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MODAL ANALYSIS OF THE FGM BEAMS WITH SPATIAL VARIATION OF MATERIAL PROPERTIES UNDER LONGITUDINAL VARIABLE ELASTIC WINKLER FOUNDATION

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Abstract. *A differential equation of the homogenized functionally graded material (FGM) beam deflection will be presented which will be used for free vibration analysis of the beams with continuous longitudinal and transversal variation of material properties. The FGM beams with continuous spatial variation of material properties of double symmetric cross-section have been transformed into the multilayer FGM beams. Symmetrical layering according the neutral plane in transversal direction is assumed: the corresponding layers have the same height and material properties. The material properties vary continuously in longitudinal direction, but they are constant along the height and width of competent layers. The FGM beams are considered to be resting on longitudinal variable (Winkler) elastic foundation. The first and second order beam theories have been used for establishing the kinematics and kinetic equations. Not only the shear force deformation effect and the effect of consistent mass distribution and mass moment of inertia have been taken into account, but also the effect of large axial forces have been considered. Numerical experiments were performed to calculate the eigenfrequencies and corresponding eigenmodes of chosen FGM beams. The solution results are discussed and compared with those obtained using a very fine mesh of two-dimensional solid finite elements. The effects of material properties variation, layering fineness, the shear force correction factor and large axial forces have been evaluated and discussed.*

1 INTRODUCTION

Nowadays, the interaction between FGM beams and elastic foundations is an important issue in the study of beam behavior. Ying et al. [1] presented exact two-dimensional elasticity solution for the bending and free vibration of FGM beams on a Winkler – Pasternak foundation. The beam is assumed orthotropic at any point, and the material properties are taken as exponential functions of the thickness coordinate. The elastic foundation modules have been assumed as a constant. Pradhan et al. [2] presented thermo-mechanical vibration analysis of FGM beams resting on a variable Winkler foundation and two-parameter elastic foundation. The FGM material properties of these beams are assumed to be varying in thickness direction. In [3], the free vibration analysis of FGM beams has been presented. Spatial variation of material properties of these beams has been assumed. The shear force deformation effect and the effect of consistent mass distribution and mass moment of inertia have been taken into account. In [4], the paper [3] was extended by the effect of large axial forces. The solution results confirmed a very strong effect of large axial forces: the tensional axial force increased and the compression axial force decreased the level of eigefrequencies significantly. Many other authors investigate with free vibration of FGM beams with constant or transversal variation of material properties, e.g. [5], [6], [7]. In [8], the nonlinear dynamic analysis of partially supported beam-columns on non-linear elastic foundation including shear deformation effect has been discussed.

In [9], a differential equation of the homogenized functionally graded material (FGM) beam deflection has been proposed to be in a further free vibration analysis of multilayer and sandwich FGM beams of rectangular cross-section. The FGM beam was considered to be resting on longitudinal variable (Winkler) elastic foundation. In the contribution [10], an extension of paper [9] was presented, where not only the shear force deformation effect, the effect of consistent mass distribution as well as mass moment of inertia have been taken into account, but also the effect of large axial forces was considered.

In this contribution, which is a continuation of the paper [10], the differential equations of the 1^{st} and 2^{nd} order beam theory of the homogenized functionally graded material (FGM) beam deflection will be presented which will be used in the modal analysis of FGM beams with continuous longitudinal and transversal variation of material properties. The FGM beam with spatial variation of material properties will be transformed to multilayer beam. Symmetric layering along the neutral plane in transversal direction is assumed. The material properties vary continuously in longitudinal direction, but they are constant along the height and width of each layer. The multilayer beam will be then transformed to one layer homogenized beam with longitudinal variation of effective material properties. The FGM beams are considered to be resting on longitudinal variable (Winkler) elastic foundation. Numerical experiments were performed to calculate the eigenfrequencies and corresponding eigenmodes of chosen FGM beams with continuous spatial variation of material properties. The shear correction function has been derived from which the average shear correction factor has been calculated. The effects of the average shear correction factor, the large axial force, the variation of material properties and the layering fineness have been studied and compared with those obtained using a very fine mesh of a plane solid finite elements.

2 MATERIAL PROPERTIES HOMOGENIZATION AND DIFFERENTIAL EQUATION DERIVATION

Let as consider a two nodal straight beam element with predominantly rectangular cross-sectional area A and quadratic moment of inertia I (Fig. 1).

The composite material of this beam/link arises from mixing two components (matrix and fibres) that are approximately of the same geometrical form and dimensions (e.g. by powder metallurgy or plasma spraying).

Both, the fibre volume fraction $v_f(x,y)$ and the matrix volume fraction $v_m(x,y)$ are chosen as polynomial functions of x with continuous and symmetrical variation through its height h in respect to the neutral plane of the beam. The volume fractions are assumed constant through the cross-sectional depth b. At each point of the beam it holds: $v_f(x,y)+v_m(x,y)=1$. The values of the volume fractions at the nodal points are denoted by indices i and j.

The material properties of the constituents (fibres - $p_f(x,y)$ and matrix - $p_m(x,y)$) can vary analogically (in dependence on inhomogeneous temperature field for example) as defined by the variation of volume fractions. Homogenization of the material properties (the reference volume is the volume of the whole beam) will be done in two steps. In the first step, the real beam (Figure 1a) will be transformed to a multilayered beam (Figure 1b). Material properties of the layers will be calculated via the extended mixture rules [11]. Each layer will have constant volume fractions and material properties of the constituents through the beam height. They are calculated as an average value from boundary values of the respective layer. Polynomial variation of these parameters will appear in the longitudinal direction. Sufficient accuracy of the substitution of the continuous lateral variation of material properties with the layer-wise constant lateral distribution of material properties will be reached when the division to layers is fine enough. In the second step, the effective longitudinal material properties of the homogenized beam will be derived using laminate theory. These homogenized material properties are constant through the beam height but they vary continuously along the longitudinal beam axis. Finally, a differential equation for calculation of the effective beam variables for homogenized beam (Figure 1c) will be established.

One thin layer of the composite or FGM is depicted in Figure 2 with a constant rectangular cross-section considered. The layer length is L. Longitudinal variation of the constituent volume fractions and longitudinal variation of the constituent elasticity modules will be assumed. These parameters will be considered constant along the layer height and width. The fibers (constituent 1) volume fraction $v_f(x)$ is chosen as a polynomial function of x :

$$v_f(x)=1-v_m(x)=v_{fi}\eta_{vf}(x)=v_{fi}\left(1+\sum_k \eta_{vfk}x^k\right) \tag{1}$$

Then the matrix (constituent 2) volume fraction $v_m(x)$ is

$$v_m(x)=1-v_f(x)=v_{mi}\eta_{vm}(x)=v_{mi}\left(1+\sum_k \eta_{vmk}x^k\right) \tag{2}$$

where v_{fi} and v_{mi} are the fibre and the matrix volume fractions at node i and $\eta_{vf}(x)$ and $\eta_{vm}(x)$ are the polynomials of fibre and matrix volume fractions variation, respectively. Constants η_{vfk} and η_{vmk}, (k = 1,…,r), and the order r of these polynomials depends on the fibre and matrix volume fractions variation.

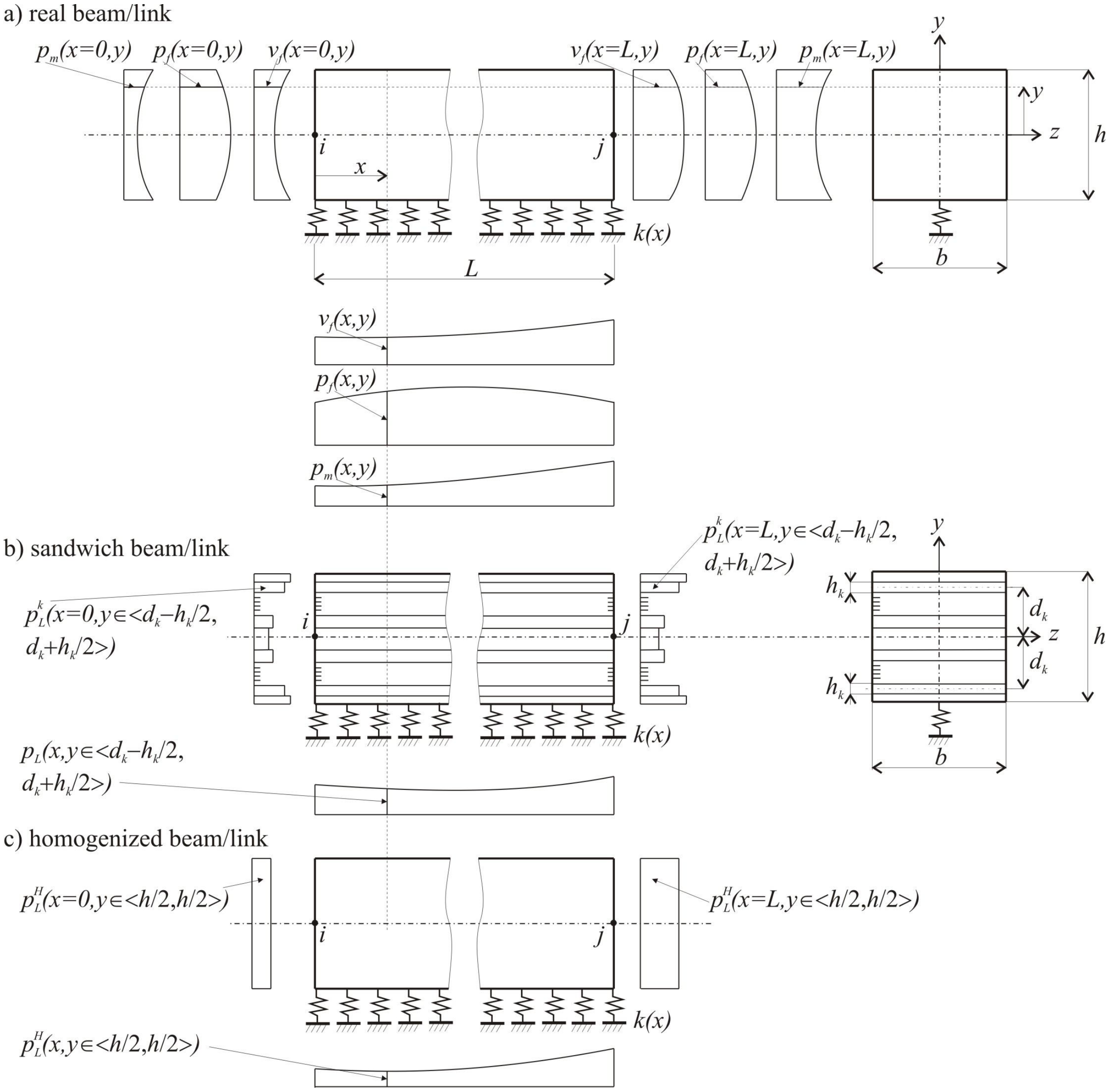

Figure 1: FGM beam with spatial variation of the material properties.

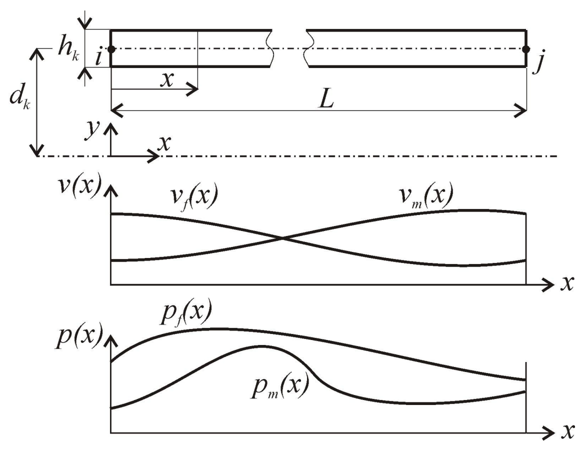

Figure 2: One thin layer of FGM.

Also the fibre material property $p_f(x)$ and the matrix material property $p_m(x)$ are chosen as polynomial functions of x :

$$p_f(x) = p_{fi}\eta_{pf}(x) = p_{fi}\left(1 + \sum_k \eta_{pfk}x^k\right) \tag{3}$$

$$p_m(x) = p_{mi}\eta_{pm}(x) = p_{mi}\left(1 + \sum_k \eta_{pmk}x^k\right) \tag{4}$$

where p_{fi} and p_{mi} are the fibre and matrix material properties at node i and $\eta_{pf}(x)$ is the polynomial of fibre material property variation. Its constants η_{pfk} , where $k = 1,\ldots, r$, and the order r of this polynomial depend on the fibre material property variation. $\eta_{pm}(x)$ is the polynomial of matrix material property variation. The constants η_{pmk} , where $k = 1,\ldots, s$, and the order s of this polynomial depend on the matrix material property variation.
Then the effective material property of the composite one-layer beam is given by

$$p_L(x) = v_f(x)p_f(x) + v_m(x)p_m(x) \tag{5}$$

Similarly to (4), we can write

$$p_L(x) = p_{Li}\eta_{p_L}(x) \tag{6}$$

Here, p_{Li} is the effective longitudinal material property at node i, and

$$\eta_{pL}(x) = \frac{p_L(x)}{p_{Li}} \tag{7}$$

is the polynomial of effective longitudinal material property variation.
The indices are: $p \equiv E$ and ρ - for elasticity modulus and mass density, respectively.
Expressions (5 - 7) can be used in the effective material properties calculation for single- layer FGM beams.
Let us replace the initial beam (Fig. 1a) by a multilayered beam (Fig. 1b). Lamination is symmetric according to the geometry of the layers and material properties as well. This symmetry allows the application of the elementary theory of the homogeneous isotropic beam for all solutions when the material properties are replaced by their effective values [12]. From the mechanical coupling point of view, axial loading is not coupled with transversal loading.
Individual layers are built of composite layers with longitudinal variation of volume fractions and material properties of the constituents as described above.
Homogenization of material properties of the multilayered beam will be done using the theory of laminates [12], [13], [14]. In this way we get one layer beam with longitudinal variation of homogenized longitudinal material properties. Main dimensions of the beam such as the length L, height h and width b remain conserved.

a) Homogenized elastic properties

If we denote the effective longitudinal elasticity modulus of a chosen layer with superscript k, then, according to (6), it holds

$$E_L^k(x) = E_{Li}^k\eta_{E_L^k}(x) \tag{9}$$

Index $k \in \langle 1, n \rangle$ represents the layer number in the upper and lower symmetrical part of the beam/link. The number of layers of the symmetrical part is n. If the cross-sectional area of the k^{th} layer is A^k, then the volume fraction of the pair of these symmetrical areas is $v^k = 2A^k/A$.

Then, the effective longitudinal elasticity modulus for axial loading of the homogenized sandwich can be derived using the expression

$$E_L^{NH}\left(x\right)=E_{Li}^{NH}\eta_{E_L^{NH}}\left(x\right)=\sum_{k=1}^{n}v^k E_L^k\left(x\right) \tag{10}$$

where $E_{Li}^{NH}=\sum_{k=1}^{n}E_{Li}^k v^k$ is the effective longitudinal elasticity modulus for axial loading of the homogenized beam at node i, and $\eta_{E_L^{NH}}\left(x\right)=E_L^{NH}\left(x\right)/E_{Li}^{NH}$ is the polynomial of its variation. This effective longitudinal elasticity modulus has to be used for calculation of an axial free vibration of the FGM beam.

According to the notation in Figure 1, the effective longitudinal elasticity modulus for flexural loading of the homogenized beam of rectangular cross-section has been derived [15]:

$$E_L^{MH}\left(x\right)=\frac{12}{h^3}\sum_{k=1}^{n}\left(\frac{h_k^3}{6}+2d_k^2h_k\right)E_L^k\left(x\right)=E_{Li}^{MH}\eta_{E_L^M}\left(x\right) \tag{11}$$

where E_{Li}^{MH} is the value of the effective longitudinal elasticity modulus for flexural loading of the homogenized beam/link at node i, and $\eta_{E_L^{MH}}=E_L^{MH}\left(x\right)/E_{Li}^{MH}$ is the polynomial of its longitudinal variation. This effective longitudinal elasticity modulus has to be used in the calculation of a flexural free vibration of the FGM beam.

In a similar way, the effective longitudinal elasticity modules can be derived for beams with different other types of cross-sections.

b) Homogenized mass density

If we denote ρ with superscript *k* the effective longitudinal mass density of a chosen layer *k*, then, according to (5), (6) and (7), it holds

$$\rho_L^k\left(x\right)=\rho_{Li}^k\eta_{\rho_L^k}\left(x\right) \tag{12}$$

Index *k*, the cross-sectional area of the the k^{th} layer A^k, and the volume fraction $v^k = 2A^k/A$ have the same meanings as in the elastic properties derivation.

Accordingly, the homogenized effective longitudinal mass density is

$$\rho_L^H\left(x\right)=\rho_{Li}^H\eta_{\rho_L^H}\left(x\right)=\sum_{k=1}^{n}v^k\rho_L^k\left(x\right) \tag{13}$$

where $\rho_{Li}^H=\sum_{k=1}^{n}\rho_{Li}^k v^k$ is the effective longitudinal mass density at node i of the homogenized beam, and $\eta_{\rho_L^H}\left(x\right)=\rho_L^H\left(x\right)/\rho_{Li}^H$ is the polynomial of its longitudinal variation.

All the homogenized effective longitudinal properties are denoted by upper right index H in this chapter. As assumed, their variation along the homogenized beam is polynomial.

Sufficient accuracy of substituting the continuous lateral variation of material properties by a layer-wise constant lateral distribution of material properties will be reached when the layer-division is fine enough. The constant value of the material property in the assumed layer at position x will be calculated as a mean value from its values at the top and the bottom of this layer. The same approach will be used in the calculation of the components volume fractions in the competent layer.

Variations of the homogenized beam properties and the loading are shown in Fig. 3.

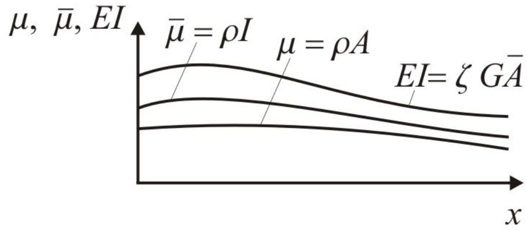

Figure 3: Variation of the beam parameters.

$\mu = \rho A$ is the mass distribution, $\bar{\mu} = \rho I$ is the mass moment of inertia distribution, $\rho = \sum_{j=0}^{p_\rho} \rho_j a_j = \rho_L^H(x)$ is the homogenized varying mass density (also defined by (13), indices *L, H* are omitted for simplicity), *A* is the cross-sectional area, *I* is the moment of inertia, $EI = \sum_{j=0}^{p_e} EI_j a_j = B$ is the varying bending stiffness caused by varying elasticity modulus $E \equiv E_L^{MH}(x)$, $G\bar{A} = \sum_{j=0}^{p_G} G\bar{A}_j\, a_j$ is the varying shear stiffness with $G = G_L^H(x)$, and $\varsigma = \dfrac{EI}{G\bar{A}} = const.$, $k(x)$ is the longitudinally varying Winkler elastic foundation modulus, and finally, $a_j = \dfrac{x^j}{j!}$ is the polynomial function. The same longitudinal polynomial variation of the homogenized shear modulus *G* has been assumed as stated by the homogenized elasticity modulus *E*. The assumption $\varsigma = \dfrac{EI}{G\bar{A}} = const.$ is a simplification which holds exactly only for single-layer FGM beams with only longitudinal variation of material properties. In addition, the condition $\dfrac{E}{G} = const.$ must be fulfilled. For the initial beam (Fig. 1a) both the shear correction factor $k^s = \dfrac{\bar{A}}{A} = k^s(x)$ and consecutive the factor $\varsigma = \varsigma(x)$ are functions of *x*. In our consideration, the parameters k^s and ς will be assumed as average values of their functional variation. The assumption $\varsigma = \varsigma(x)$ would make the differential equation of the beam deflection very complicated. In the following derivation of the differential equation, indices *L, M, H* have been also omitted for simplicity.

In Fig. 4 the distributed loads and the nodal internal forces are shown at the beam increment *dx* in the deformed configuration of the beam increment. In Fig. 5 the internal forces and nodal displacements at position *x* are shown.

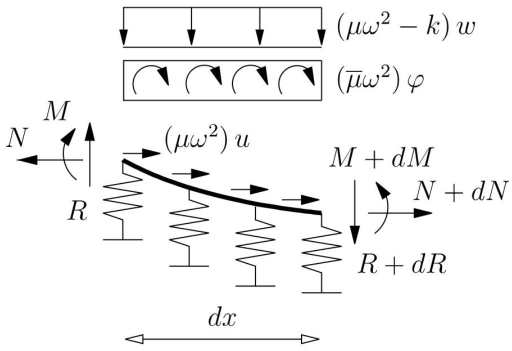

Figure 4: Distributed loads and internal forces in *R* – formulation.

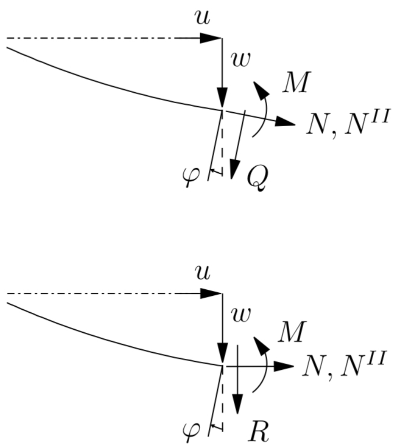

Figure 5: Internal forces and displacements at point *x* in *Q*- and *R* – formulation, respectively.

Here, *M* is the bending moment, *N* is the axial force, *Q* is the shear force, *R* is the transversal force, φ is the angle of cross-section rotation, *w* is the deflection perpendicular to the un-deformed longitudinal beam axis and *u* is the longitudinal displacement, ω is the circular frequency. The inertia forces in *x* – direction has not been assumed in the following considerations. N^{II} is the 2nd order beam theory constant axial force, which is of a system character and it is a known variable.

The differential equilibrium equations of a harmonic free vibration of 2nd order beam theory according to the un-deformed longitudinal beam axis (the *R* - formulation) are (the superscript "′" denotes the differentiations *d/dx*):

$$R' = \left(k - \omega^2 \mu\right) w \tag{14}$$

$$M' = Q + \overline{\mu}\omega^2 \varphi \tag{15}$$

The dependence between the shear and transversal force can be expressed as [16]: $Q = R - N^{II} w'$.

The kinematical differential equations are:

$$\varphi' = -\frac{M}{B} \tag{16}$$

$$w' = \varphi + \frac{Q}{G\overline{A}} = \varphi + \varsigma \frac{Q}{B} \tag{17}$$

After some mathematical operations the homogeneous differential equation of the 4th order of the homogenized functionally graded material (FGM) beam deflection has been obtained. Its form is:

$$\eta_4 w'''' + \eta_3 w''' + \eta_2 w'' + \eta_1 w' + \eta_0 w = 0 \tag{18}$$

After the boundary conditions application, the equation (18) can be used for the modal analysis of the multilayer FGM beams resting on the elastic Winkler foundation, with above mentioned effects.
The non constant parameters of the differential equation of the 4th order (18) are:

$$\eta_0 = \left(\omega^2 \mu - k\right)\left(\varsigma B\left(B'' - \omega^2 \overline{\mu}\right) - \varsigma B'^2 + B^2\right)^2 + \varsigma B k \Omega B'' - \varsigma \omega^2 B \mu \Omega B'' + \varsigma B \Omega B'\left(\omega^2 \mu' - k'\right)$$
$$- \varsigma B k \Gamma B' + \varsigma k \Omega B'^2 + \varsigma \omega^2 B \Gamma \mu B' - \varsigma \omega^2 \mu \Omega B'^2 - \varsigma B^2 \Omega k'' + \varsigma B^2 \Gamma\left(k' - \omega^2 \mu'\right) + \varsigma \omega^2 B^2 \Omega \mu''$$

$$\eta_1 = -B\left(\Omega\left(B'\left(\omega^2\left(-\left(2\mu + \varsigma\mu\right)\right) + \varsigma k + 2N^{II}\right) + \varsigma N^{II}\left(B''' - \omega^2 \overline{\mu'}\right)\right) + \varsigma N^{II} \Gamma \Delta\right) +$$
$$\varsigma N^{II} B'\left(\Omega\left(\omega^2 \overline{\mu} + B''\right) - \Gamma B'\right) + B^2\left(\Gamma\left(\omega^2\left(-\left(\overline{\mu} + \varsigma\mu\right)\right) + \varsigma k + N^{II}\right) + \Omega\left(\omega^2\left(\overline{\mu'} + 2\varsigma\mu'\right) - 2\varsigma k'\right)\right)$$

$$\eta_2 = B\left(B\left(\Omega\left(\omega^2\left(\overline{\mu} + \varsigma\mu\right) + B'' - \varsigma k - N^{II}\right) - \Gamma B'\right) + \Omega\left(\varsigma N^{II} \omega^2 \overline{\mu} - 2\varsigma N^{II} B'' + 2B'^2\right) + \varsigma N^{II} \Gamma B'\right)$$

$$\eta_3 = B\left(\Omega B'\left(4B + \varsigma N^{II}\right) - B\Gamma\left(B + \varsigma N^{II}\right)\right) \tag{19}$$

$$\eta_4 = B^2 \Omega\left(B + \varsigma N^{II}\right)$$

$$\Gamma = \varsigma B'\left(\omega^2 \overline{\mu} + B''\right) + B\left(\varsigma \omega^2 \overline{\mu'} - \varsigma B''' - 2B'\right)$$

$$\Omega = \varsigma B\left(\omega^2 \overline{\mu} - B''\right) + \varsigma B'^2 - B^2$$

$$\Delta = \omega^2 \overline{\mu} - B''$$

After integration of the differential equation the transfer relations will be obtained for the vertical displacement, angle of cross-section rotation, bending moment and transversal force. The 4 transfer relations have been transformed to the transfer matrix method form:

$$\begin{bmatrix} w(x) \\ \varphi(x) \\ M(x) \\ R(x) \\ 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w_i \\ \varphi_i \\ M_i \\ R_i \\ 1 \end{bmatrix} \tag{20}$$

The kinematical and kinetic variables at node *i* are denoted by index "*i*" in (20). The parameters of the transfer matrix are not expressed in detail here from space spending point of view. After implementation of the applied boundary conditions, the homogenous equations system will be obtained for the calculation of internal forces and deformations. Hence, eigenvalue problems can be solved for circular eigenfrequencies and subsequently, the eigenfrequencies $f_n = \omega_n / 2\pi$ can be obtained (index *n* denotes the frequency number).

If the axial force N^{II} is set to zero in equations (14) – (20), then $R = Q$ and the equation of the 1st order beam theory has been obtained.

3 NUMERICAL EXPERIMENTS

3.1 Modal analysis of the FGM beam: Case I

A FGM beam has been considered as shown in Fig. 6. Its square cross-section is constant with height h = 0.2 m and width b = 0.2 m. Length of the beam is L = 3 m. Material of the beam consists of two components: NiFe – named as a matrix and denoted with index *m* ; Tungsten – named as a fibre and denoted with index *f*.

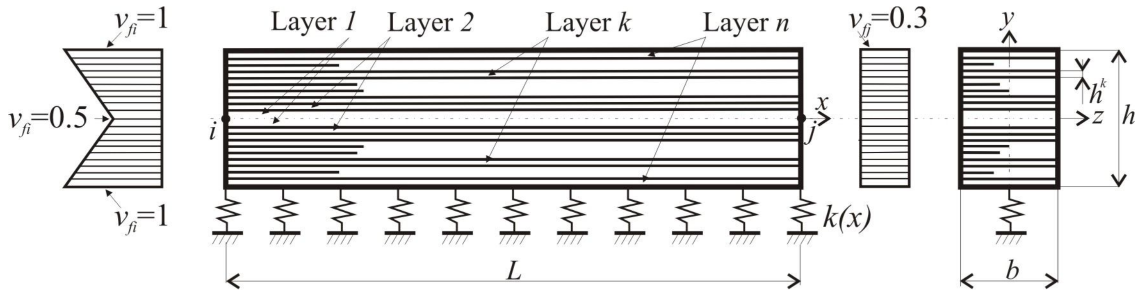

Figure 6: Composite beam with spatial variation of material properties.

Material properties of the components are constant and their values are: Tungsten (fibres) – elasticity modulus E_f = 400 GPa, mass density $\rho_f = 19300$ kg·m^{-3}; NiFe (matrix) – elasticity modulus E_m = 255 GPa, mass density $\rho_m = 9200$ kg·m^{-3} [11]. The cross-sectional area A = 0.04 m^2; the moment of inertia $I = bh^3/12 = 0.2^4/12$ m^4; and the reduced cross-sectional area $\overline{A} = k^s A$ [m^2]. The shear correction factor k^s can be computed so that the strain energy of the classical transverse shear stress equals the strain energy computed using the first order shear deformation theory [12], [17]]. According the notation in Fig. 1, it is possible to extend the calculation of shear stresses for a symmetrically layered beam with longitudinal variation of material properties of the layers [15]. The shear stress at point (*x,y*) of the relevant layer is a function of position of this point and it varies through the height and length of the layer. There is no jump of the shear stresses on the interface of the layers. Especially, for a rectangular cross-section – see Fig.1, the shear stress in the k[th] layer for $k \in \langle 1, n \rangle$ and $y \in \langle d_k - h_k / 2, d_k + h_k / 2 \rangle$ is

$$\tau_k(x, y) = \frac{Q(x)}{D(x)} \left(\frac{E_L^k(x)}{2} \left(\left(d_k + \frac{h_k}{2} \right)^2 - y^2 \right) + \sum_{j=k}^{n} \frac{E_L^{j+1}(x)}{2} \left(\left(d_{j+1} + \frac{h_{j+1}}{2} \right)^2 - \left(d_j + \frac{h_j}{2} \right)^2 \right) \right) \tag{21}$$

where $Q(x)$ is the shear force, $D(x)$ is the effective bending stiffness of the multilayer FGM beam at position x [15]. Using the expression (21) for the calculation of the strain energy in the cross-sectional area of the multilayered beam and putting it equal to the strain energy of the first order shear deformation theory, the shear correction factor function $k^s(x)$ (which is originally studied in [18]) can be calculated (because of longitudinal variation of the material properties in the layers, the shear correction factor is a function of x as well). Its average value

$$k^{sm} = \frac{1}{L} \int k^s(x) dx \tag{22}$$

has been used in calculation of the constant parameter ς included in the differential equation (19):

$$\varsigma = \frac{EI}{k^{sm} GA} \tag{23}$$

If the shear force effect is neglected, the factor $\varsigma = 0$. If the non-continuous distribution of the shear stresses (21) and the longitudinal variation of material properties in the layers is not accounted, then $k^s = 0.833334$. Otherwise, the shear correction function and the average shear correction factor (22) depend on the layering fineness. By increasing numbers of layers, the average shear correction factor will converge to the value which approximately fits the continuous variation of the material properties in transversal and longitudinal direction.

The fiber volume fraction, in this case, varies linearly and symmetrically according to the x – z neutral plane in transversal direction at node i ($v_{fi} \in \langle 0.5, 1.0 \rangle$), and continues linearly in the longitudinal direction to the constant value at node j.

Both halves of the height h of this beam have been divided symmetrically to the beam neutral plane to n = 5, and n = 10 layers in such a way that all the layers have the same thickness h^k, where $k \in \langle 1, n \rangle$. The transversally constant fiber volume fractions of the assumed layers have been calculated from their values at the top and the bottom of the competent layer.

Pairs of the symmetrical layers in all cases were built as a mixture of the two components. The volume fraction of the components is constant along the height and width of the competent layer but it varies linearly along the layer length:

$$v_f^k(x) = v_{fi}^k \left(1 + \eta_1^k x \right) = 1 - v_m^k(x) \tag{24}$$

where index k is the number of the symmetrical layers, v_{fi}^k is the volume fraction of the fibres in the k^{th} layer at node i, and η_1^k is a parameter of variation of the fibres volume fraction. The list of these parameters for n = 5 is given in Table 1. The fibres volume fraction in the layers at node j is a constant and it is equal to $v_{fj}^k = 0.3$, $v_m^k(x)$ is the matrix volume fraction in layer k.

Layer k	1	2	3	4	5
v_{fi}^{k} (-)	0.55	0.65	0.75	0.85	0.95
η_1^k (-)	-4.545	-5.384	-6.000	-6.470	-6.842

Table 1: Parameters of the fibres volume fractions variation for n = 5.

With expressions (9) and (12) the effective longitudinal elasticity modules and the mass densities of the layers can be calculated. Subsequently, the effective elasticity modulus for transversal loading, the effective mass density and the average shear stress correction factor of the homogenized multilayer beam have been calculated by using of the expressions (10), (11) (13) and (22), and we have got:

a) for $n = 5$ layers
$E_L^{MH}(x) = 381.15 - 27.55x$ [GPa]; $\rho_L^H(x) = 16775.0 - 1515.0x$ [kgm^{-3}]; k^{sm} = 0.836594 [-];
b) for $n = 10$ layers
$E_L^{MH}(x) = 381.69 - 27.73x$ [GPa]; $\rho_L^H(x) = 16775.0 - 1515.0x$ [kgm^{-3}]; k^{sm} = 0.836798 [-].

The shear stress correction functions $k^s(x)$ have non-linear longitudinal distribution (see Fig. 9): its value is equal to 0.83934 for $n = 5$, and 0.839714 for $n = 10$ at point *i*; and it is equal to 0.833334 at point *j* for the both divisions (the material properties are the same at this point in each layer).
If the homogenized material properties have been calculated by direct integration method (without division into layers), following parameters have been obtained:

$E_L^{MH} = 381.875 - 27.79x$ [GPa] $\qquad$ k^{sm} = 0.836868

As can be seen, the homogenized material properties converge very fast to the values obtained by the direct integration methods.
The ratio $\dfrac{E_L^{MH}(x)}{G_L^H(x)} = 2.6$ has been used for the effect of shear forces assumption in this example. Longitudinal distributions of the homogenized elasticity modulus and the effective elasticity modules in the respective layers for n = 5 are shown in Fig. 7.

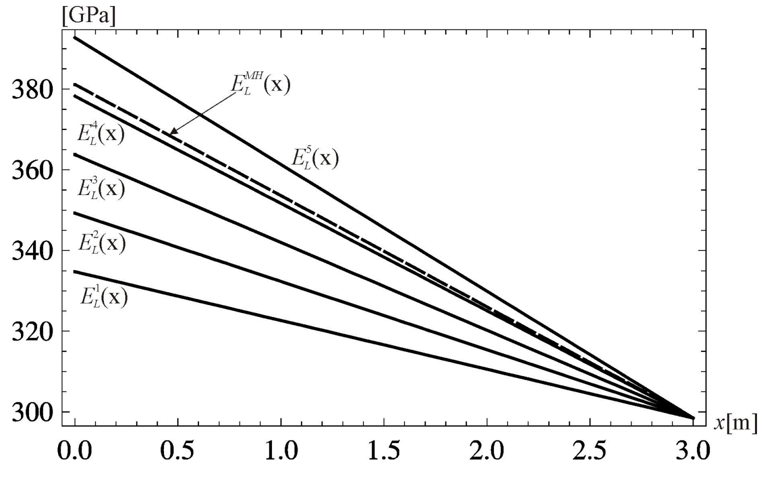

Figure 7**:** Longitudinal distributions of the homogenized elasticity modulus and the effective elasticity modules in respective layers for n = 5

Longitudinal distributions of the mass density in the homogenized beam and the effective mass densities distribution in respective layers of the multilayer beam for n = 5 are shown in Fig. 8.

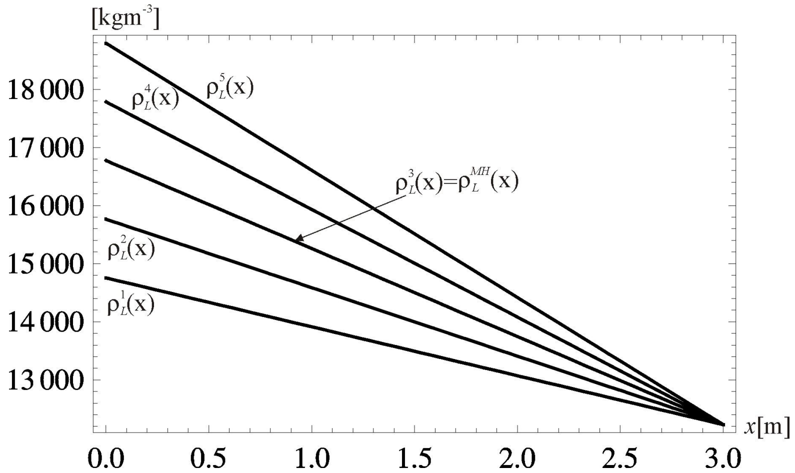

Figure 8. Effective mass density distribution along the homogenized beam and the effective elasticity mass density in respective layers for n = 5.

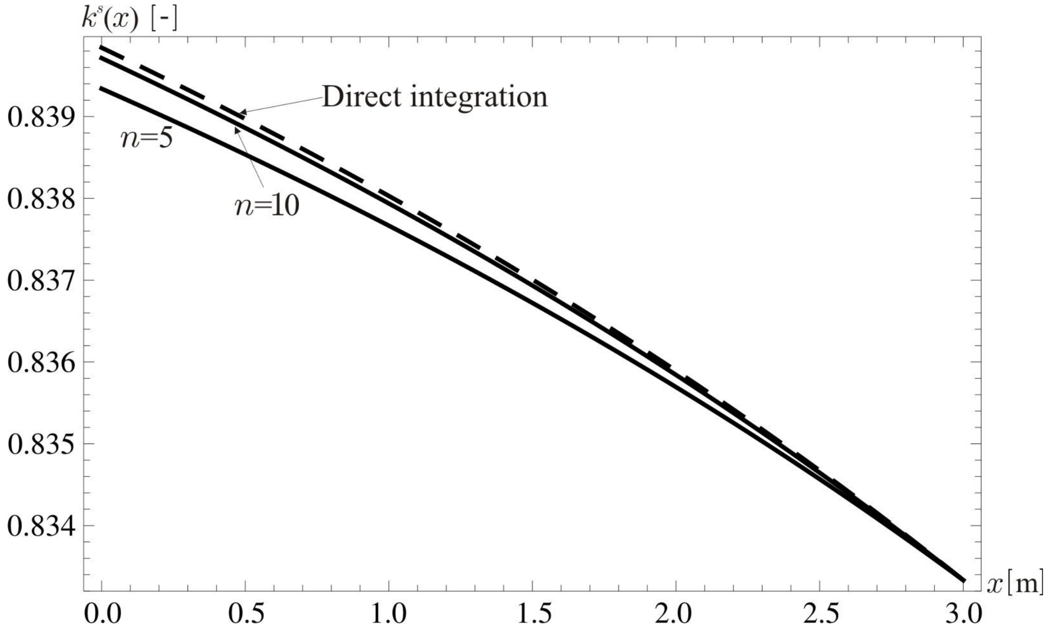

Figure 9**:** Shear correction function for n = 5 and n =10.

The varying Winkler elastic foundation modulus is chosen as a linear function: $k(x) = 5000 - 1000x$ [kN/m^2].

The homogenized simply supported beam (Fig. 6) has been studied by modal analysis. The first three bending eigenfrequencies have been found using the differential equation (18) and the appropriate boundary conditions. The buckling critical forces have been calculated by [19], and they are: $N_{Ki}^{II} = -52172.0$ kN for $n = 5$ layers; and $N_{Ki}^{II} = -52209.7$ kN for $n = 10$ layers. The axial force (tension and compression) $N^{II} \equiv N$ will be chosen as a part of the critical buckling force N_{Ki}^{II}. In all cases, the applied axial force N was smaller than the critical buck-

ling force. The effects of the shear stress correction factor and the large axial force have been studied and evaluated.
The same problem has been solved using a very fine mesh – 12000 of 2D PLANE42 elements and 401 COMBIN14 elements of the FEM program ANSYS [20] where each element has material properties equal to their value at the (x,z) location of the element center. The results of ANSYS as well as the results of the differential equations solution (using the software MATHEMATICA [21] and denoted by DIF) are presented in Tables 2 – 5. The shear force deformation effect and the effect of axial force (tension and compression) on the eigefrequencies have been studied.
The Table 2 presents the eigenfrequency results for tensional axial force when the average shear correction factors k^{sm} have been considered by DIF solution. In the second column of this table, the 1st order beam theory results obtained by DIF solution are presented.

$N^{II} = N$ [kN]		1st order DIF $N = 0$	DIF $N =$ 13000	ANSYS $N =$ 13000	DIF $N =$ 26000	ANSYS $N =$ 26000	DIF $N =$ 39000	ANSYS $N =$ 39000
f_1 [Hz]	$n = 5$	49.93	55.81	56.33	61.11	61.58	66.00	66.41
	$n = 10$	49.95	55.82		61.13		66.01	
f_2 [Hz]	$n = 5$	190.10	196.49	197.50	202.68	203.63	208.68	209.58
	$n = 10$	190.18	196.56		202.75		208.75	
f_3 [Hz]	$n = 5$	413.10	419.69	418.34	426.18	424.81	432.57	431.17
	$n = 10$	413.27	419.85		426.34		432.72	

Table 2: Eigenfrequencies in the Case I for tensional axial force with the shear correction factors

The Table 3 presents the eigenfrequency results for compression axial force when the average shear correction factors k^{sm} have been considered by DIF solution.

$N^{II} = N$ [kN]		DIF –13000	ANSYS –13000	DIF –26000	ANSYS –26000	DIF –39000	ANSYS –39000
f_1 [Hz]	$n = 5$	43.27	43.95	35.37	36.19	25.09	26.21
	$n = 10$	43.29		35.40		25.13	
f_2 [Hz]	$n = 5$	183.49	184.60	176.63	177.78	169.49	170.69
	$n = 10$	183.57		176.71		169.58	
f_3 [Hz]	$n = 5$	406.41	405.08	399.60	398.27	392.67	391.33
	$n = 10$	406.57		399.77		392.84	

Table 3: Eigenfrequencies in the Case I for compression axial force with the shear correction factors

Table 4 and Table 5 show the DIF solution results when the shear correction has not been considered.

$N^{II} = N$ [kN]		1st order DIF $N = 0$	DIF $N =$ 13000	DIF $N =$ 26000	DIF $N =$ 39000
f_1 [Hz]	$n = 5$	50.20	56.04	61.33	66.00
	$n = 10$	50.22	56.06	61.35	66.21
f_2 [Hz]	$n = 5$	194.24	200.49	206.55	212.44
	$n = 10$	194.32	200.57	206.62	212.21
f_3 [Hz]	$n = 5$	432.49	438.76	444.93	451.02
	$n = 10$	432.66	438.92	445.09	451.18

Table 4: Eigenfrequencies in the Case I for tensional axial force without the shear correction factors

$N^{II} = N$ [kN]		DIF –13000	DIF –26000	DIF –39000
f_1 [Hz]	$n = 5$	43.58	35.75	25.62
	$n = 10$	43.60	35.77	25.66
f_2 [Hz]	$n = 5$	187.79	181.10	174.15
	$n = 10$	187.86	181.18	174.23
f_3 [Hz]	$n = 5$	426.14	419.68	413.13
	$n = 10$	426.31	419.85	413.30

Table 5: Eigenfrequencies in the Case I for compression axial force without the shear correction factors

Dependence of the 1st eigenfrequency on the applied axial force (with and without the shear correction factor and for $n = 10$) is shown in Fig. 9

As we can see in Tab. 2 to Tab. 5, and in Fig. 9, the solution results obtained by our new differential equation agree very well with the ones obtained by very fine mesh of the PLANE42 solid elements. As expected, inclusion of the shear force deformation effect makes the results more accurate. But this effect is not significant in this example. This effect will grow with higher variation of non-linearities in material properties in transversal and longitudinal direction. As expected, increasing compression axial force decreases the values of eigenfrequency and, on the other hand, increasing tensional force increases the eigenfrequency. The influence of the large axial force is significant in calculated cases. This fact can be utilized by the control and tuning of dynamic properties of beam structures.

Foundation elastic properties have not been studied in this contribution.

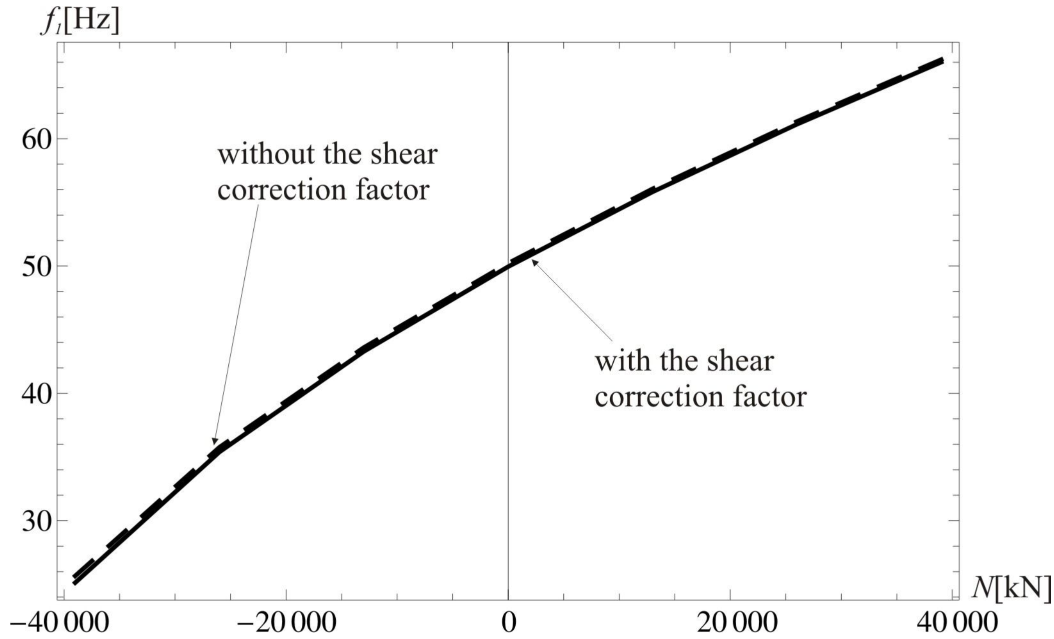

Figure 9: Dependence of the eigenfrequencies in the Case I for n = 10.

3.2 Modal analysis of the FGM beam: Case II

An FGM beam has been considered as shown in Fig. 10. Its geometry and material properties of the constituents are the same as in Case I. Only the variation of fibre volume fractions has been changed.

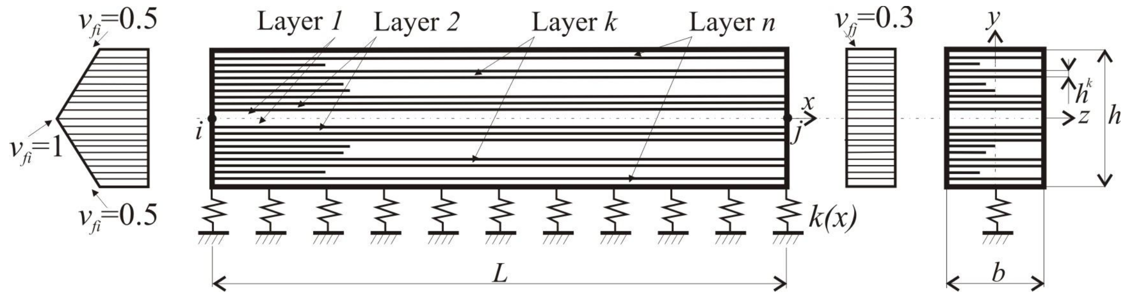

Fig. 10. FGM beam with spatial variation of material properties

The fibre volume fraction at node i varies linearly and symmetrically according to the neutral plane in the transversal direction ($v_{fi} \in \langle 1.0, 0.5 \rangle$) and continuous linearly in the longitudinal direction to the value $v_{fj} = 0.3$.

By the same approach described in chapter 2 the fibres volume fraction have been obtained - see Table 6.

Layer k	5	4	3	2	1
v_{fi}^k (-)	0.55	0.65	0.75	0.85	0.95
η_1^k (-)	-4.545	-5.384	-6.000	-6.470	-6.842

Table 6: Parameters of the fibres volume fractions variation for n = 5

With expressions (9) and (12) the effective longitudinal elasticity modules and the mass densities of the layers can be calculated. Following, the effective elasticity modulus for transversal loading, the effective mass density and the average shear stress correction factor of the homogenized multilayer beam have been calculated using expressions (10), (11) (13). The effective elasticity modulus and the average shear correction factor of the homogenized multilayer beams for transversal loading have been calculated via expressions (11) and (29), and we have got:

a) for $n = 5$ layers
$E_L^{MH}(x) = 345.88 - 15.79x$ [GPa]; $\rho_L^H(x) = 16775.0 - 1515.0x$ [kgm^{-3}]; k^{sm} = 0.828612 [-];
b) for $n = 10$ layers
$E_L^{MH}(x) = 345.74 - 15.74x$ [GPa]; $\rho_L^H(x) = 16775.0 - 1515.0x$ [kgm^{-3}]; k^{sm} = 0.829386 [-].

The shear stress correction function $k^s(x)$ has non-linear longitudinal distribution (see Fig. 11): its value is equal to 0.824329 for $n = 5$ and 0.82579 for $n = 10$ at point *i*; and it is equal to 0.833334 at point *j* for the both divisions (the material properties are the same at this point in each layer).

The ratio $\dfrac{E_L^{MH}(x)}{G_L^H(x)} = 2.6$ has been used to express the effect of shear forces assumption in this example.

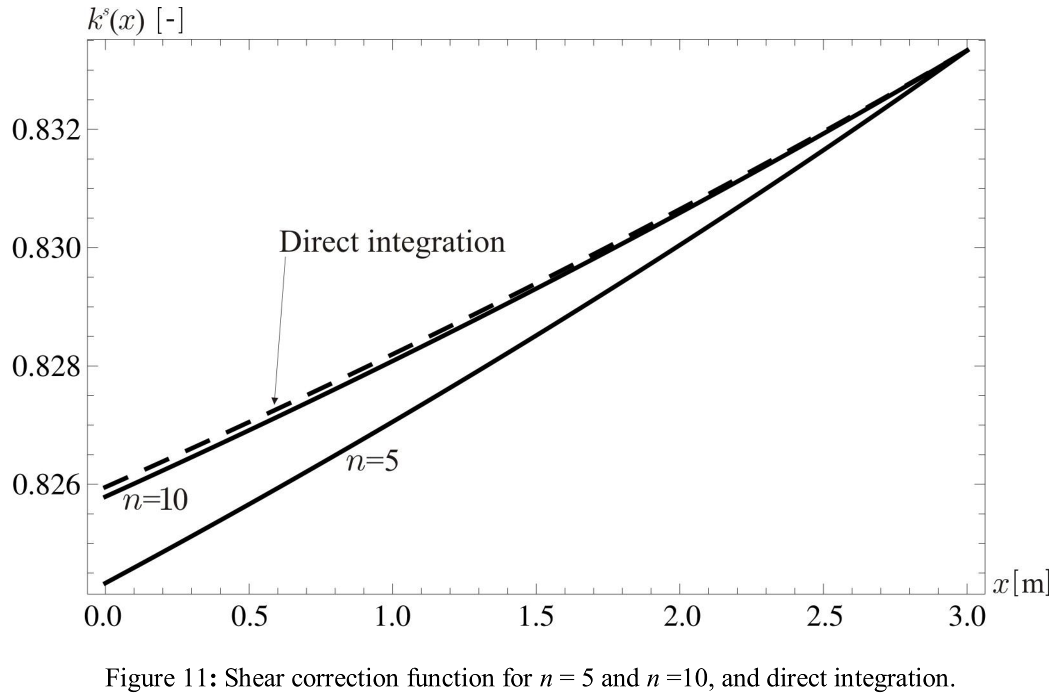

Figure 11**:** Shear correction function for n = 5 and n =10, and direct integration.

Homogenized material properties obtained by direct integration (without division into layers) give following parameters:

$$E_L^{MH}(x) = 345.625 - 15.70x \text{ [GPa]} \qquad k^{sm} = 0.829478$$

The above listed parameters show that the division of the beam into $n = 10$ layers gives enough accurate results which agree very well with the ones obtained by the direct integration method.

The ratio $\frac{E_L^{MH}(x)}{G_L^H(x)} = 2.6$ has been used to express the effect of shear forces assumption in this example.

Longitudinal distributions of the homogenized elasticity modulus and the effective elasticity modules in respective layers for $n = 5$ are shown in Fig. 12.

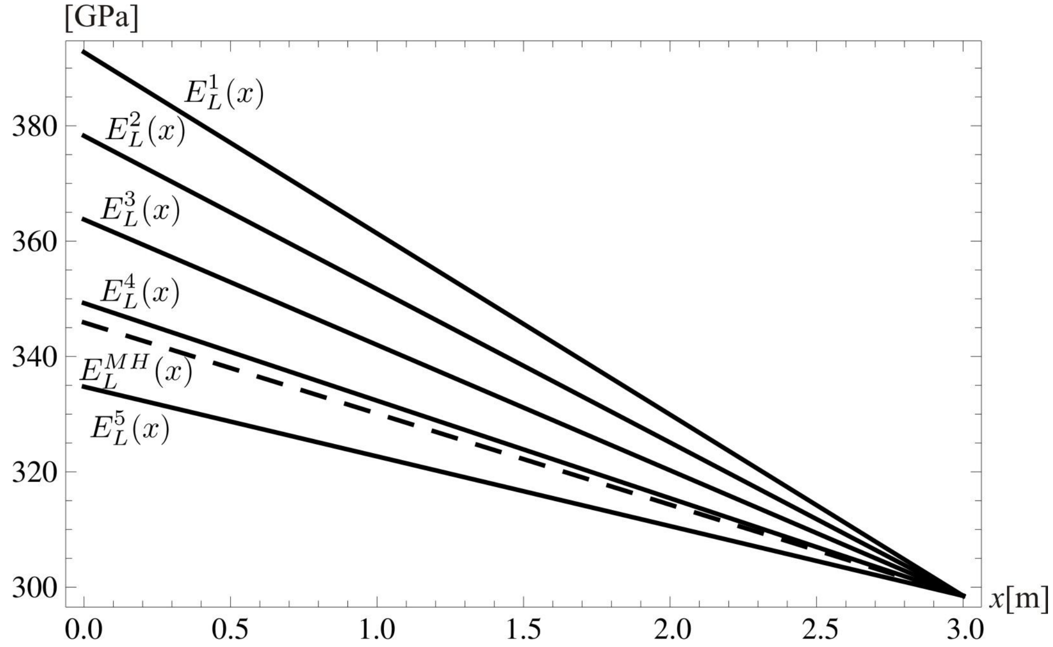

Fig. 12. Longitudinal distributions of the homogenized elasticity modulus and the effective elasticity modules in respective layers for *n* = 5.

Longitudinal distributions of the homogenized elasticity modules for division into n = 10 layers are shown in Fig. 13.

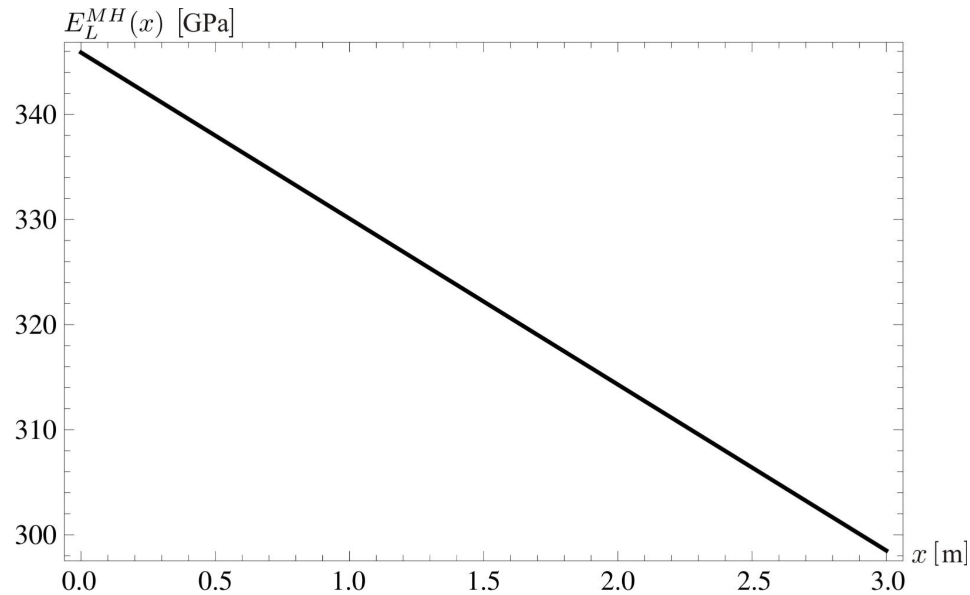

Fig. 13. Longitudinal distributions of the homogenized elasticity modules for *n* = 10 layers.

The homogenized mass density has been calculated by expressions (12) and (13): $\rho_L^H(x) = 16775.0 - 1515.0x$, [kg·m^{-3}]; this value is the same for any number of divisions n as it was in the Case I. Distribution of the mass density in the layers is similar to the one displayed in Fig. 8 but the ordering of layers is in contrary series. The same varying Winkler elastic foundation has been chosen as in the Case I.
The homogenized simply supported beam (Fig. 10) has been studied using modal analysis. The first three bending eigenfrequencies have been found using the differential equation (18) and the appropriate boundary conditions. The buckling critical forces have been calculated by [19], and they are: $N_{Ki}^{II} = -49711.3$ kN for $n = 5$ layers; and $N_{Ki}^{II} = -49701.6$ kN for $n = 10$ layers. The axial force (tension and compression) $N^{II} \equiv N$ will be chosen as a part of the critical buckling force N_{Ki}^{II}. In all cases, the applied axial force N was smaller than the critical buckling force. The effects of the shear stress correction factor and the large axial force have been studied and evaluated. The same problem has been solved using a very fine mesh – 12000 of 2D PLANE42 elements and 401 COMBIN14 elements of the FEM software ANSYS [20] – with a variation of material properties (in competent layers) as displayed in Fig. 7 and Fig. 8. The results from ANSYS as well as the results of the differential equations solution (using the software MATHEMATICA [21] and denoted by DIF) are presented in Tables 7 – 10. The shear force deformation effect and the effect of axial force (tension and compression) on the eigefrequencies have been investigated.
The Table 7 presents the eigenfrequency results for tensional axial force when the average shear correction factors k^{sm} have been considered in the DIF solution. In the second column of this table, the 1st order beam theory results obtained by DIF solution are presented.

$N^{II} = N$ [kN]		1st order DIF $N = 0$	DIF $N =$ 13000	ANSYS $N =$ 13000	DIF $N =$ 26000	ANSYS $N =$ 26000	DIF $N =$ 39000	ANSYS $N =$ 39000
f_1 [Hz]	$n = 5$	48.72	54.72	55.27	60.12	60.620	65.07	65.525
	$n = 10$	48.71	54.71		61.12		65.07	
f_2 [Hz]	$n = 5$	185.18	191.72	193.22	198.05	199.49	204.17	205.56
	$n = 10$	185.16	191.71		198.04		208.75	
f_3 [Hz]	$n = 5$	402.31	409.06	409.96	415.70	416.58	422.24	423.09
	$n = 10$	402.28	409.03		415.68		422.22	

Table 7: Eigenfrequencies in the Case II for tensional axial force with the shear correction factors

The Table 8 presents the eigenfrequency results for compression axial force when the average shear correction factors k^{sm} have been considered in the DIF solution.

$N^{II} = N$ [kN]		DIF –13000	ANSYS –13000	DIF –26000	ANSYS –26000	DIF –39000	ANSYS –39000
f_1 [Hz]	$n = 5$	41.87	42.59	33.65	34.53	22.62	23.86
	$n = 10$	41.86		33.64		22.61	
f_2 [Hz]	$n = 5$	178.39	180.00	171.34	172.99	163.98	165.69
	$n = 10$	178.38		171.32		163.96	
f_3 [Hz]	$n = 5$	395.44	396.38	388.45	389.39	381.33	382.27
	$n = 10$	395.41		388.42		381.30	

Table 8: Eigenfrequencies in the Case II for compression axial force with the shear correction factors

Table 9 and Table 10 show the DIF solution results when the shear correction has not been considered.

$N^{II} = N$ [kN]		1st order DIF $N = 0$	DIF $N =$ 13000	DIF $N =$ 26000	DIF $N =$ 39000
f_1 [Hz]	$n = 5$	48.98	54.95	60.33	65.27
	$n = 10$	48.97	54.94	60.33	65.26
f_2 [Hz]	$n = 5$	189.25	195.65	201.85	207.86
	$n = 10$	189.23	195.63	201.83	207.84
f_3 [Hz]	$n = 5$	421.37	427.79	434.11	440.34
	$n = 10$	421.32	427.74	434.06	440.29

Table 9: Eigenfrequencies in the Case II for tensional axial force without the shear correction factors

$N^{II} = N$ [kN]		DIF –13000	DIF –26000	DIF –39000
f_1 [Hz]	$n = 5$	42.17	34.03	23.17
	$n = 10$	42.17	35.02	25.16
f_2 [Hz]	$n = 5$	182.62	175.75	168.59
	$n = 10$	182.60	175.72	168.56
f_3 [Hz]	$n = 5$	414.85	408.23	401.50
	$n = 10$	414.80	408.18	401.45

Table 10: Eigenfrequencies in the Case II for compression axial force without the shear correction factors

Dependence of the 1st eigenfrequency on the applied axial force (with and without the shear correction factor and for $n = 10$) is shown in Fig. 14.
As we can see in Tab. 7 to Tab. 10 and in Fig. 14, the effects and conclusions are similar to the ones in the previous Case I. In addition, also the changed variation of material properties influenced the eigenfrequencies of the FGM beams. The eigenfrequencies are higher in the Case I due a variation of material properties making the beam stiffer.

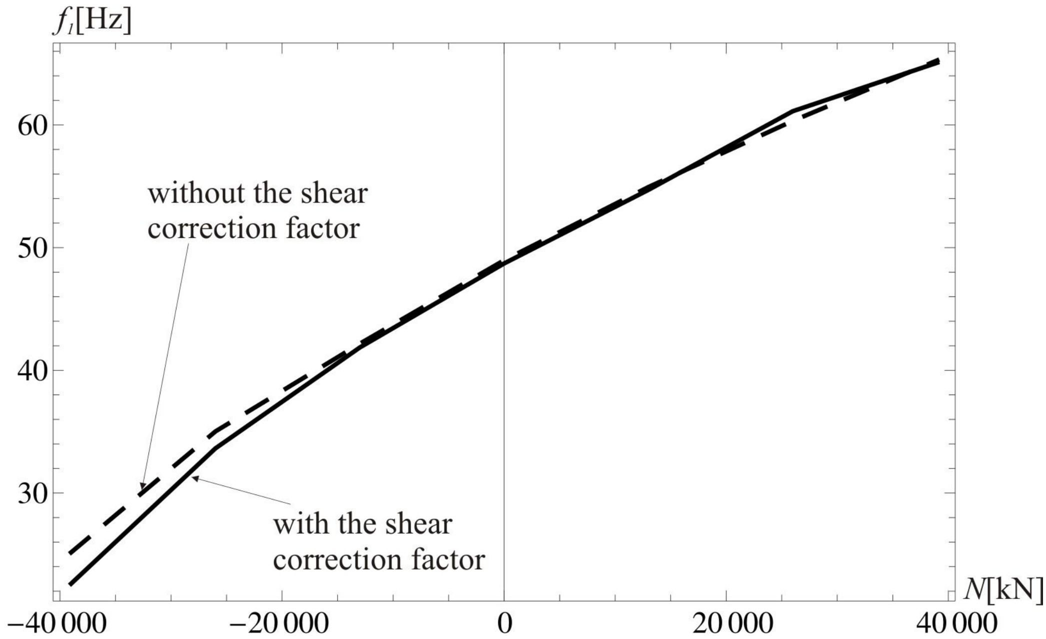

Figure 14: Dependence of the eigenfrequencies on the axial force in the Case II for n = 10.

4 CONCLUSIONS

Differential equation of the homogenized functionally graded material (FGM) beam deflection was established for 2nd the order beam theory, which has been used in a modal analysis of FGM beams of rectangular cross-sections with continuous spatial variation of material properties. The FGM beams are considered to be resting on longitudinally variable (Winkler) elastic foundation. Homogenization of material properties has been done by the multilayer method and by direct integration. By the multilayer method, the FGM beam with continuous spatial variation of material properties has been transformed into a multilayer beam. Symmetrical layering along the neutral plane in transversal direction has been assumed: the corresponding layers having the same height and material properties variation. The material properties vary continuously in longitudinal direction but they are constant along the height and width of each layer. By the second one the direct integration of the varying material properties over the height and length of the beam has been used. Increasing numbers of layers make the homogenized material properties converge to the ones obtained by a direct integration.

Not only the shear force deformation effect and the effect of consistent mass distribution and mass moment of inertia but also the effect of a large axial force has been taken into account. Large axial force has a system character: if set to zero in the derived differential equation, the 1st order beam theory differential equation will be obtained. The shear correction function has been derived and an average shear correction factor has been calculated to express the shear force effect. Effects of variation of elastic foundation properties are not subject of this contribution.

Numerical experiments were performed to calculate the eigenfrequencies of the chosen FGM beams. The solution results are discussed and compared with those obtained using a very fine mesh of two-dimensional solid finite elements.

The main conclusions that can be drawn from this investigation are:

- the large axial force has a significant effect on the eigenfrequency value also in the case of a beam resting on an elastic foundation;
- increasing compression axial force decreases the eigenfrequency and increasing tensional force increases the eigenfrequency;

- inclusion of the shear force deformation effect makes the results more accurate (the shear correction function has been derived for calculation of the average shear correction factor);
- not only transversal but also longitudinal variation of material properties affects the dynamic properties of the FGM beams;
- the results obtained by solution of the differential equation agree very well with these obtained by an FEM solution using a very fine mesh of the solid finite elements.

The presented differential equation is suitable for an effective modal analysis of beams with continuous spatial variation of material properties resting on a Winkler elastic foundation with longitudinally varying material properties. Results of such analyses can be used not only as a benchmark solution in comparison of results obtained by other numerical method but also in modal analysis of the real FGM beams.

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