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HERMITE POLYNOMIAL CHAOS EXPANSION METHOD FOR STOCHASTIC FREQUENCY RESPONSE ESTIMATION CONSIDERING MODAL INTERMIXING

H. Yu[1], **F. Gillot**[1], **M. Ichchou**[1]

[1]Laboratoire de Tribologie et Dynamique des Systèmes
Ecole Centrale de Lyon, 36 Avenue Guy de Collongue, Ecully 69134, France
e-mail: {hang.yu, frederic.gillot, mohamed.ichchou}@ec-lyon.fr

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Abstract. *The frequency response shows a big significance in estimating the structural response of linear systems subjected to dynamic loads. Its quantification is decided by the modal properties (e.g. natural frequencies and mode shapes), which are always involved in the random eigenvalue problem regarding uncertain structures. The conventional way to determine the associated stochastic frequency response (SFR) is the directly sample-based method (DSBM), which relays on Monte Carlo sampling technique and deterministic modal analysis by FEA. However, the accuracy of the DSBM is at the cost of efficiency since a large number of modal analyses are required. In this work, the Hermite polynomial chaos expansion method (HPCEM) is implemented to solve the random eigenvalue problem. The coefficients are estimated by the least square method (LSM) considering its robustness and facility. Furthermore, the modal intermixing problem is particularly considered whereby the mathematical limitation and physical validity of the HPCEM are specified. In order to fulfill the practical requirements, a univariable based strategy is proposed to avoid this issue. This strategy in conjunction with the modal assurance criterion (MAC) provides a quantitative way to define the modal intermixing. The fast modal solutions by the HPCEM will be helpful to control the dynamic response within certain frequency interval of interest during the design phase, on one hand; on the other hand, the HPCEM can also be used to evaluate the dynamic response and thereafter to improve efficiency of reliability analysis. Numerical investigations of the SFR provide the comparisons between the results obtained by the HPCEM and DSBM, which demonstrate the efficiency and accuracy.*

1 INTRODUCTION

Uncertainties are always taken into account for modern systems since there is no adequate information to precisely specify the structural parameters, boundary conditions and excitations, the randomness of which can be described by random variables or stochastic processes. Furthermore, the model uncertainties induced from the mathematical-mechanical modeling process has to be dealt with an alternative developed by Soize [17, 18] in the light of the random matrix theory. In the context of design or maintenance, how to determine the effects of uncertainties has been paid close attention to for decades.

In practical applications, complex structures or systems are always concerned. The procedure to obtain the structural response (other quantities such as the stress and deformation are also included) relays exclusively on the finite element analysis (FEA). One of the most attractive structural response of dynamical systems is the frequency response, which can provide the information of the response within certain excitation frequency interval of interest. It will be helpful to control the amplitude of the response in the design phase [19–22], regardless of deterministic systems or random systems. In cases where the uncertainties are considered, in conjunction with Monte Carlo sampling technique, huge amounts of deterministic FE analyses have to be executed so as to quantify the statistics of the structural response. The way mentioned above to determine the associated SFR is the DSBM, which is related to MCS. However, this method will not be tractable in engineering due to computationally intensive numerical simulation.

For linear structures, the frequency response is the function of the modal properties, i.e. natural frequencies, mode shapes and modal damping. As long as the statistics of the modal properties can be decided, the randomness of the SFR are readily characterized. To gain the statistics of the modal properties, the random eigenvalue problem is usually involved. From the literatures, the perturbation method [15, 23] is a classic method to solve this problem. Nevertheless, the achieved accuracy will only be satisfied for random inputs (e.g. material properties) with small coefficients of variation (COV), say less than 5% [23]. To this end, an alternative named the HPCEM [4, 5, 15] is developed to solve the random eigenvalue problem.

In current work, the HPCEM is implemented to calculate the eigenvalues and eigenfactors defined in [11]. The LSM is implemented to determine the associated coefficients. The DSBM based on Matlab in conjunction with FE-softwares is applied to prepare the small number of samples for the LSM. Moreover, the modal intermixing [14] or modal interaction [15, 16] problem is particularly considered whereby the mathematical limitation and physical validity of the HPCEM are specified. Regarding the practical applications, it is dispensable to be avoided. However, none of the above works provide an effective way to remove this issue. In this case, a univariable based strategy is proposed, the kernel of which is to keep the random eigenvalues and eigenvetors with small variability comparing to the mean model. The consistency is delineated by the MAC (modal assurance criterion) factor. In this strategy, we can check which random variable can cause modal intermixing and then avert the modal intermixing by reducing the associated variance.

In section 2, the SFR representation of multi-degree structures is addressed. Section 3 specifies the DSBM and HPCEM to solve the random eigenvalue problem and discusses the approaches to calculate the associated coefficients. In Section 4, we state the strategy to avoid the modal intermixing problem. To show the efficiency and accuracy, in Section 5, two numerical examples are investigated, which are demonstrated by comparing the results obtained by the DSBM. At last, we summarize the main findings of this work in Section 6.

2 SFR REPRESENTATION

The SFR of a linear multi-degree structure can be calculated by the modal superposition principle which is written as:

$$d_{rk}(\boldsymbol{\theta},\omega)=\sum_{j}^{N\ll n}\frac{\phi_{rj}(\boldsymbol{\theta})\phi_{kj}(\boldsymbol{\theta})\omega^{l}}{\omega_{j}^{2}(\boldsymbol{\theta})+i\cdot 2\zeta_{j}(\boldsymbol{\theta})\omega_{j}(\boldsymbol{\theta})\omega-\omega^{2}}, \tag{1}$$

where $d_{rk}(\boldsymbol{\theta},\omega)$ denotes the SFR at the rth degree of freedom (DOF) under the excitation at the kth DOF; $\boldsymbol{\theta}$ indicates the set of random structural parameters; ω is the excitation frequency; $\omega_j(\boldsymbol{\theta})$ is associated with the jth random natural frequency, i.e. $\lambda_j=\omega_j^2(\boldsymbol{\theta})$ is the associated random eigenvalue; $\phi_{kj}(\boldsymbol{\theta})$ represents the jth random eigenvector (mode shape) at the kth DOF; $l=0,1,2$ represent the displacement, velocity and acceleration SFR, respectively; n is the number of the DOF, while N is the number of the modes of interest.

High non-linearity has been observed in Eq. (1) so that no simple linearly approximate relation between the SFR and random parameters can be found. To this end, more attempts to solve the random eigenvalue problem have been made since the SFR is expressed explicitly by the natural frequencies and mode shapes.

3 RANDOM EIGENVALUE PRROBLEM

3.1 DSBM

Regarding the random eigenvalue problem, the simply numerical method is the DSBM. In this work, the general process is realized by Matlab and FE softwares shown in Fig.1. Matlab takes charge of generating random variables (random inputs) and FE softwares provide modal solutions. The advantages of this method are those:

- Matlab can offer commonly used random variables based on Monte Carlo sampling technique. That means we can get almost any unimodal distributed random variable of interest efficiently and accurately.
- The modal analysis is the basic module of most FE softwares. No matter how complex and large the structure is, modal solutions of FE softwares could be reasonably precise.
- There is no assumption during the whole process. Hence, all the results are accurate which can be treated as calibrations.

Because FE softwares are usually the deterministic solvers, each time they can only carry out one modal analysis associated with one set of random variables. It is obvious that, in order to gain a large number of modal solutions, the process must be repeated for many times shown in Fig. 1. As a result, this method is computationally expensive, which is not practical in engineering.

3.2 Approximate sample-based method

3.2.1 HPCEM

The generalized random eigenvalue problem of undamped multi-degree structures is given by:

$$[K(\boldsymbol{\theta})]\{\phi(\boldsymbol{\theta})\}=\lambda(\boldsymbol{\theta})[M(\boldsymbol{\theta})]\{\phi(\boldsymbol{\theta})\}, \tag{2}$$

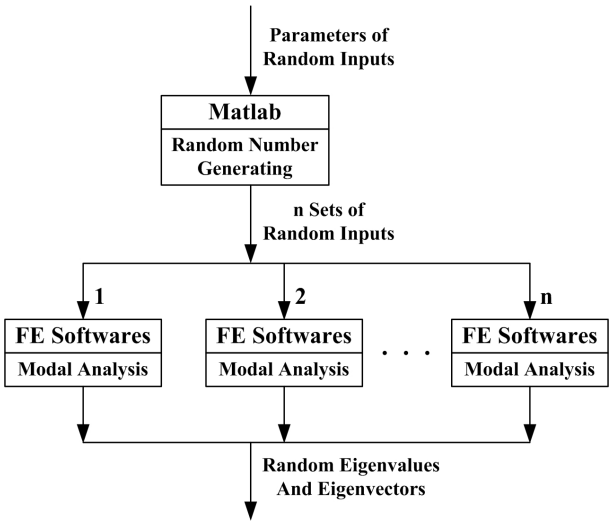


Figure 1: DSBM

where $[K(\boldsymbol{\theta})]$ and $[M(\boldsymbol{\theta})]$ are random stiffness and mass matrices; $\lambda(\boldsymbol{\theta})$ and $\{\phi(\boldsymbol{\theta})\}$ represent random eigenvalue and eigenvector respectively. Solving a random eigenvalue problem, the statistics of the eigenvalue and eigenvector must be characterized by the randomness of the structures, i.e the random inputs $\boldsymbol{\theta}$. Based on the theorem of Gameron-Martin [6], a square integrable random variable can be represented by HPC with an acceptable convergence. In case that the convergence rate is substantially slow, extensions to other orthogonal polynomials (named Wiener-Askey polynomial chaos) representation could be used according to the work [1,4,5]. Furthermore, Roger Ghanem et al [2] used HPC to express the stiffness matrix, random eigenvalues and eigenvectors to solve the random eigenvalue problem (see Eq. (2)). Inspired by these works, the jth random eigenvalue and eigenvector can reasonably be formed as:

$$\lambda_j = \sum_{p=0}^{\infty} a_p H_p(\boldsymbol{\xi}), \qquad \{\phi\}_j = \sum_{p=0}^{\infty} b_p H_p(\boldsymbol{\xi}), \tag{3}$$

where $H_p(\boldsymbol{\xi})$ is the multidimensional HPC of order p in standard normal space in which $\boldsymbol{\xi}$ represents the standard normal variables, a_p and b_p are the constant coefficients. HPC has the properties:

$$H_0 \equiv 1, \qquad \langle H_p = 0 \rangle, \qquad p > 0, \qquad \langle H_p H_q \rangle = \delta_{pq} \left\langle H_p^2 \right\rangle, \tag{4}$$

where δ_{pq} is the Kronecker delta and $\langle \cdot, \cdot \rangle$ denotes the ensemble average. This is the inner product in the Hilbert space of Gaussian random variables defined by:

$$\langle f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) \rangle = \int f(\xi) g(\boldsymbol{\xi}) W(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{5}$$

The weight $W(\boldsymbol{\xi})$ function is multidimensional Gaussian joint probability density function. Notice that Hermite polynomials are functions of standard normal random variables $\boldsymbol{\xi}$. In the original random eigenvalue problem (see Eq. (2)), random solutions are determined by random variables $\boldsymbol{\theta}$. $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$ are usually not the same. Nonetheless, implementation of transformation techniques, such as Rossenblatt transformation [7] and Nataf model in [8], the random variables $\boldsymbol{\theta}$ can be transformed to standard normal random variables $\boldsymbol{\xi}$ with one to one mapping: $\theta_i \rightarrow \xi_i$. Then the Eq. (2) will be rewritten as:

$$[K(\boldsymbol{\xi})]\{\phi(\boldsymbol{\xi})\} = \lambda(\boldsymbol{\xi})[M(\boldsymbol{\xi})]\{\phi(\boldsymbol{\xi})\}, \tag{6}$$

Some analytical transformations between random variables of common univariate distributions and standard normal variables have been listed in [9, 10]. From Eq. (1), the SFR is the function of natural frequencies ω_j, and the eigenfactor defined as $\phi_{rj}\phi_{kj}$ [11]. Debusschere et al [12] indicated that multiplication of two random variables expanded by polynomial chaos can also be expanded by the same polynomial chaos. This proposition can save the CPU time. Moreover, the elements ϕ_{rj} and ϕ_{kj} from the same eigenvector which insinuates that they have the same converged rate. Consequently, we can use the HPC expansion to represent the eigenfactor directly:

$$\phi_{rj}\phi_{kj} = \sum_{p=0}^{\infty} c_p H_p(\boldsymbol{\xi}), \tag{7}$$

3.2.2 Coefficients calculation

When implementing HPC, one of the critical issues is to accurately estimate the coefficients. In the expressions of random eigenvalues and eigenfactors (See Eq. (3) and Eq. (7)), the number of the coefficients is infinite. This formula is not tractable in practice. Generally, the truncation formula is usually used. The representations of eigenvalues and eigenvectors are rewritten as:

$$\lambda_j = \sum_{p=0}^{P} a_p H_p(\boldsymbol{\xi}), \qquad \phi_{rj}\phi_{kj} = \sum_{p=0}^{P} c_p H_p(\boldsymbol{\xi}), \tag{8}$$

The total number of polynomial chaos is $N_{pc} = P + 1$ which is determined by the dimension N_{rv} of random variables $\boldsymbol{\xi}$ and the highest order N_{ho} of the HPC.

$$N_{pc} = \frac{(N_{rv} + N_{ho})!}{N_{rv}! N_{ho}!}, \tag{9}$$

Several methods can be exploited to compute the coefficients:

- Orthogonalized method [2]. This method makes use of the orthogonality of the polynomial chaos basis. Taking the inner product in Eq. (8) with H_p, we have

$$a_p = \frac{< \lambda_j H_p >}{< H_p^2 >}, \qquad c_p = \frac{< \phi_{rj}\phi_{kj} H_p >}{< H_p^2 >}, \tag{10}$$

  The denominators in the above expressions have been calculated in [3]. However, the numerators are usually evaluated by MCS. To keep the coefficients accurate, a large quantity of samples are needed. In that case, we prefer the DSBM since the orthogonalized method takes no advantage of economy.

- Probabilistic collocation method [9, 13]. As the name defined, the collocation points are specially selected which correspond to the roots of the Hermite polynomial of one degree higher than the maximum order of current HPC expansion. With the increase of the number of random inputs and the order of the expansion, the number of available collocation points increasing exponentially. For example, in Eq. (9), for the case $N_{rv} = 5, 6$ and $N_{ho} = 2, 3$, the number of the collocation is $(N_{ho} + 1)^{N_{rv}} = 243, 4096$, whereas $N_{pc} = 21, 84$ respectively. As long as the selected number of the collocation points is equal to the number of HPC expansion terms, we could obtain the coefficients by solving a set of equations. However, different combination of collocation points may result in different coefficients. This imposes the instability of the collocation method.

- Least square method (LSM). This method belongs to parametric data fitting methods which will be representative for the whole sample population rather than precisely fitting each sample point. From this point of view, as long as the HPC expansion model can contain all information of random outputs, least square method is the first choice thanks to its facilities and efficiency. Let$\{a\} = [a_1, ...a_p, ...a_{N_{pc}}]^T$, $\{c\} = [c_1, ...c_p, ...c_{N_{pc}}]^T$ and $[H] = \{H_1, ..., H_p, ..., H_{N_{pc}}\}$. Consequently, the least square solutions of the coefficients with respect to the j^{th} set of random eigenvalues and eigenfactors are

$$\{a\} = ([H]^T[H])^{-1}[H]^T\{\lambda_j\}, \qquad \{c\} = ([H]^T[H])^{-1}[H]^T\{\phi_{rj}\phi_{kj}\} \tag{11}$$

  Here $\{\lambda_j\}$ and $\{\phi_{rj}\phi_{kj}\}$ are the m-dimension vectors, in which m denotes the number of random input samples; $[H]$ is the matrix of dimension of $m \times N_{pc}$. **For this work, LSM is utilized**. That signifies when we need M random eigenpairs, only $m \ll M$ are needed to determine the HPC representation, with which the M random eigenpairs are readily gained.

Practically, no matter what method is applied to calculate the coefficients, another critical issue is that HPC expansion can represent the random eigenvalue reasonably. In the next section, the mathematical limitation and physical validity of the HPCEM will be discussed.

4 MODAL INTERMIXING

4.1 Problem description

In random structures, the modal intermixing [14] or modal interaction [15, 16] phenomenon is usually observed when using MCS to model the uncertainties. That means, from one simulation to next, the adjacent modes may exchange whereby the random eigenvectors associated with the same order, actually, contain more than one mode. Generally, different modes behave physically different. This is exacerbated for those structures with closed space eigenvalues as the variances of random inputs increase. That is, mathematically, because the area of probability overlapping of adjacent modes become larger and larger.

Modal intermixing can strongly affect the accuracy of the HPCEM. In fact, the HPC expansion is originated to represent the solutions of random differential equations. That signifies one deterministic HPC expansion just refers to the random solutions of one family. Accordingly, for random eigenvalue problem, the accuracy of HPCEM is guaranteed by averting the modal intermixing problem. Such mathematical limitation seems to restrict wide applications of the HPCEM for random eigenvalue problem. However, this limitation has great significance in practice.

In engineering, when concerning the randomness in structures, the mean model is usually treated as a benchmark or reference. The modal behaviors of uncertain structures should be consistent with the benchmark. In that case, the modal intermixing is indispensable to avoid, the interpretation of which is that the variabilities of random eigenvalues and eigenvector should be limited in a reasonably acceptable bounds.

4.2 MAC factor

As indicated above, the variabilities of random eigenvalues and eigenvectors around the ones of mean model ought to be small in order to remove the modal intermixing issue. If this property holds, the random mode shape can be described as a linear combination [15, 19] of N mode shapes of mean model. The jth eigenvector of simulation k is formulated as:

$$\{\phi\}_j^{(k)} = \sum_{i=1}^{N} A_i^{(k)} \{\phi\}_i^{(0)}, \qquad 1 \leq j \leq N, \tag{12}$$

where N is the number of modes considered in Eq. (1); $A_i^{(k)}$ denotes the constant weight coefficient; $\{\phi\}_i^{(0)}$ represents the ith eigenvector of the mean model. When the behaviors of random models are identical with the ones of mean model, the effects of the other modes can be ignored, which is interpreted as the most contribution to the sample $\{\phi\}_j^{(k)}$ is from the jth mode of the mean model. In this situation, the last equation in the approximate formula is:

$$\{\phi\}_j^{(k)} \approx A_j^{(k)} \{\phi\}_j^{(0)}, \qquad A_j^{(k)} > 0, \tag{13}$$

Obviously, the angle between eigenvector $\{\phi\}_j^{(k)}$ and $\{\phi\}_j^{(0)}$ will meet $\alpha_j^{(k)} \approx 0$. In conjunction with Eq. (13), the direction cosine of these two eigenvectors has the property:

$$cos(\alpha_{jj}^{(k)}) = \frac{\{\phi\}_j^{(k)} \cdot \{\phi\}_j^{(0)}}{\left\|\{\phi\}_j^{(k)}\right\| \left\|\{\phi\}_j^{(0)}\right\|} \approx 1, \tag{14}$$

Here, $\{\phi\}_j^{(k)} \cdot \{\phi\}_j^{(0)}$ denotes the dot product and $\left\|\{\phi\}_j^{(0)}\right\|$ is the Euclidean norm. Eq. (14) is the condition to prevent the random models from the modal intermixing. It is actually a variant of the Modal Assurance Criterion (MAC) [24]. The corresponding MAC factor is defined by:

$$f_{MAC_{jj}}^{(k)} = \frac{(\{\phi\}_j^{(k)} \cdot \{\phi\}_j^{(0)})^2}{\left\|\{\phi\}_j^{(k)}\right\|^2 \left\|\{\phi\}_j^{(0)}\right\|^2} = cos^2(\alpha_{jj}^{(k)}) \approx 1 \tag{15}$$

The MAC is the criterion to check the consistency between two modes. When the value approaches unity, the consistency is well observed; on the contrary, the value is smaller than 1, the behaviors show the violation. The modal intermixing is the right phenomenon from one simulation to another that reflects the violated modes against the mean modes. Therefore, based on Eq. (15), the MAC factor is an indicator of the modal intermixing:

$$\begin{array}{ll} f_{MAC_{jj}}^{(k)} \approx 1, & \text{no modal intermixing} \\ f_{MAC_{jj}}^{(k)} < 1, & \text{modal intermixing} \end{array} \tag{16}$$

4.3 Univariable based strategy

The randomness in modal properties is induced from the random inputs. We have to note that the information of underlying physical models is usually insufficient so that uncertainties are always assumed to follow certain distribution. Mean values can be obtained readily according to the knowledge of the problems right in hand. On the contrary, it is difficult to get the precise variance since the number of sampling tests is practically small. For that reason, by means of simulation techniques, we need to carefully select the proper variance to fulfill the practical requirements, i.e. no modal intermixing.

So as to select the proper variance, a univariable based strategy is proposed to check which random input can cause the modal intermixing and determine what variance should be used. In this strategy, only one parameter is treated as random variable; others are deterministic. Special random inputs around the mean value are picked out to capture the associated modal solutions.

Let μ_θ and σ_θ be the mean and standard deviation (SD) of random variable θ. Then, the random inputs are written as:

$$\{\tilde{\theta}\} = [\mu_\theta \pm 2\sigma_\theta \quad \mu_\theta \pm 3\sigma_\theta \quad \mu_\theta \pm 4\sigma_\theta \quad \mu_\theta \pm 5\sigma_\theta]^T; \tag{17}$$

where $\{\tilde{\theta}\}$ are the selected random inputs of dimension 8. The last four elements of $\{\tilde{\theta}\}$ suggest the very rare realizations. If all the associated eigenvectors especially the ones corresponding the rare random inputs can meet the condition in Eq. (15), σ_θ is the proper SD; otherwise, a smaller SD will be used to check until the condition is satisfied. An infeasible variance is always chosen as the initial one so that the final choice will be as large as possible. Because only 8 modal analyses are required for each candidate variance, regardless of simple systems or complex systems, this strategy will be computationally efficient. Furthermore, this strategy is insensitive with distributions of random inputs since the special random inputs are determined by the mean and SD but the types of distributions. For those reasons, this strategy is very powerful to avoid the modal intermixing.

5 NUMERICAL EXAMPLES

In the following, two models are investigated to demonstrate the efficiency and accuracy of this procedure to determine the statistics of the SFR. Random structural parameters are modeled by normal, lognormal and Gamma distribution respectively for both examples.

Due to the similarity of the results, only the cases associated with normal distribution are specified. In Section 5.1, a 2-D bridge-like structure: simply supported beam is regarded. For this model, no modal intermixing problem is taken into consideration as it belongs to the structure with largely-spaced frequencies. A building-like structure: fixed support plate-beam is investigated in Section 5.2. The modal intermixing problem must be considered due to its high symmetry.

5.1 Bridge-like structure: simply supported beam

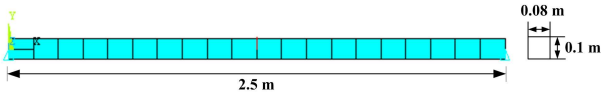


Figure 2: Beam model

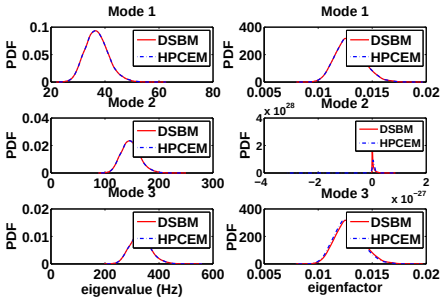
A 2-D simply supported beam shown in Fig. 2. The first three modes have been used to estimate the SFR. The corresponding natural frequencies of the mean model are $\omega_1 = 321.0$ rad/s, $\omega_2 = 921.9$ rad/s and $\omega_3 = 2065.9$ rad/s. Tow deterministic parameters are: Poisson's ratio $\nu = 0.3$, modal damping ratio coefficient $\zeta = 0.05$; the other parameters are independent random variables, the mean values of which are $\rho = 7.8 \times 10^3\ kg/m^3$, $E = 2.0 \times 10^{11}$ Pa, $L = 2.5$ m, $H = 0.1$ m, $W = 0.08$ m for density, Young's modulus, length, height and width respectively. Two levels of coefficients of variance (COV), $COV = 5\%$ and $COV = 10\%$, are considered associated with each distribution. All following results are related to the node in the middle.

5.1.1 PDFs investigations

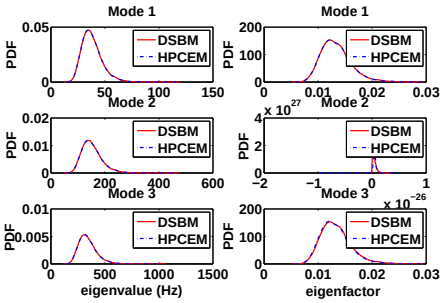
Consider normal distribution. Fig. 3(a) and Fig. 3(b) show the probability density functions (PDFs) of eigenvalues and eigenfactors for the first three modes. The perfect consistence be-

tween the distributions obtained by HPCEM and DSBM implies that HPCEM can reflect the statistics characteristics correctly. The largest error of the PDF estimation occurs for the second eigenfactor. However, it is emphasized that the magnitude is very small which contributes almost zero to the SFR. Therefore, such bad approximation of Mode 2 can be ignored. Moreover, larger COVs of eigenvalues and eigenfactors are observed as to the random inputs with large COVs. The approximate method such as perturbation method always has difficulty in dealing with larger COV problems, while the HPCEM can provide the acceptable results. Therefore, the HPCEM is a powerful tool to solve such problems.

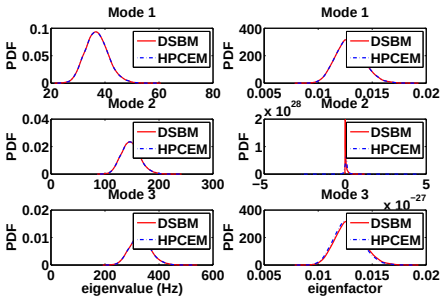
It is revealed that the choice of the current order of HPC seems not sufficient since differences exist. Note that increasing the order of HPC is profitable in improving the accuracy. But we have to pay attention that higher order HPC signifies the more computational expense. Accordingly, there should be a trade-off between the efficiency and accuracy. The similar results of PDFs can be seen from the rest four figures in Fig. 3 corresponding the other two distributions.



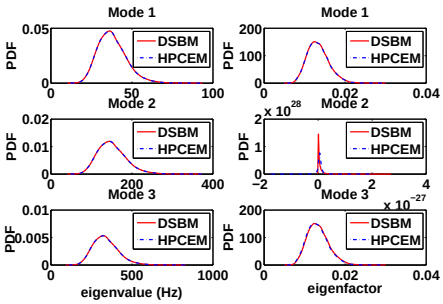
(a) Normal: $COV = 5\%$



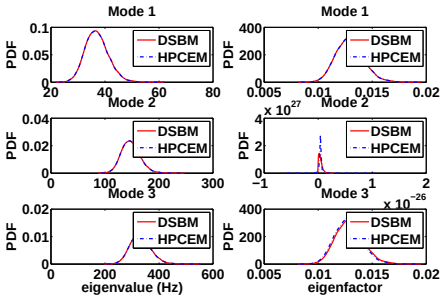
(b) Normal: $COV = 10\%$



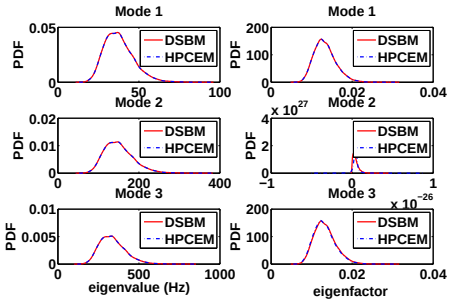
(c) Lognormal: $COV = 5\%$



(d) Lognormal: $COV = 10\%$



(e) Gamma: $COV = 5\%$

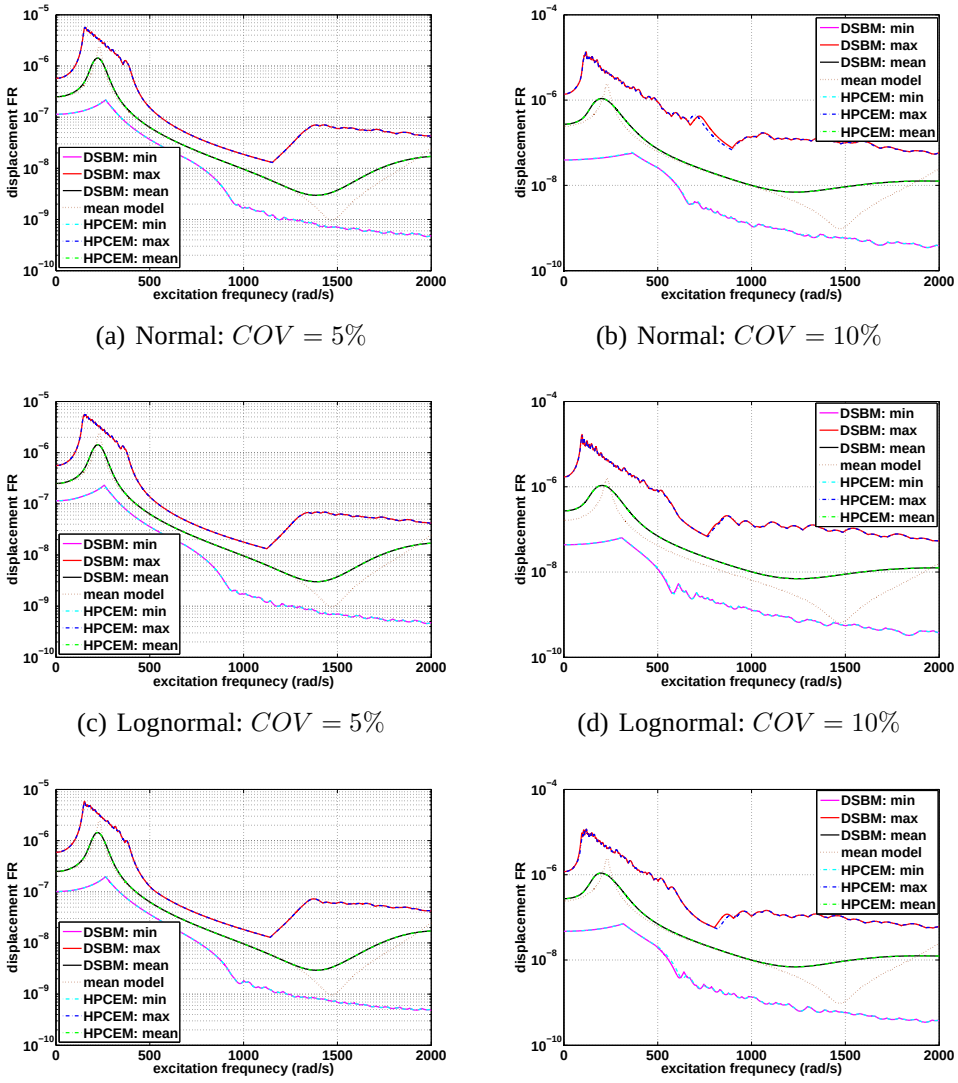


(f) Gamma: $COV = 10\%$

Figure 3: PDF of eigenvalues and eigenfactors based on 10000 samples

5.1.2 SFR investigations

Fig. 4 shows a comparison of the statistics of the SFR between the DSBM and HPCEM, in which the maximum, minimum, mean value of the SFR are very close to each other, respectively. The differences are observed for larger COV (see Fig. 4(b), 4(d) and 4(f)). However, the mean value comparisons represent little difference, which is very significant for reliability analysis. For smaller COV, the maximum is smaller.



(a) Normal: $COV = 5\%$ (b) Normal: $COV = 10\%$

(c) Lognormal: $COV = 5\%$ (d) Lognormal: $COV = 10\%$

(e) Gamma: $COV = 5\%$ (f) Gamma: $COV = 10\%$

Figure 4: SFR based on 10000 samples for beam

The fourth-order HPC expansion are utilized to represent the eigenvalues and eigenfactors for all considered distributions with both COV levels. According to Eq. (9), the number of coefficients is 126. After numerical examinations, 400 samples are chosen to calculate these coefficients, which is more or less 3 times the coefficients.

In the following, a comparison of average computational time will be specified. So as to obtain the results in Fig. 4, 10^4 modal analyses are executed in the context of DSBM. Yet, for the HPCEM, only 400 modal analyses are carried out by DSBM and plus 10s to determine the

coefficients and approximate 10^4 sets of eigenpairs. In conclusion, HPCEM only needs about 4% CPU time of the one needed by DSBM. Efficiency is strongly improved.

It is evident that the HPCEM could be a general method to deal with eigenvalue problem since good results are obtained for the normal, lognormal and Gamma distribution respectively. From the above, HPCEM has the capacities to estimate the SFR effectively, which is of big value concerning reliability analysis since it can save a lot of time from generating a large quantity of samples. Moreover, for structures with larger COV, it turns out that HPCEM is a powerful tool. We also find that the LSM works well in estimating the coefficients of HPC model.

5.2 Building-like structure: plate-beam

In this example, four parameters are selected as random inputs. They are: density, mean value $\mu_\rho = 7.8 \times 10^3$, $COV = 10\%$; Young's modulus, mean value $\mu_E = 2.0 \times 10^{11}$, $COV = 10\%$; beam length, mean value $\mu_{BL} = 3$, $COV = 1.75\%$; plate thickness, mean value $\mu_{PT} = 0.02$, $COV = 2\%$. The cases with random inputs following lognormal, normal and Gamma distribution are studied respectively. More information about this model are as followed: Poisson's ratio $\nu = 0.3$, modal damping ratio coefficient $\zeta = 0.01$, the height of beam is 0.04, the width of beam is 0.04 and the length of plate (square) is 5. All the unites belong to SI. Ten modes are used to calculated the displacement response. The natural frequencies of the mean model are $\omega_1 = 1.30$ rad/s, $\omega_2 = 1.31$, $\omega_3 = 1.87$, $\omega_4 = 3.99$ rad/s, $\omega_5 = 4.01$ rad/s, $\omega_6 = 5.78$ rad/s, $\omega_7 = 6.48$ rad/s, $\omega_8 = 6.50$ rad/s, $\omega_9 = 9.56$ rad/s and $\omega_{10} = 10.86$ rad/s. 294 nodes are in the FE model which is shown in Fig. 5. The results are related to node 14 in direction X. Because of high symmetry in this model, modal intermixing must be considered. The COVs mentioned above are the proper ones selected by the univariable based strategy.

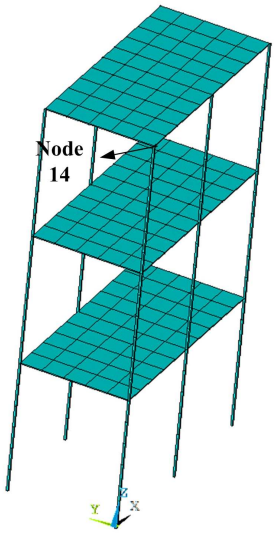
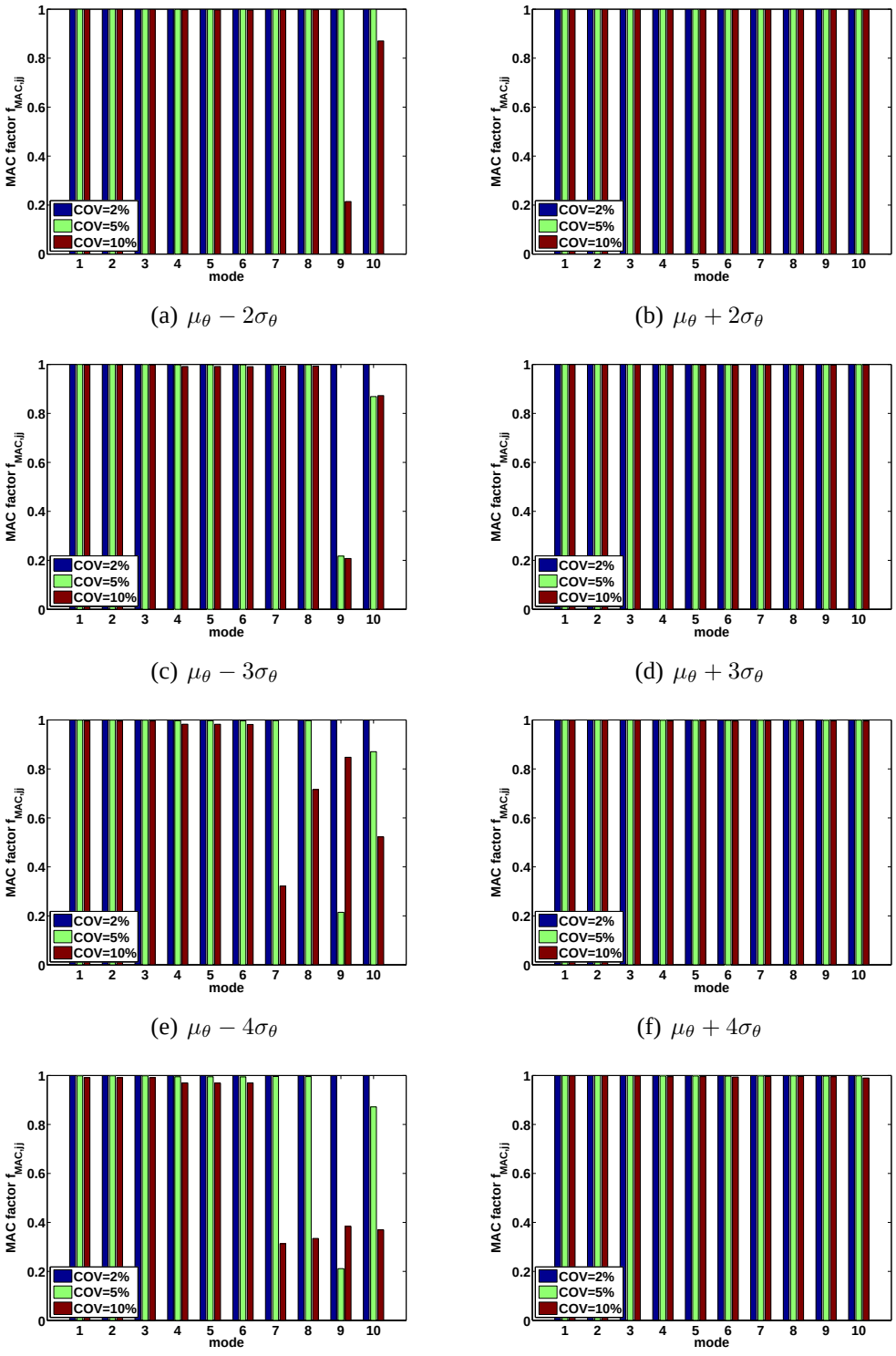


Figure 5: Plate-Beam model

5.2.1 Modal intermixing

Take the random parameter plate thickness as the example. Fig. 6 schematically describes the MAC factors with respect to 8 special random inputs. It is found that the modal intermixing always concerns the random model whose plate thickness is less than the mean value. The

explanation is that as the thickness decreases, the flexibility increases since the plate belongs to thin walled structures. The associated structural behaviors can be different from the reference (mean model) more easily. Yet, as to thicker plates, the modes have a large probability to agree with the ones of the mean model.



(a) $\mu_\theta - 2\sigma_\theta$ (b) $\mu_\theta + 2\sigma_\theta$

(c) $\mu_\theta - 3\sigma_\theta$ (d) $\mu_\theta + 3\sigma_\theta$

(e) $\mu_\theta - 4\sigma_\theta$ (f) $\mu_\theta + 4\sigma_\theta$

(g) $\mu_\theta - 5\sigma_\theta$ (h) $\mu_\theta + 5\sigma_\theta$

Figure 6: MAC factor comparison for 8 special random inputs

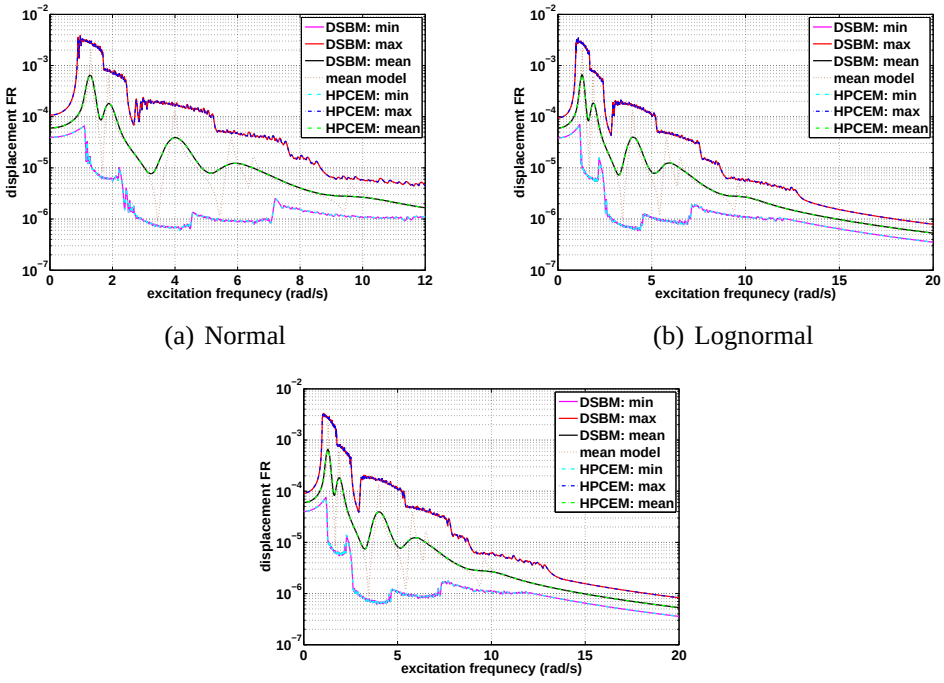
From Fig. 6, it is observed that the modal intermixing always occurs with respect to higher modes, i.e. mode 7, 8, 9 and 10. Moreover, the larger the variance is, the more probable the modal intermixing is induced. To that end, it is reasonable to avert the modal intermixing by reducing the variance.

It is also discovered that when we increase the COVs of density and Young's modulus to 30%, no modal intermixing occurs, the MAC factors of which is always similar to Fig. 6(b). In that case, the random inputs like them can be seen as the insensitive parameters; On the contrary, in cases where random inputs with large COV causing the modal intermixing can be defined as the sensitive parameters, e.g. plate thickness.

5.2.2 Cases study with different distributions

As the probabilistic information of the random inputs are priory unknown, one can choose any distribution to describe the randomness as long as the engineering requirements are fulfilled. From this point of view, like what we have done in Section 5.1, different distribution assumptions are applied so that the universality of the HPCEM is revealed again.

The accuracy investigation is shown by the comparison of the frequency response in Fig. 7 for all the three distributions. Probabilistic results (mean) obtained by HPCEM and DSBM agree with each other well. It is observed that only 6 peaks of the mean model can be told off. That implies there exists very close frequencies.



(a) Normal (b) Lognormal

(c) Gamma

Figure 7: SFR based on 10000 samples of Plate-Beam

For this model, the fourth-order HPC expansion is utilized. Only 200 samples are needed to calculate the 70 coefficients by LSM. We have to note that the number of samples is again about 3 times of that of the coefficients. Considering the situations in Section 5.1, we recommend that the number of the samples equals more or less 3 times the number of the coefficients in order to determine the HPC model. The efficiency is still shown by the average CPU time. The DSBM

needs 10^4 modal analyses, while the HPCEM only requires 200 modal analyses plus 10s to gain 10^4 sets of eigenvalues and eigenfactors. Almost 98% CPU time is saved.

6 CONCLUSIONS

This work concentrates on estimating the statistics of the SFR. The HPCEM is applied to solve the involved random eigenvalue problem, the coefficients are evaluated by the LSM. The mathematical limitation and physical validity are specially discussed by considering the modal intermixing problem, which is always encountered when using Monte Carlo sampling technique. The univariable based strategy with conjunction with the MAC is proposed to avoid the modal intermixing problem by reducing the variances of the random inputs. It is helpful to select the proper variances since no adequate information is a priori in the physical models. Comparing with the DSBM, the HPCEM is an efficient method to obtain random eigen-solutions. Several factors have significant influences on the accuracy and efficiency of HPCEM:

- The modal intermixing problem should be avoided. In cases where this problem exists, the physical behaviors associated with individual random model may differ from the mean model (benchmark) and the HPCEM in not acceptable to solve the random eigenvalue problem.
- The proper number of samples used for LSM for the HPC coefficients estimation ought to be proper. In this work, according to the simulation examinations, the number of the samples is recommended as about 3 times the number of the coefficients.
- The dimension of the underlying physical (the number of DOF) systems has a large influence of the efficiency. The larger a structure is, the more time the modal analysis needs.
- The efficiency is highly improved compared with the DSBM. However, as the number of random inputs increase,
  - the dimension of the HPC expansion increase. It signifies more time are needed to construct the HPC model which is always obtained by the expensive symbolic calculation.
  - more coefficients have to be evaluated by the LSM. In that case, more samples have to be prepared for the least square solution.

Note that the HPCEM can effectively obtain the variability of the random eigenvalues and eigenfactors, and thereafter the SFR. This achievement is very valuable for reliability analysis as well as reliability based design optimization since reliability analysis has to be carried out during each function evaluation of the optimization procedure.

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