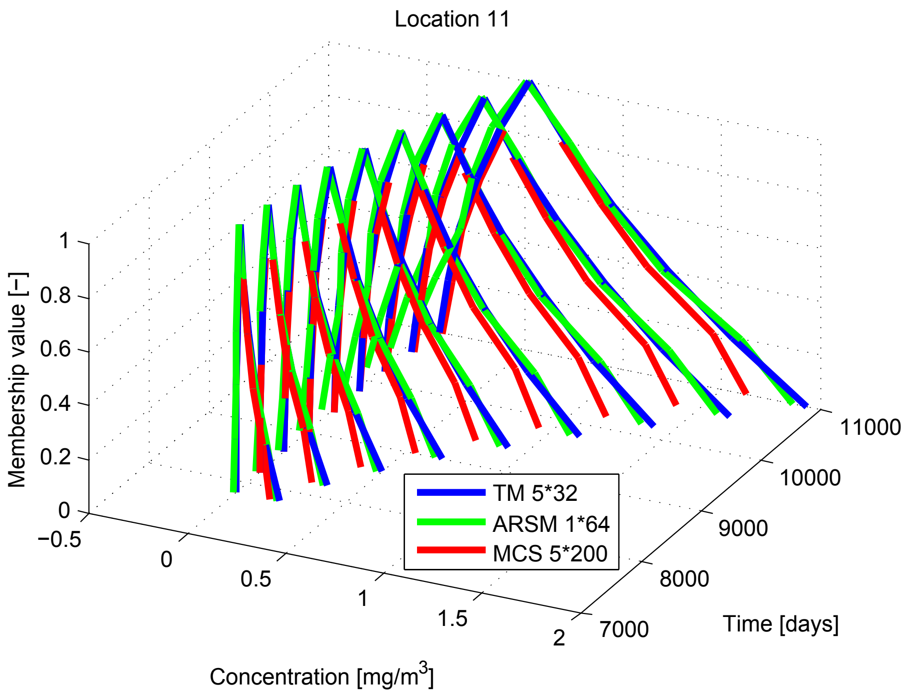

Figure 5: The fuzzy concentration at the river, 3 m below the surface.

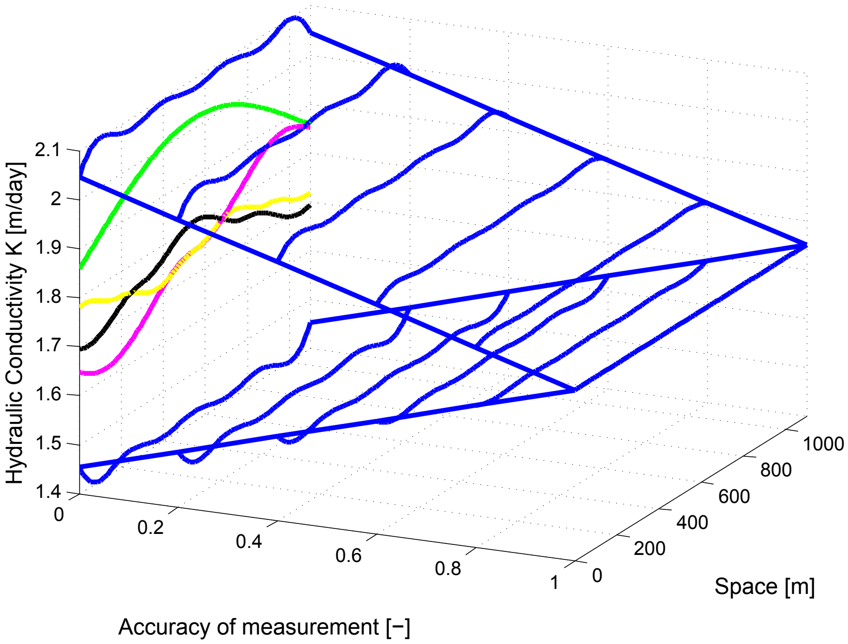

Figure 6: The spatial fuzzy number in blue and the four base vectors for membership level 0 in colours for the hydraulic conductivity of material layer 1, including spatial uncertainty.

- The upper bound $\overline{x}(\mathbf{r})$ and lower bound $\underline{x}(\mathbf{r})$ are given by a constant, namely the maximum and minimum of the interval given in table 1.

- The base vectors are derived from an exponential autocovariance function

$$C_{HH}(x_1, x_2) = e^{-\frac{|x_1 - x_2|}{L}} \tag{12}$$

as described in section 3.2. The first four eigenfunctions are used. Since a limited number of base vectors is used, the upper and lower bound on the uncertainty are not exactly satisfied throughout the domain. A scaling factor to adjust the maximal possible value of the interval field in the spatial domain to the requested bounds is applied. For a correlation length $L = 500$ m, the resulting base vectors are shown in 6.

To check the influence of taking into account the spatial uncertainty, two analyses (TM and ARSM) with the non-spatial uncertainty (i.e. uncertain, but homogeneous hydraulic conductivity of material layer 1) are performed as well. In total, the following types of analyses were performed:

- non-spatial uncertainty, Reduced Transformation Method (TM) with 5 alpha-cuts, resulting in 11 samples.

- non-spatial uncertainty, optimisation on a Kriging response surface (ARSM) that was built using 6 initial latinhypercube samples and 12 additional samples.

- spatial uncertainty, with correlation length between 500 and 2000 m, optimisation on a Kriging response surface (ARSM) that was built using 20 initial latinhypercube samples and 30 additional samples.

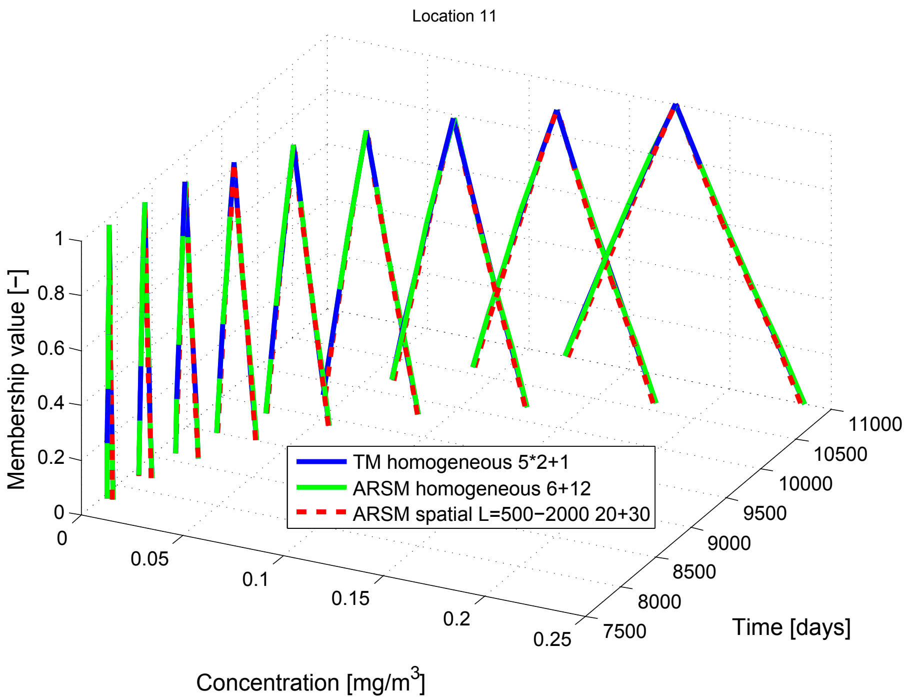

Figure 7: The fuzzy concentration at the river, 3 m below the surface.

Figure 7 shows the fuzzy concentration through time for location 11 (at the river, 3 m below the surface). The influence of taking into account the spatial uncertainty results in slightly narrower fuzzy numbers. This suggests that assuming homogeneity for the hydraulic conductivity of material layer 1 gives conservative bounds on the contaminant's concentration for the studied case.

5.4 Input uncertainty elasticity of the output

By performing an additional point measurement to determine the hydraulic conductivity in one location, the uncertainty on the contaminant's concentration will be reduced. To determine the optimal measurement location an input uncertainty elasticity of the output is calculated. The following assumptions, referring to section 4.2 and figure 8, are made:

- The magnitude of the reduction of the uncertainty for the considered measurement location is a design parameter. By selecting a more accurate measurement device, the uncertainty remaining after measurement is a choice of the expert. In figure 8 the influence of an increasing measurement accuracy on the bounds and the base vectors is shown.
- The magnitude of the reduction of the uncertainty in the local influence zone is an uncertainty. $f_{dif}(\mathbf{r})$ increases from the value at the measurement location to the reference value at the end of the local influence zone. In the presented analysis a quadratic function of the distance to the measurement location is chosen.
- The magnitude of the local zone of influence is an uncertainty. What is the extent of the influence of a measurement in one location on the rest of the spatial domain? Since a comparison between the input uncertainty elasticities of the output for different locations is of interest, the magnitude of this local zone of influence is chosen to be a fixed value. In

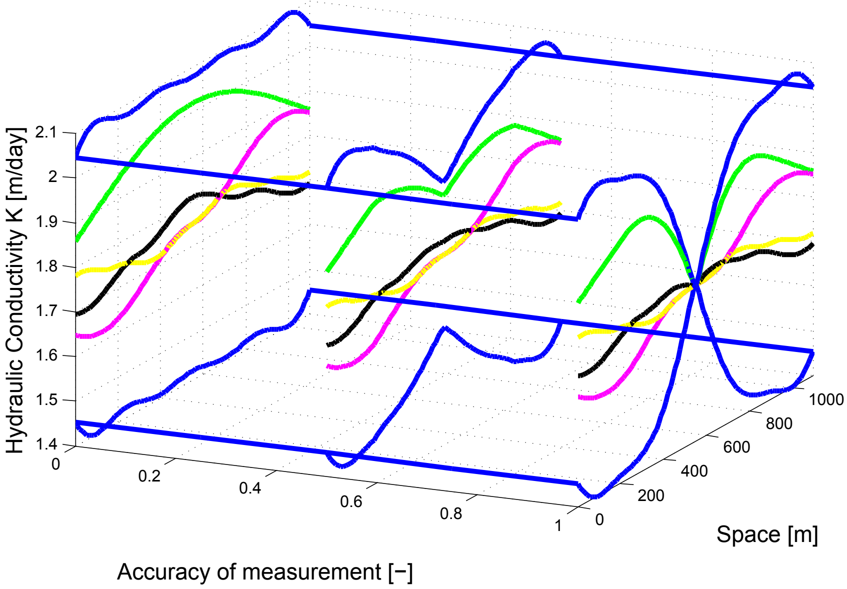

Figure 8: The influence of a measurement in the middle of the domain on the bounds of the uncertainty (in blue) and the base vectors (in colours), with a zone of influence of 330 m to both sides of the measurement location.

Design Parameter	Sampled values
Measurement location	110, 330 and 550 m from river
Measurement accuracy	50% reduction and 100% reduction of uncertainty
Uncertain Parameter	**Range**
Extent of influence	value chosen is 330 m.
Correlation length	$\lfloor 500\ 2000 \rceil$ m

Table 2: Parameters in the input uncertainty elasticity of the output analysis.

figure 8 the bounds and base vectors for an influence up to 330 m to both sides is shown, as it is used in the analysis.

- The actual outcome of the measurement gives a value for $f_{mid}(\mathbf{r})$ in the measurement location. Untill the measurement is done, this is also an uncertainty that influences the actual bounds on the output uncertainty. In the presented analysis the value of $f_{mid}(\mathbf{r})$ is considered a constant and unchanged by a measurement.

To summarize: the measurement location and the accuracy of the measurement are design parameters, whereas the influence of the measurement and the spatial correlation length are uncertainties. For the influence of the measurement a fixed magnitude is assumed and the correlation length is modelled as an interval. The values used in the analysis are presented in table 2. For a choice of the design parameters, the uncertainty analysis was carried out using the ARSM method with 20 inital samples and 30 additional samples. The results are presented in figure 9. The bounds on the contaminant's concentration are presented for location 11 (at the river, 3 m below the surface) at the end time of the simulation (approx. 30 years) as a function of the measurement location and the accuracy of the measurement. Based on this information the input uncertainty elasticity of the output is calculated using equation (11) with X and Y respectively the range on the hydraulic conductivity and the range on the contaminant's concentration. The results are presented in table 3, the reference is the range on the uncertainty

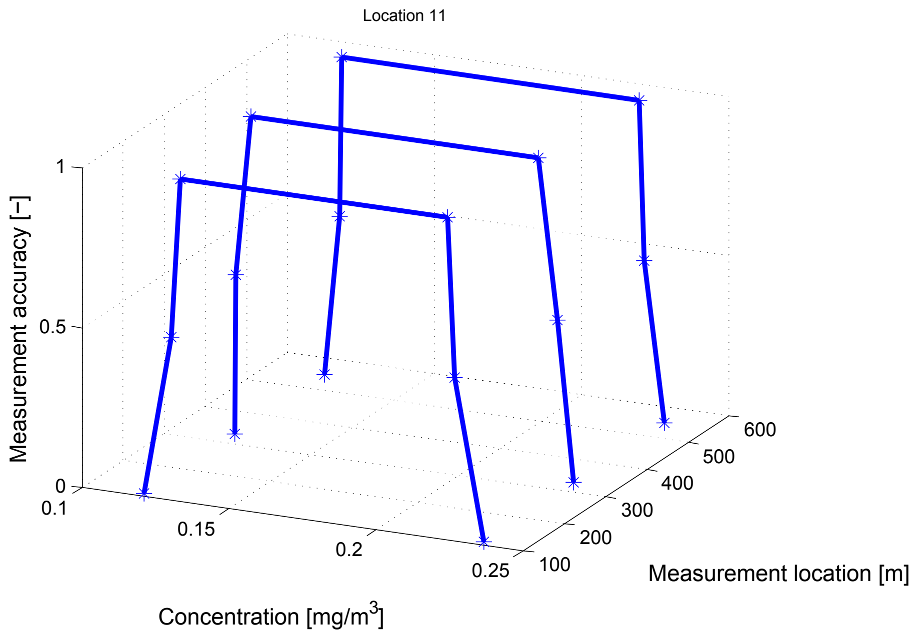

Figure 9: The results of the uncertainty analysis to determine the input uncertainty elesticity of the output

Measurement location [m]	110	330	550
Measurement accuracy [-]			
0.5	0.33	0.10	0.20
1.0	0.21	0.15	0.13

Table 3: The input uncertainty elasticity of the output.

before measurement. From this table 3 it becomes clear that performing an input uncertainty reduction (i.e. a measurement of the hydraulic conductivity in material layer 1) at 110 m from the river provides the greatest reduction in uncertainty on the output (i.e. the concentration of the contaminant at the considered location and time). Furthermore, for this measurement location increasing the uncertainty reduction from 50% to 100% will not decrease the uncertainty on the output with the same amount. In other words, a measurement with an uncertainty reduction of 50% will have a $\frac{0.33}{0.21} \simeq 1.5$ times higher relative uncertainty reduction on the output than a measurement with an uncertainty reduction of 100%. For a measurement at 330 m from the river the inverse is true: the extra effort of reducing the input uncertainty from 50% to 100% gives a 1.5 times higher relative uncertainty reduction on the output. Based on this information and knowledge of the actual costs of a measurement campaign an informed decision can be made concerning where and how accurate to measure.

6 CONCLUSION

From a methodological point of view, the paper introduces interval fields as an easy conceptual tool to deal with spatial uncertainty. The implementation of the interval field based on correlation length is made possible by deriving certain base vectors from the random field expansion technique. This allows for taking into account uncertainty on the correlation length. Furthermore, the concept of input uncertainty elasticity of the output is introduced in a spatial uncertainty context.

From an applied point of view, the paper shows the applicability of the interval field to a geohydrological problem of realistic complexity. The adaptive response surface technique proves to be very useful in practice. Certainty on the value of the correlation length often is a problem. The feasibility of dealing with the correlation length as an interval is shown. The concept of input uncertainty elasticity of the output in a spatial uncertainty context is proven to be usefull to determine the optimal location of a measurement (i.e. an uncertainty reduction) to reduce uncertainty on the output.

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