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# NON-PERIODIC HOMOGENIZATION OF THE ELASTIC WAVE EQUATION FOR WAVE PROPAGATIONS IN COMPLEX MEDIA

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**Abstract.** *When considering numerical acoustic or elastic wave propagation in media containing small heterogeneities with respect to the minimum wavelength of the wavefield, being able to upscale physical properties (or homogenize them) is valuable, for mainly two reasons: first, replacing the original discontinuous and very heterogeneous media by a smooth and more simple one, is a judicious alternative to the necessary fine and difficult meshing of the original media required by many wave equation solvers; second, it helps to understand what properties of a medium are really "seen" by the wavefield propagating through it, which is an important aspect in an inverse problem approach. We present here a solution to solve this up-scaling problem for non-periodic complex media with rapid variations in all directions based an a non-periodic homogenization procedure. We first present a pedagogical introduction to non-periodic homogenization in 1-D, allowing to find the effective wave equation and effective physical properties of the wave equation in a highly heterogeneous medium. It can be extended from 1D to a higher space dimension and a special care of boundary conditions is required. This development can be seen as an extension of the classical two-scale periodic homogenization theory applied to the wave equation for non-periodic media. To validate this development, we then present two examples of wave propagation in 2D complex elastic models: a geometrically square model with random heterogeneities, and the Marmousi2 model. A reference solution is computed with the Spectral Element Method with meshes honoring all interfaces. Furthermore, we compare the results obtained in the homogenized model and in a low-pass filtered model with respect to the reference solution.*

## 1 Introduction

Being able to model and understand wave propagation in complex media is a constant concern for seismologist and the exploration community. In the recent years, advances in numerical methods have allowed to model full seismic waveform in complex media. Among these advances in numerical modeling, the introduction of the Spectral Element Method (SEM) in seismology has been particularly interesting [1]. This method has the advantage to be accurate for all type of waves and all type of media, as long as an hexahedral mesh, on which the method relies, can be designed such that all physical discontinuities are honored. In realistic media, the design of such a mesh is often impossible. For such a case, a smooth effective elastic media would solve the problem by removing the discontinuities of the elastic model while keeping the waveforms intact. Actually this problem is not limited to SEM and can be seen as a particular case of a more general problem in seismology: when an elastic model contains details much smaller than the wavelength, the model can be up-scaled consistently with the wave equation in some specific cases only: the layered media, and the periodic ones. In other words, the effective medium of a given general elastic model for a given minimum wavelength is unknown. For layered media, the up-scaling solution is known since the early work of Backus [2]. More recently, this order 0 (when referred to the homogenization theory) pioneer work has been extended to higher order, but still in the layered media case [3]. For elastic models with fast variations in several directions, the problem has been addressed for long with the two scale homogenization [4, ] but is limited to the periodic media case.
In this work, we go beyond the non-periodic layered case and the high dimension periodic case with an up-scaling tool based on non-periodic homogenization.

## 2 Theory

We first present the homogenization method in a simple 1D periodic case and in a simplified manner. We consider a scalar wave propagating in a infinite elastic bar with $\ell$-periodic elastic property $E(x)$ and density $\rho(x)$. We assume the existence of a minimum wavelength $\lambda_m$ for the propagating wavefield and that $\varepsilon := \frac{\ell}{\lambda_m} << 1$. The equation of motion, driving the displacement $u^\varepsilon$, and the constitutive relation in the bar are

$$\begin{aligned} \rho^\varepsilon \partial_{tt} u^\varepsilon - \partial_x \sigma^\varepsilon = f^\varepsilon \\ \sigma^\varepsilon = E^\varepsilon \partial_x u^\varepsilon \end{aligned} \tag{1}$$

The classical periodic homogenization procedure to solve the above problem is the following:

1. The fast variable $y = \frac{x}{\varepsilon}$ is introduced;
2. The cell property $\rho(y) := \rho^\varepsilon(\varepsilon y)$ and $E(y) := E^\varepsilon(\varepsilon y)$ are defined;
3. As $\varepsilon \to 0$, $y$ and $x$ are treated as independent variables implying $\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}$;
4. Solutions to (1) are sought as asymptotic expansions in $\varepsilon$:

$$\begin{aligned} u^\varepsilon(x,t) &= \sum_{i\geq 0} \varepsilon^i u^i(x, y = \frac{x}{\varepsilon}, t) \\ \sigma^\varepsilon(x,t) &= \sum_{i\geq -1} \varepsilon^i \sigma^i(x, y = \frac{x}{\varepsilon}, t) \end{aligned}$$

5. Finally, injecting the previous expansions in 1 the series of equation to be solved for each $i$ are:

$$\begin{aligned} \rho\partial_{tt}u^i + \partial_x\sigma^i + \partial_y\sigma^{i+1} &= f^i \\ \sigma^i &= E(\partial_x u^i + \partial_y u^{i+1}) \end{aligned} \tag{2}$$

Defining, for any function $h(x, y)$, $\lambda_m$-periodic in $y$, the cell average

$$\langle h \rangle (x) = \frac{1}{\lambda_m}\int_0^{\lambda_m} h(x, y)dy\,,$$

and solving the series of equations (2) up to the first order, we find that $u = \langle u^0 \rangle + \varepsilon \langle u^1 \rangle$ and $\sigma = \langle \sigma^0 \rangle + \varepsilon \langle \sigma^1 \rangle$ are solutions of the following effective wave equation:

$$\langle \rho \rangle \partial_{tt}u - \partial_x\sigma = f\,, \qquad \sigma = E^*\partial_x u$$

where $E^* = \langle E(1 + \partial_y\chi^1)\rangle$ is the effective elastic parameter and where $\chi^1$ is the first order periodic corrector. $\chi^1$ is solution of the cell problem:

$$\partial\left[E(1 + \partial_y\chi^1)\right] = 0\,. \tag{3}$$

The final order 1 solution can be obtained with

$$u^\varepsilon = (1 + \varepsilon\chi^1\partial_x)u + O(\varepsilon^2)$$

It appears that:

- in this simple 1D case, we can find an analytical solution to the cell problem leading to $1/E^* = \langle 1/E \rangle$. This result is similar to the Backus (1962)'s result. There is not such an analytical solution for higher spatial dimensions.
- at the order 0, the solutions do not depend on the microscopic scale $(y)$ . This is still true in 2D and 3D for $u^0$ but not for $\sigma^0$;
- at order $> 0$: the boundary conditions change (e.g. Neumann condition becomes Dirichlet to Neumann);
- at order $> 1$: the effective equation changes (it is not the classical wave equation anymore);

**Moving to the non-periodic case**

When dealing with non-periodic media, the classical periodic homogenization can still be applied for the whole bar, but, then obtaining a simple constant effective media, is not really interesting. To keep the ideas of periodic homogenization and allowing a more complete effective medium, we introduce an arbitrary scale separation around a given wavelength $\lambda_0$. All scales smaller that $\lambda_0$ are considered as small scales and scales larger than $\lambda_0$ are considered as large scales. We define $\varepsilon_0 := \lambda_0/\lambda_m$. A spatial filter operator (a low-pass filter) is introduced:

$$\mathcal{F}^{\varepsilon_0}(h)(x) = \int h(x')w_{\varepsilon_0}(x - x')\,, dx'$$

where $w_{\varepsilon_0}$ is a low-pass filter wavelet with a wave-number cutoff around $1/\lambda_0$ (see Fig. 1). $\mathcal{F}^{\varepsilon_0}$

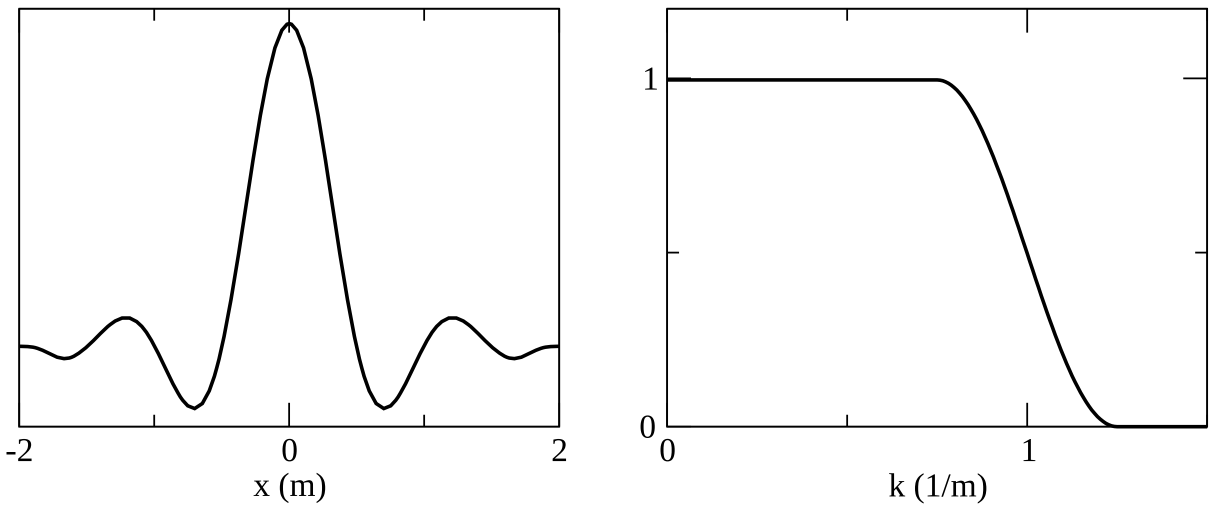


Figure 1: $w$ wavelet example, in the space (left plot) and in the wave-number (right plot) domains, with a cutoff at $\simeq 1m^{-1}$.

allows to piratically define the slow ($x$ part) from the fast ($y$ part) variations. The difficulty is to define an $E^{\varepsilon_0}(x,y)$ allowing to separate the scales according to $\mathcal{F}^{\varepsilon_0}$. In the 1D case [3], this can be done using

$$\frac{1}{E^{\varepsilon_0}}(x,y) = \mathcal{F}^{\varepsilon_0}\left(\frac{1}{E}\right)(x) + \left(\frac{1}{E} - \mathcal{F}^{\varepsilon_0}\left(\frac{1}{E}\right)\right)(y) \tag{4}$$

In that case, we can readily show that

$$\frac{1}{E^{\varepsilon_0 *}} = \left\langle \frac{1}{E^{\varepsilon_0}} \right\rangle = \mathcal{F}^{\varepsilon_0}\left(\frac{1}{E}\right)$$

This effective $E^{\varepsilon_0 *}$ is smooth but allows to capture the whole wavefield for small enough $\varepsilon_0$. At this point, most of the periodic development is still valid and non-periodic correctors can be computed.

**Going to higher dimensions**

For the periodic case, 2D/3D homogenization technique exists and can be applied to the wave equation without specific difficulty. For non periodic media, the generalization of the 1D case previously presented is difficult because no analytical solution to the cell (3) problem does exist and a direct construction similar to (4) is not possible. We have nevertheless developed an un-direct construction of the elastic tensor $\mathbf{c}^{\varepsilon_0}(\mathbf{x},\mathbf{y})$ allowing non-periodic homogenization in a spatial dimension higher than 1. In the next section are shown two examples of applications in the case of the P-SV wave propagation in 2D.

**Two examples**

## 2.1 Random square example

The first model is a randomly generated 2D elastic medium. It consists of a $30 \times 30\,km^2$ square matrix of $300 \times 300$ elements of constant elastic properties surrounded by a $10\,km$ thick strip of constant elastic properties corresponding to P and S wave velocities of $5\,km^{-1}$ and

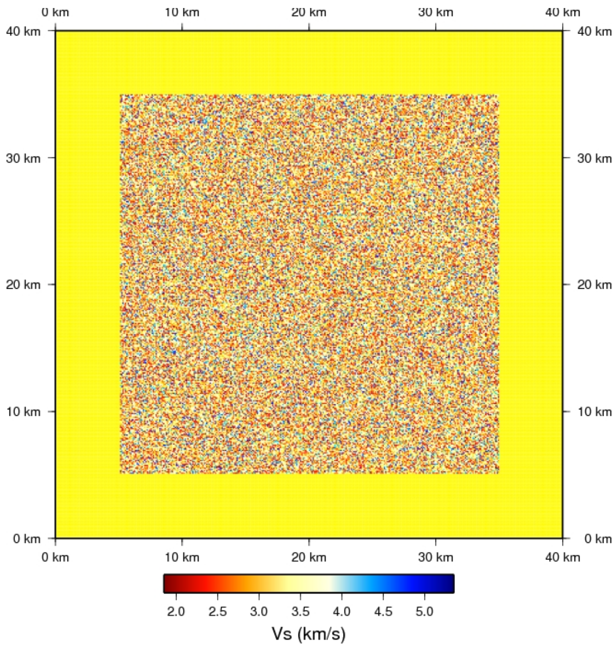


Figure 2: "Random square" model. Only $V_s$ is presented in this plot.

$3.2\,km^{-1}$ respectively and a density of $3000\,kg\,m^{-3}$ (see Fig. 2). In each element of the matrix, the constant elastic properties and density are generated independently and randomly within $\pm 50\%$ of the outer square elastic values. We wish to propagate waves induced by an explosion located at $\mathbf{x}_0 = {}^t(20\,km, 20\,km)$ (center of the square), the source time dependence being described by a Ricker wavelet (i.e. second derivative of a Gaussian function) with a central frequency of 1.5 Hz (corresponding roughly to a corner frequency of 3.6 Hz). Ignoring the fluctuation of velocity in the inner square and far away enough from the source, we can estimate the minimum wavelength $\lambda_m$ of the wavefield generated by the explosion, to be roughly equal to $800\,m$. To obtain the promised accuracy of the SEM, we must generate a mesh based on square elements that honors all physical discontinuities. In this case, the geometry is so simple that the mesh generation is trivial, nevertheless, it imposes $100 \times 100\,m^2$ elements in the random matrix. Knowing that a degree 4 spectral element (a tensorial product of degree 4 polynomial basis) can roughly handle one wavelength per element, the mesh is oversampling the wavefield by a factor 8 in each direction leading to a factor 512 in numerical cost (a factor 8 in each direction and a factor 8 in time to match the Newmark time marching scheme stability condition). For this simple 2-D case, this factor 512 can be handle and this allows to compute a reference solution. Nevertheless, one can imagine that for a 3-D case, meshing the original model can quickly be out of reach for a reasonable computing power and the temptation would be high to either use a mesh that doesn't honor the physical interfaces or to simplify the model.

We choose an $\varepsilon_0 = 0.4$ for the homogenization procedure (which means we filter out all oscillations in the medium that are twice smaller than the wavelength). The $V_s$ and the total anisotropy (we define the total anisotropy as $\max\{|\mathbf{c}^* - \mathbf{c}^*_{iso}|\}/\max\{|\mathbf{c}^*|\}$ where $\mathbf{c}^*_{iso}$ is the closest isotope elastic tensor to $\mathbf{c}^*$) of the homogenized media are shown Fig. 3. In Fig. 4 are shown the reference solution, the result of a run in the homogenized model and in a naively "filtered model" (we defined the filtered model as $\mathbf{c}^f = \mathcal{F}^{\varepsilon_0}(\mathbf{c})$. The is a low-pass filtered version of the original model) for a receiver located in ${}^t(37\,km, 20\,km)$. It can seen that the homogenized solution is very accurate compared to the "filtered" solution.

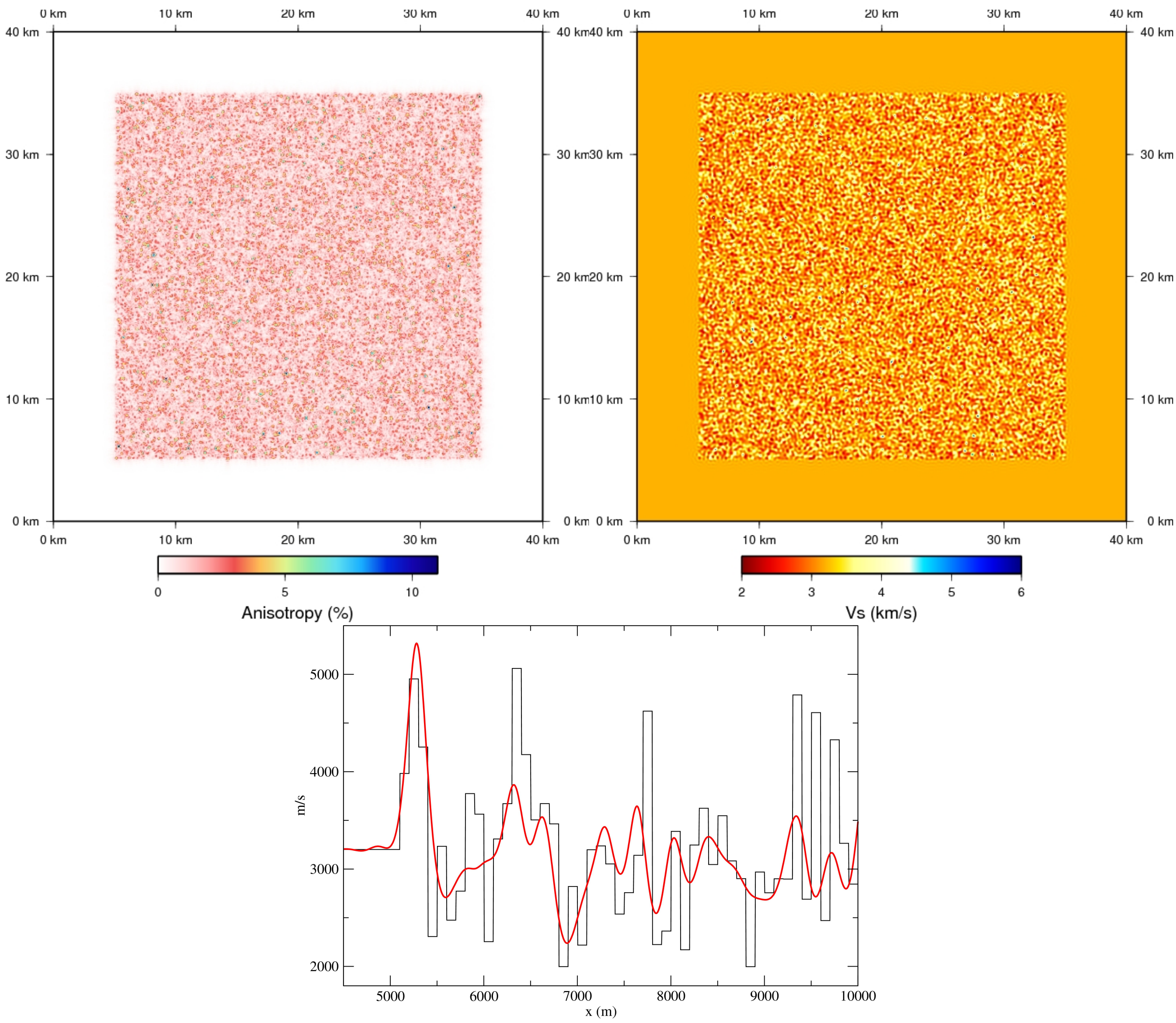


Figure 3: One of the $V_s$ (top left plot) and total anisotropy (top right plot) of the homogenized model of the random square model. The bottom plot is a cut along $z = 32\ km$ in the original $V_s$ (black line) and one of the $V_s$ of the homogenized model (red line).

## Marmousi2 model

Marmousi2 is a 2D elastic geological model derived by G.S. Martin from the original Marmousi model designed by the IFP (Fig. 5). The model contains thin (down to some meters) and complex structures. Even for a 2D model, the quadrangular mesh required for the Spectral Element Method is difficult to design and leads to a significant numerical cost (the reference solution computed here lasted for 7 days using 64 CPU). A sample of the mesh is shown Fig. 6, top plot. Here, the source is an explosion with, in time, a Ricker wavelet with central frequency of 6Hz (15Hz corner) leading to a minimum wavelength varying from 20m at the top of the model to 230m at the bottom. A snapshot of the propagating kinetic energy is shown in Fig. 8. The non-periodic homogenization is performed using a corner wavenumber of 0.017m$^{-1}$ which implies a $\varepsilon_0$ varying from 3 at the top of the model to 0.25 at the bottom. One of the $S$ velocity and the total anisotropy of the homogenized model are plotted in Fig. 7. For the homogenized model, the SEM mesh is a trivial regular mesh (see Fig. 6, bottom plot) leading to a lower numerical cost (1 hour using 64 CPU).

An example of traces is shown in Fig. 9. It appears that, even if the $\varepsilon_0$ is very poor at the

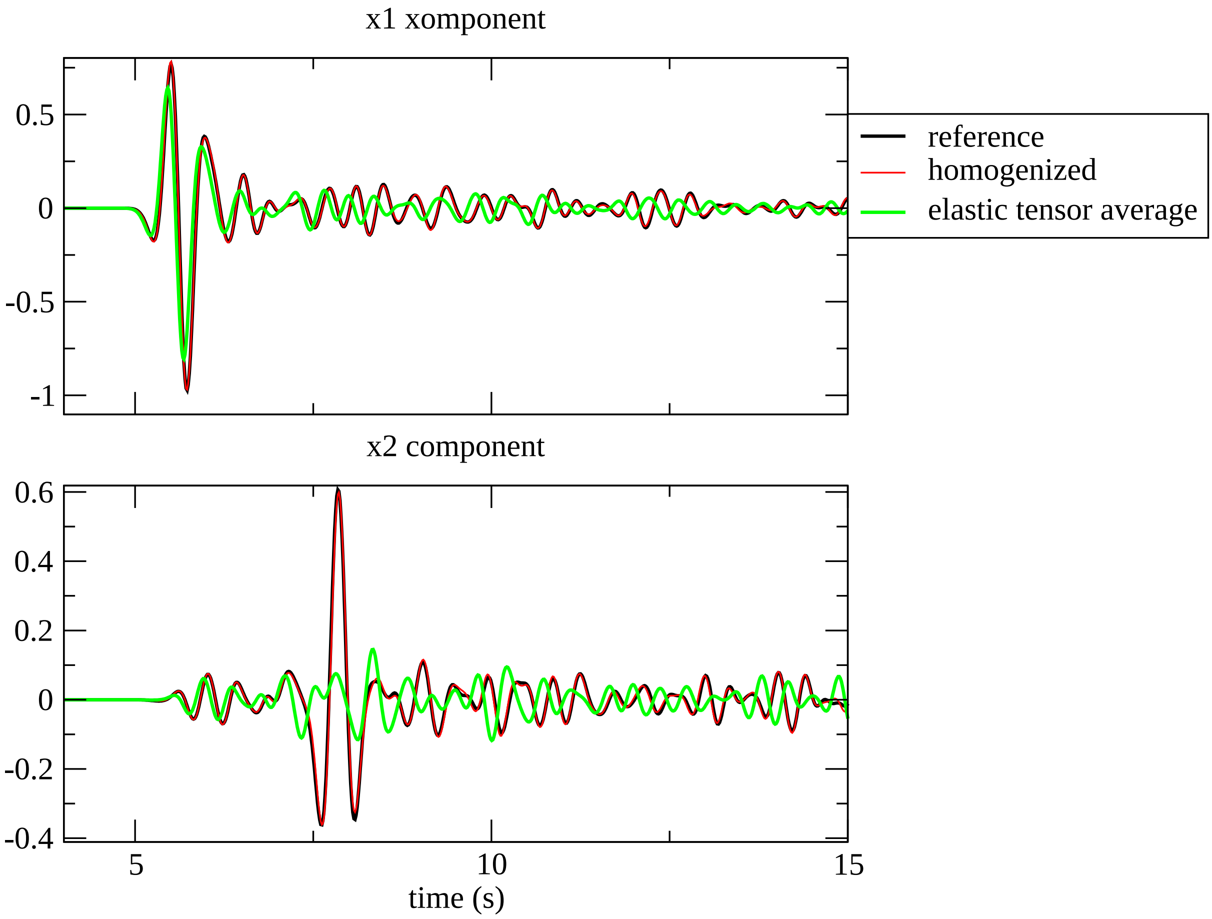


Figure 4: Traces (top: horizontal component, bottom: vertical component) for a receiver located in the middle of the model. The reference solution (black line), the homogenized solution (red line) and the "filtered" solution (green line, see text) are represented.

top of the medium, the seismograms obtained using the homogenization procedure are in very good agreement with the reference solution. The difference between those solutions and the "filtered" one is nevertheless less spectacular than for the random square example: this is due to the heterogeneities spectrum of the Marmousi2 model which has little power in the domain of the high wavenumbers.

## 3 Conclusions

We have presented a homogenization process for the wave equation allowing to up-scale 2D/3D non-periodic elastic models. This is a significant improvement of previous works which were limited to the layered non-periodic media case or to the 2D/3D periodic media case (more results can be found in [6, 7, 8]). To obtain a complete 2D/3D up-scaling tool, issues like boundary conditions in non-periodic 2-D/3-D cases (1-D case have been solved, [3, 5]) remain to be treated. This work should be useful for both forward and inverse problems.

## 4 Acknowledgments

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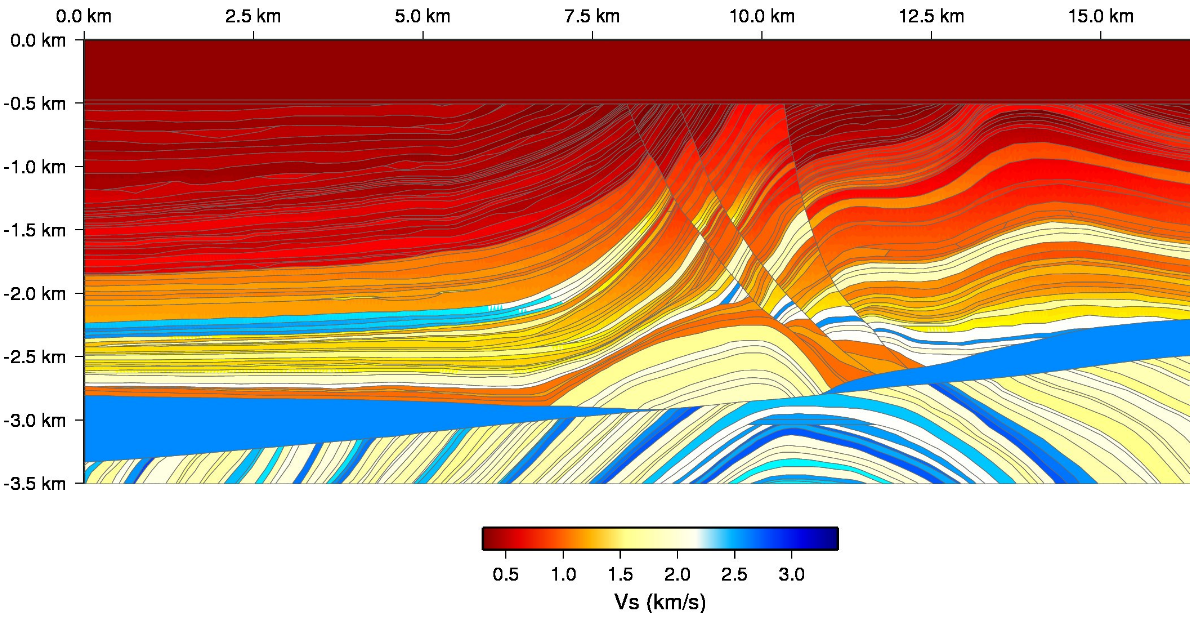


Figure 5: S velocity of the Marmousi2 model. The liquid layer has been replaced by a solid layer with slow $S$ wave velocity.

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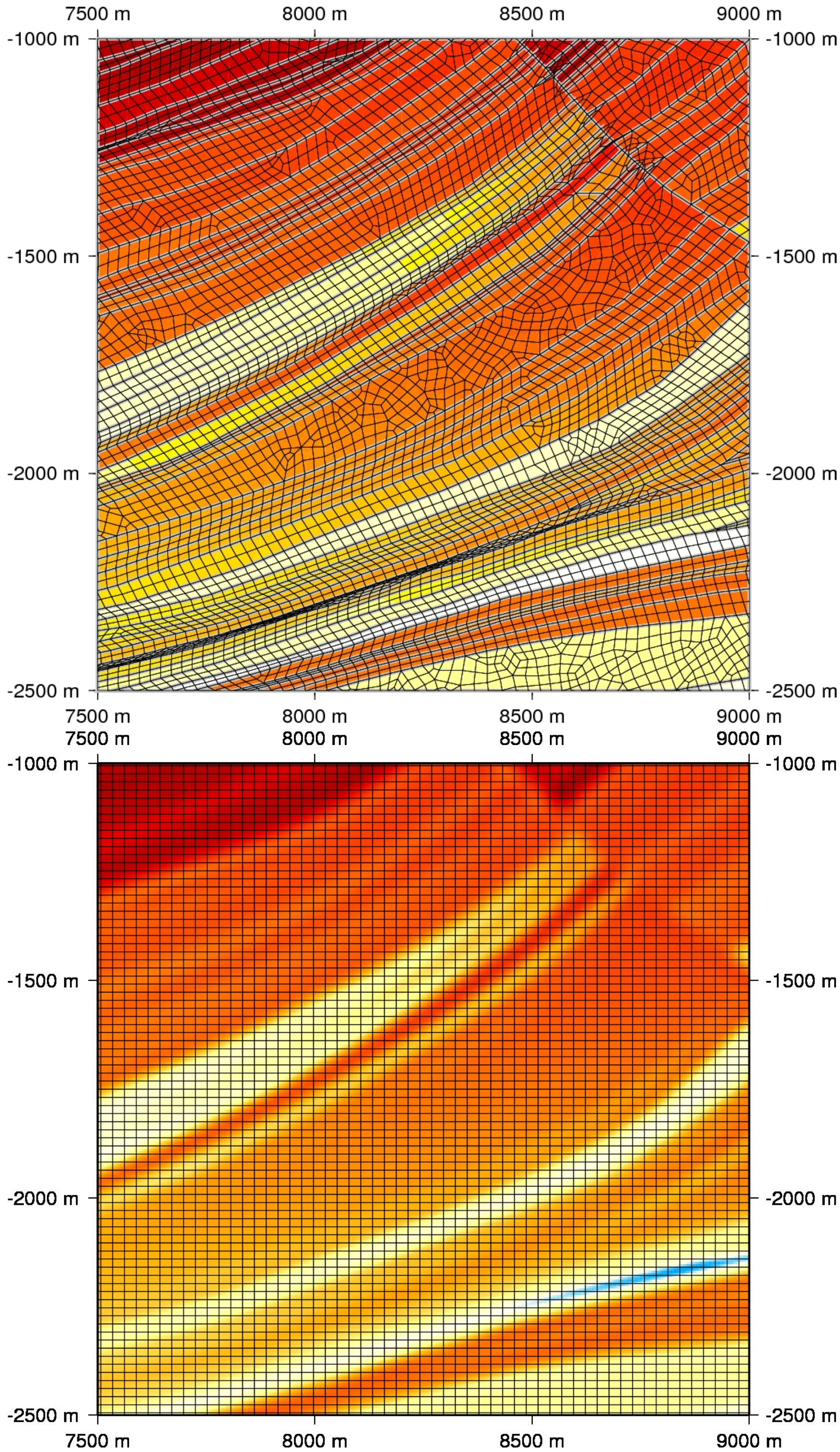


Figure 6: Sample of the SEM mesh for the original marmousi2 model (top plot) and for the homogenized Marmousi2 model (bottom plot). The back ground color is $V_s$.

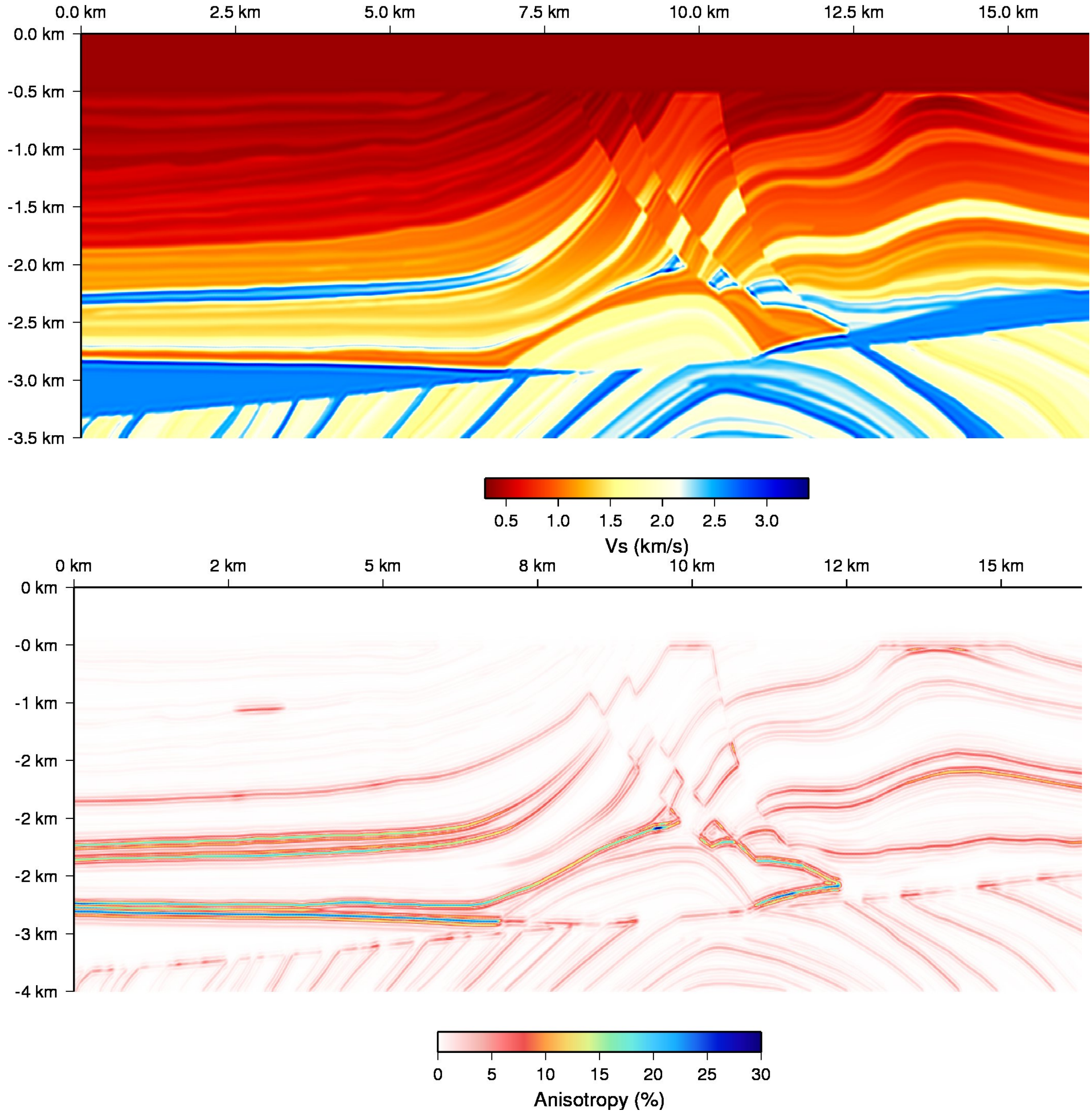


Figure 7: One of the $S$ velocity (top plot) and the total anisotropy (bottom plot) of the homogenized marmousi2 model

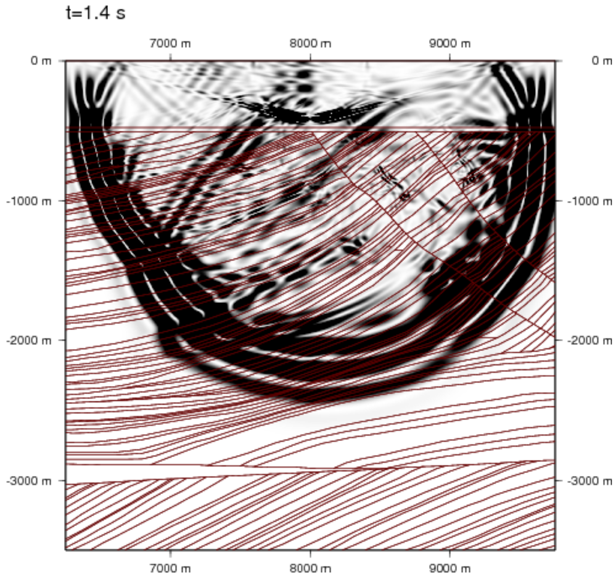


Figure 8: Snapshot of the propagating kinetic energy in marmousi2 at $t = 1.4$ s

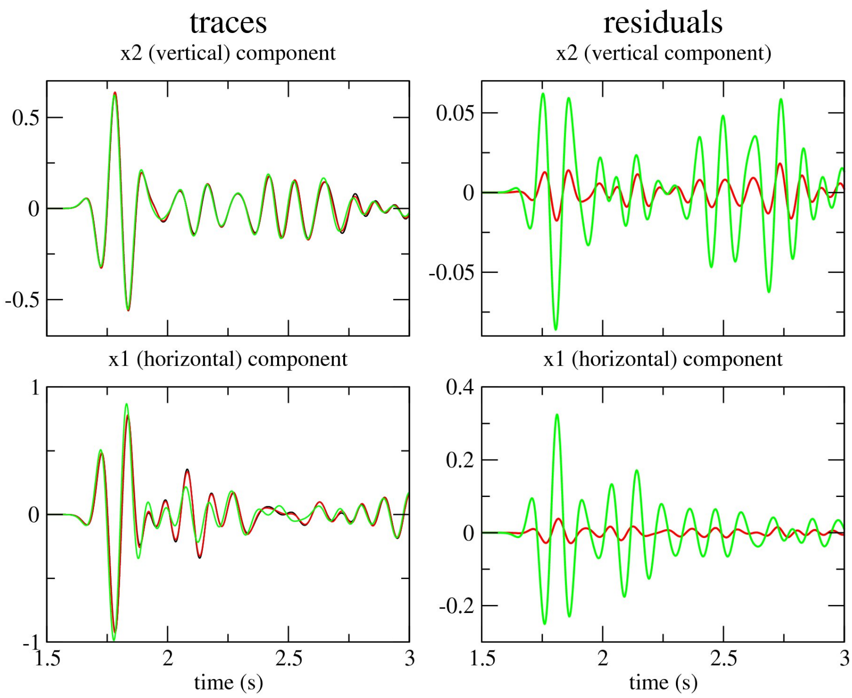


Figure 9: Traces (top: vertical component, bottom: horizontal component) for a receiver located at the middle of the model. The reference solution (black line), the homogenized solution (red line) and the "filtered" solution are represented (green line). On the right column are shown the residuals (the differences between the approximate solutions and the reference solution)