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AN EFFICIENT USE OF THE SYMBOLIC SPLINE-BASED DIFFERENTIAL QUADRATURE METHOD IN VIBRATION ANALYSIS OF SHELLS

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Abstract. *The paper presents the differential quadrature method (DQM) based on a modified spline interpolation and the application of the method in vibration analysis of laminated, composite shells. The goal of the modification of the spline interpolation is to improve the rate of convergence and preserve the stability of the method. The modification changes the definition of the end conditions for the spline interpolation. Two types of the end conditions are combined and appropriately applied at the stage of the determination of the weighting coefficients for the DQM. With the aid of the symbolic computation the weighting coefficients can be successfully determined for any spline degree. The efficiency of the method is examined by the example of the free vibration of composite, conical shells. The influence of the modified end conditions, number of nodes and spline degree on the convergence and accuracy is studied. The achieved results are compared with the results obtained using conventional DQM and using other numerical techniques.*

1 INTRODUCTION

More and more advanced mechanical structures as well as new construction materials require to develop computational methods that allow efficiently to estimate mechanical properties. Some of the mechanical structures can be efficiently analyzed by analytical methods but usually numerical methods give more versatility. Among the latter, the finite element method seems to be the most common and it is applied to many challenging problems. This method belongs to low order numerical techniques, what means that accurate numerical results can be achieved using large number of nodes. It requires much computational effort. In the case of eigenvalue problem, where the number of nodes corresponds to the number of obtained eigenpairs, only the lowest eigenpairs are of the much interest in mechanical problems. However, to compute these eigenpairs with acceptable accuracy using low order techniques, one has to aggregate and solve large sets of equations, unlike applying methods that use high order approximation. One of them is differential quadrature method (DQM) [1], which allows to obtain very accurate results using few discrete points. This feature as well as simple formulation cause the DQM to be applied in many fields of mathematics and mechanics [2].

However, the DQM has some drawbacks resulting from the assumed approximation of the sought solution. One of these drawbacks is computational instability. To overcome the problem, new approaches have been developed. In one of them, only some neighbouring nodes are used to approximate a derivative [3]. In others, variable order approach is applied [4] or B-Spline interpolation is used [5,6] instead of conventional polynomial. The latest advances in the DQM are presented in [7].

In [8] another idea to improve the stability of the method is proposed. In this idea, spline functions, considered as *n* degree polynomials, defined separately in each subinterval of the domain are used. In this manner, the solution is approximated. The developed algorithm requires to use symbolic manipulations at a initial step of computation, therefore in present paper the method is referred to Symbolic Spline-based Differential Quadrature Method (SSDQM). This algorithm allows to determine the weighting coefficients for the derivative approximation in the DQM with the use of any spline degree, unlike the technique that uses B-Spline functions [5,6]. In the latter, the formulas based on quintic and sextic B-Splines are developed so far. Obtained results indicate that the SSDQM provides a balance between low order methods and high order ones. It is characterized by less rate of convergence then the conventional DQM, but ensures computational stability, what is clearly seen in dynamic problems [9]. The SSDQM gives also reasonable results using uniform point distribution, what is usually impossible in the case of the conventional DQM. Accurate results are also obtained for problems, where the DQM fails [10].

In order to improve the rate of convergence of the SSDQM a modification is proposed in this paper. This modification changes the formulation of the end conditions for the spline interpolation. The improvement is shown by the example of the free vibration of laminated, orthotropic, conical shells.

The layout of the paper is as follows. In section 2, the short description of the DQM and the SSDQM is presented. In the latter, the way of the determination of the weighting coefficients is shown. In section 3, the approach to improve the SSDQM is proposed. In section 4, the presented idea is tested on the example of the free vibration analysis of the conical shells. On the basis of the results some concluding remarks on the convergence and accuracy are drawn.

2 SYMBOLIC SPLINE-BASED DIFFERENTIAL QUADRATURE METHOD

The basic idea of the DQM lies in the fact that the spatial derivative of a function at a given point is approximated by a linear weighted sum of the function values at all discrete points in the domain along the coordinate lines. It can be put as follows

$$\frac{d^r f(x)}{dx^r}\Big|_{x=x_i} = \sum_{j=1}^{N} a_j^{(r)}(x_i) f(x_j) = \sum_{j=1}^{N} a_{ij}^{(r)} f_j \quad i=1,\dots,N \tag{1}$$

where N denotes the number of grid points and $a_{ij}^{(r)}$ are the weighting coefficients of the rth order derivative.

In the case of the multidimensional problem, the derivatives with respect to other spatial variables are approximated in the similar manner and the approximation of the mixed derivatives is done with the use of the weighting coefficients determined separately for derivatives with respect to appropriate variables.

Applying the governing equation with the derivatives described by Eq. (1) at each interior point of the domain and implementing boundary conditions, one obtains a set of algebraic equations. A key stage of the method is to determine the weighting coefficients. These coefficients depend on the way the sought solution is approximated. Therefore, they influence the convergence, accuracy and stability of the method.

In the conventional approach, the solution is approximated by the interpolation polynomial. The use of the Lagrange base functions and the recurrence relationships [11] allows efficiently to determine the weighting coefficients in this approach. The method provides very high rate of convergence and accuracy, when appropriate point distribution is applied. When the number of points is too large or the points are uniformly distributed then the method fails.

2.1 Spline interpolation in the DQM

The definition of the interpolation function depends on whether the spline degree n is odd or even. If the spline degree is odd then the interpolation function has the following form

$$f(x) \approx \left\{ s_i(x),\, x \in [x_i, x_{i+1}],\ i=1,\dots,N-1 \right\} \tag{2}$$

where N is the number of nodes and the ith spline section $s_i(x)$ can be written as

$$s_i(x) = \sum_{j=0}^{n} c_{ij}(x-x_i)^j \tag{3}$$

In order to determine the interpolation function, the $(n+1)\cdot(N-1)$ coefficients c_{ij} in Eq. (3) have to be calculated. To this end, the interpolation conditions and the derivative continuity conditions at the nodes are used. They constitute the set of $(n+1)\cdot N - 2n$ equations. The detailed description of these equations is presented in [8]. To complete the set of equations, the n-1 end conditions have to be introduced. They can be defined in different forms. The most common form is obtained by equating some high order derivatives to zero at the end points – so-called the natural end conditions

$$s_1^{(k)}(x_1) = 0,\ s_{N-1}^{(k)}(x_N) = 0,\ \ k = \frac{n+1}{2},..,n-1 \tag{4}$$

or by requiring the nth order derivative to be continuous at some nodes at the left and right end of the domain – so-called the not-a-knot end conditions

$$s_i^{(n)}(x_{i+1}) = s_{i+1}^{(n)}(x_{i+1}), \quad s_{N-1-i}^{(n)}(x_{N-i}) = s_{N-i}^{(n)}(x_{N-i}), \quad i = 1,\ldots,\frac{n-1}{2} \tag{5}$$

If the spline degree is even, the auxiliary spline knots have to be introduced in order to define the sufficient number of conditions. The spline knots are introduced at the midpoints of the existing nodes as follows

$$z_1 = x_1, \ z_{i+1} = \frac{1}{2}\left[x_i + x_{i+1}\right], \ i = 1,..,N-1, \ z_{N+1} = x_N \tag{6}$$

The interpolation function is defined between these auxiliary points by the formula

$$f(x) \approx \left\{ s_i(x), \ x \in \left[z_i, z_{i+1}\right], \ i = 1,..,N \right\} \tag{7}$$

Each spline segment $s_i(x)$ is expressed by Eq. (3). In order to calculate the $(n+1)\cdot N$ spline coefficients c_{ij} in Eq. (3) one uses the interpolation conditions at the nodes as well as at the spline knots and the derivative continuity conditions at the spline knots. It gives $(n+1)\cdot N - n$ equations. To complete the set of equations, the n end conditions are introduced in two different forms, similarly to Eqs. (4) or (5)

$$s_1^{(k)}(x_1) = 0, \ s_N^{(k)}(x_N) = 0, \quad k = \frac{n}{2},\ldots,n-1 \tag{8}$$

$$s_i^{(n)}(z_{i+1}) = s_{i+1}^{(n)}(z_{i+1}), \quad s_{N-i}^{(n)}(z_{N+1-i}) = s_{N+1-i}^{(n)}(z_{N+1-i}), \quad i = 1,\ldots,\frac{n}{2} \tag{9}$$

Equation (8) represents the natural end conditions and Eq. (9) – the not-a-knot end conditions.

2.2 Weighting coefficients in the SSDQM

Using equations discussed in section 2.1 one can express the spline coefficients c_{ij} as a function of node distribution and unknown values of the solution f_i at the nodes. It can be put in general form as

$$c_{ij} = \sum_{k=1}^{N} C_{ijk}(x_1,\ldots,x_N)\cdot f_k, \quad i = 1,\ldots,\overline{N}, j = 0,\ldots,n \tag{10}$$

where $\overline{N} = N-1$ when n is odd and $\overline{N} = N$ when n is even.

In order to determine the weighting coefficients for the SSDQM one should calculate an appropriate order derivative of the interpolation function (2) or (7)

$$f^{(r)}(x) \approx \left\{ s_i^{(r)}(x), \ i = 1,..,\overline{N} \right\} \tag{11}$$

where

$$s_i^{(r)}(x) = \sum_{j=r}^{n}\left(c_{ij}\cdot(x-x_i)^{j-r}\cdot\frac{j!}{(j-r)!}\right) \tag{12}$$

and evaluate it at each node, what yields

$$s_i^{(r)}(x_i) = c_{ir} \cdot r!\,, \quad i = 1,\ldots,\bar{N}$$
$$s_{N-1}^{(r)}(x_N) = \sum_{j=r}^{n} \left(c_{N-1j} \cdot (x_N - x_{N-1})^{j-r} \cdot \frac{j!}{(j-r)!} \right), \quad \text{when } n \text{ is odd} \tag{13}$$

Taking into account Eq. (10), the derivatives $s_i^{(r)}(x_i)$ and $s_{N-1}^{(r)}(x_N)$ in Eq. (13), after some algebraic manipulations, can be written as

$$s_i^{(r)}(x_i) = \sum_{k=1}^{N} \left[C_{irk} \cdot r! \right] f_k, \quad i = 1,\ldots,\bar{N}$$
$$s_{N-1}^{(r)}(x_N) = \sum_{k=1}^{N} \left[\sum_{j=r}^{n} \left(C_{N-1jk} \cdot (x_N - x_{N-1})^{j-r} \cdot \frac{j!}{(j-r)!} \right) \right] f_k \tag{14}$$

Comparing with Eq. (1), it is easy to notice that the expressions in the square brackets in Eq. (14) are the weighting coefficients $a_{ij}^{(r)}$ for the *r*th order derivative in the DQM based on the spline interpolation.

The assumed spline segments are not typical base functions, what makes impossible to determine the weighting coefficients in pure numeric approach, especially deriving explicit formulas. These coefficients can be determined with the use of symbolic-numeric manipulations, where the unknown function values are noted as symbols. Equations discussed in section 2.1 allow to obtain spline coefficients (10) in symbolic-numeric form. Then, coefficients C_{ijk} in Eq. (10) can be determined by separating numbers from appropriate symbols f_k, what allows easily to compute the weighting coefficients contained in Eq. (14). This algorithm has been implemented in Computer Algebra System – Maple [12].

3 MODIFIED SSDQM – COMBINING OF THE END CONDITIONS

The research done so far indicates that the use of different forms of the end conditions (Eqs. (4) or (5) and (8) or (9)) significantly influences the quality of the solution. The use of the not-a-knot end conditions (Eqs. (5) or (9)) rapidly improves the convergence of the method but leads to the computational instability, especially when the spline degree is high. The use of the natural end conditions (Eqs. (4) or (8)) gives stable results, however the rate of convergence is significantly weaker in this case. Therefore in this paper a modification is proposed. In this approach the assumed, small number of the not-a-knot end conditions is completed by the appropriate number of the natural end conditions. These equations form the set of the combined end conditions used for computing the spline coefficients (10) and finally the weighting coefficients.

Assuming that N_{nak} denotes the assumed number of the not-a-knot end conditions, the combined end conditions take the form

$$s_i^{(n)}(x_{i+1}) = s_{i+1}^{(n)}(x_{i+1}), \quad s_{N-1-i}^{(n)}(x_{N-i}) = s_{N-i}^{(n)}(x_{N-i}), \quad i = 1,\ldots,N_{nak}$$
$$s_1^{(k)}(x_1) = 0,\; s_{N-1}^{(k)}(x_N) = 0, \quad k = \frac{n+1}{2} + N_{nak},..,n-1 \tag{15}$$

for odd spline degree and

$$s_i^{(n)}(z_{i+1}) = s_{i+1}^{(n)}(z_{i+1}), \quad s_{N-i}^{(n)}(z_{N+1-i}) = s_{N+1-i}^{(n)}(z_{N+1-i}), \quad i = 1,\ldots,N_{nak}$$
$$s_1^{(k)}(x_1) = 0,\; s_N^{(k)}(x_N) = 0, \quad k = \frac{n}{2} + N_{nak},\ldots,n-1 \tag{16}$$

for even spline degree.
Equation (15) is introduced into the algorithm presented in section 2 instead of Eq. (4) or (5) and Eq. (16) – instead of Eq. (8) or (9). The presented formulation makes a balance between rapid convergence and the stability of the method.

4 FREE VIBRATION ANALYSIS OF CONICAL SHELLS

The presented modification is tested in the problem of the free vibration of thin, composite, orthotropic shells. The governing equations that follow from classical shell theory are taken under consideration.

Shell structures are commonly used in civil, mechanical and aerospace engineering. Various analytical [13-16] and numerical [17-20] methods have been used to analyze mechanical properties of these structures or to test novel computational procedures. The DQM has been also applied to these ends [21,22]. The known results enable to estimate the effectiveness of the approach presented in the paper in this type of mechanical problems.

In Fig. 1, the analyzed conical shell with the reference coordinate system (x, θ, z) and the components of the displacement field in appropriate directions (u, v, w) are shown.

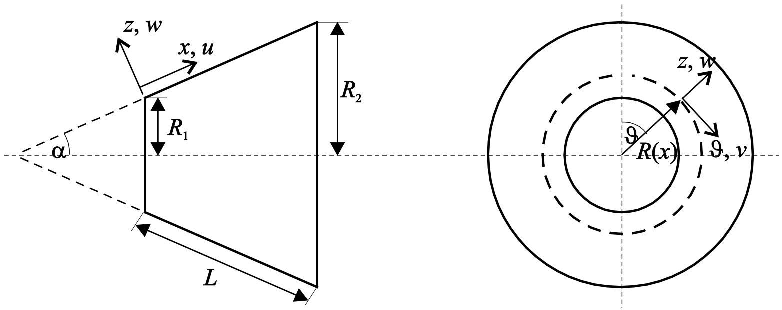

Figure 1: Geometry of truncated conical shell.

The derivation of the governing equations for the thin, composite, conical shell based on classical shell theory is presented in several papers, e.g. [21].
Taking into account that the field of displacement, in the case of the free vibration of this structure, can be expressed as

$$u = U(x)\cos(m\theta)\cos(\omega t),\ \ v = V(x)\sin(m\theta)\cos(\omega t),\ \ w = W(x)\cos(m\theta)\cos(\omega t) \quad (17)$$

where m is the wave number in the circumferential direction and ω is the circular frequency, mathematical model of the shell can be written in a compact matrix form as follows

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = -\rho h \omega^2 \begin{bmatrix} U \\ V \\ W \end{bmatrix} \quad (18)$$

In Equation (18), L_{ij} are differential operators, e.g.

$$L_{22}=\left(\frac{A_{66}\sin(\alpha)}{R}-\frac{B_{66}\sin(\alpha)\cos(\alpha)}{R^2}-\frac{4D_{66}\sin(\alpha)\cos^2(\alpha)}{R^3}\right)\frac{d}{dx}$$
$$+\left(A_{66}+\frac{3B_{66}\cos(\alpha)}{R}+\frac{2D_{66}\cos^2(\alpha)}{R^2}\right)\frac{d^2}{dx^2}-\frac{A_{22}m^2+A_{66}\sin^2(\alpha)}{R^2}$$
$$-\frac{(2m^2B_{22}-B_{66}\sin^2(\alpha))\cos(\alpha)}{R^3}-\frac{(D_{22}m^2-4D_{66}\sin^2(\alpha))\cos^2(\alpha)}{R^4},$$

These operators contain constants A_{ij}, B_{ij} and D_{ij} that denote extensional, coupling and bending stiffnesses. For a shell that is composed of different layers of orthotropic materials, these stiffnesses can be expressed as

$$A_{ij}=\sum_{k=1}^{N_l}\overline{Q}_{ij}^k(h_k-h_{k-1}),\ B_{ij}=\frac{1}{2}\sum_{k=1}^{N_l}\overline{Q}_{ij}^k(h_k^2-h_{k-1}^2),\ D_{ij}=\frac{1}{3}\sum_{k=1}^{N_l}\overline{Q}_{ij}^k(h_k^3-h_{k-1}^3) \tag{19}$$

where N_l denotes the number of layers in the shell and h_k i h_{k-1} are the distances from the shell reference surface to the outer and inner surfaces of the kth layer, respectively. $\overline{Q}_{ij}^k$ are elements of the transformed, reduced, elasticity matrix of the kth layer.

For a given layer of orthotropic material, the $\overline{\mathbf{Q}}$ matrix is a result of the transformation of the elasticity matrix $\mathbf{Q}$ between the principal material coordinates and the shell's coordinates according to the formula

$$\overline{\mathbf{Q}}=\mathbf{TQT}^T \tag{20}$$

The transformation matrix **T** can be expressed as

$$\mathbf{T}=\begin{bmatrix}\cos^2(\phi) & \sin^2(\phi) & -2\sin(\phi)\cos(\phi)\\ \sin^2(\phi) & \cos^2(\phi) & 2\sin(\phi)\cos(\phi)\\ \sin(\phi)\cos(\phi) & -\sin(\phi)\cos(\phi) & \cos^2(\phi)-\sin^2(\phi)\end{bmatrix} \tag{21}$$

where ϕ is the angular orientation of the fibers.
The material constants in $\mathbf{Q}$ are defined by Young's moduli: E_1, E_2, the shear moduli G_{12} and the Poisson's ratios: ν_{12}, ν_{21} with the following formulas

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}},\ Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}},\ Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}},\ Q_{66}=G_{12} \tag{22}$$

In the paper, the shell, which is simply supported at both edges ($R=R_1, R=R_2$) is considered. At these edges, appropriate displacement functions V, W as well as force N_x and moment M_x have to be zero, what can be put as

$$V=0, W=0 \tag{23}$$

$$U^{(1)}+a_1U+a_2V+a_3W+a_4W^{(1)}+a_5W^{(2)}=0 \tag{24}$$

$$W^{(2)}+b_1U+b_2U^{(1)}+b_3V+b_4W+b_5W^{(1)}=0 \tag{25}$$

where constants a_i and b_i are as follows

$a_1 = A_{12}\sin(\alpha)/A_{11}R$, $a_2 = B_{12}m\cos(\alpha)/A_{11}R^2 + A_{12}m/A_{11}R$,
$a_3 = A_{12}\cos(\alpha)/A_{11}R + B_{12}m^2/A_{11}R^2$, $a_4 = -B_{12}\sin(\alpha)/A_{11}R$,
$a_5 = -B_{11}/A_{11}$, $b_1 = -B_{12}\sin(\alpha)/D_{11}R$, $b_2 = -B_{11}/D_{11}$,
$b_3 = -D_{12}m\cos(\alpha)/D_{11}R^2 - B_{12}m/D_{11}R$,
$b_4 = -B_{12}\cos(\alpha)/D_{11}R - D_{12}m^2/D_{11}R^2$, $b_5 = D_{12}\sin(\alpha)/D_{11}R$

Taking advantage of the algorithm discussed in section 2, where the modification based on combining of the end conditions is introduced, the weighting coefficients $a_{ij}^{(r)}$ for the *r*th order derivative are computed. This approach is characterized by the number N_{nak}, that denotes the number of the not-a-knot type conditions contained in the set of the end conditions.
Equation (18), which is discretized using differential quadrature rules, takes the form

$$\begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} & \mathbf{P}_{13} \\ \mathbf{P}_{21} & \mathbf{P}_{22} & \mathbf{P}_{23} \\ \mathbf{P}_{31} & \mathbf{P}_{32} & \mathbf{P}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \\ \mathbf{W} \end{bmatrix} = -\rho h \omega^2 \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \\ \mathbf{W} \end{bmatrix} \tag{26}$$

where vectors **U**, **V**, **W** contain nodal function values and elements of matrices $\mathbf{P}_{lk}$ contain appropriate weighting coefficients $a_{ij}^{(r)}$, $i, j = 1,...,N$, for example

$$P_{22_{ij}} = \left(\frac{A_{66}\sin(\alpha)}{R_i} - \frac{B_{66}\sin(\alpha)\cos(\alpha)}{R_i^2} - \frac{4D_{66}\sin(\alpha)\cos^2(\alpha)}{R_i^3} \right) a_{ij}^{(1)} + \left(A_{66} + \frac{3B_{66}\cos(\alpha)}{R_i} + \frac{2D_{66}\cos^2(\alpha)}{R_i^2} \right) a_{ij}^{(2)} - \left(\frac{A_{22}m^2 + A_{66}\sin^2(\alpha)}{R_i^2} + \frac{(2m^2B_{22} - B_{66}\sin^2(\alpha))\cos(\alpha)}{R_i^3} + \frac{(D_{22}m^2 - 4D_{66}\sin^2(\alpha))\cos^2(\alpha)}{R_i^4} \right) \delta_{ij},$$

To implement boundary conditions, Eqs. (23)-(25) are discretized at both edges

$$V_k = 0, W_k = 0 \tag{27}$$

$$\sum_{j=1}^{N} a_{kj}^{(1)} U_j + a_1 U_k + a_2 V_k + a_3 W_k + a_4 \sum_{j=1}^{N} a_{kj}^{(1)} W_j + a_5 \sum_{j=1}^{N} a_{kj}^{(2)} W_j = 0 \tag{28}$$

$$\sum_{j=1}^{N} a_{kj}^{(2)} W_j + b_1 U_k + b_2 \sum_{j=1}^{N} a_{kj}^{(1)} U_j + b_3 V_k + b_4 W_k + b_5 \sum_{j=1}^{N} a_{kj}^{(1)} W_j = 0 \tag{29}$$

where $k = 1$ at the small edge (R_1) and $k = N$ at the large one (R_2).

Following the general approach to implement boundary conditions in the DQM [23], Eqs. (26)-(29) can be put as the final eigenvalue equation

$$\mathbf{A} \cdot \mathbf{Z} = -\rho h \omega^2 \cdot \mathbf{Z} \tag{30}$$

where vector **Z** contains nodal function values at the inner points
$\mathbf{Z} = [U_2,...,U_{N-1},V_2,...,V_{N-1},W_3,...,W_{N-2}]$.

The calculations are carried out for the antisymmetric cross-ply conical shell. The elementary material parameters of each layer are given as

$$E_1/E_2 = 15,\ \nu_{12} = 0.25,\ G_{12}/E_2 = 0.5 \tag{31}$$

As the result, the coefficients in Eq. (19) can be simplified as [15]

$$\begin{aligned} &A_{11} = A_{22} = h/2(Q_{11} + Q_{22}), \quad A_{12} = Q_{12}h, \quad A_{66} = Q_{66}h, \\ &B_{11} = -B_{22} = (h^2/4N_l)(Q_{11} - Q_{22}), \quad B_{12} = B_{66} = 0, \\ &D_{11} = D_{22} = (h^3/24)(Q_{11} + Q_{22}), \quad D_{12} = \tfrac{1}{12}Q_{12}h^3, \quad D_{66} = \tfrac{1}{12}Q_{66}h^3 \end{aligned} \tag{32}$$

where N_l is the number of plies.

The calculations are carried out using uniform node distribution as well as Chebyshev-Gauss-Lobatto pattern. The results are presented in the form of dimensionless frequency parameter

$$\lambda = R_2\sqrt{\rho h/A_{11}}\,\omega \tag{33}$$

The dependence on the combined end conditions (N_{nak}), spline degree (n) and the number of nodes (N) is presented in Fig. 2 (Chebyshev-Gauss-Lobatto pattern) and Fig. 3 (uniform grid). These figures show the percentage relative error $\delta = (\lambda_{SSDQM} - \lambda_{DQM})/\lambda_{DQM} \cdot 100\%$ between the results from the SSDQM and the reference value taken as the result from the DQM ($\lambda_{DQM} = 0.1779$) obtained in present work.

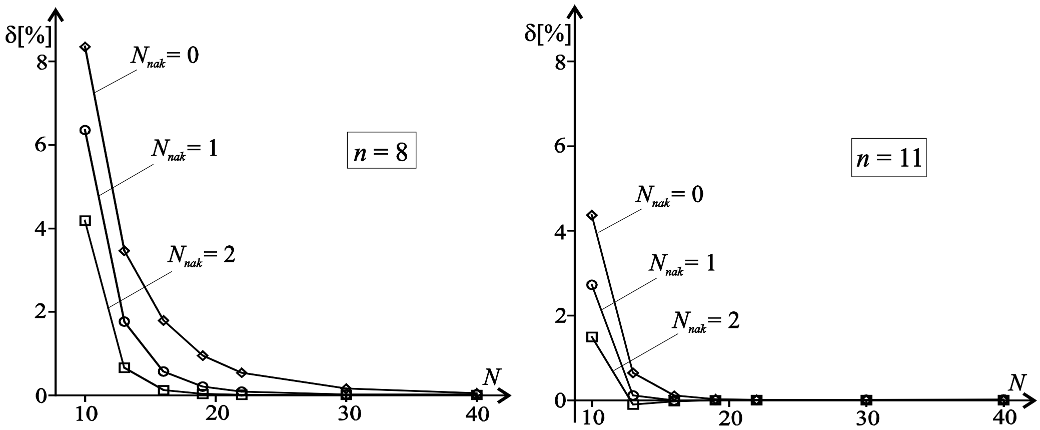

Figure 2: Error of fundamental frequency parameter λ for antisymmetric cross-plied laminated conical shell ($N_l = 2$, $h/R_2 = 0.01$, $\alpha = 30°$, $L\sin\alpha/R_2 = 0.25$, $m = 0$) – Chebyshev-Gauss-Lobatto grid.

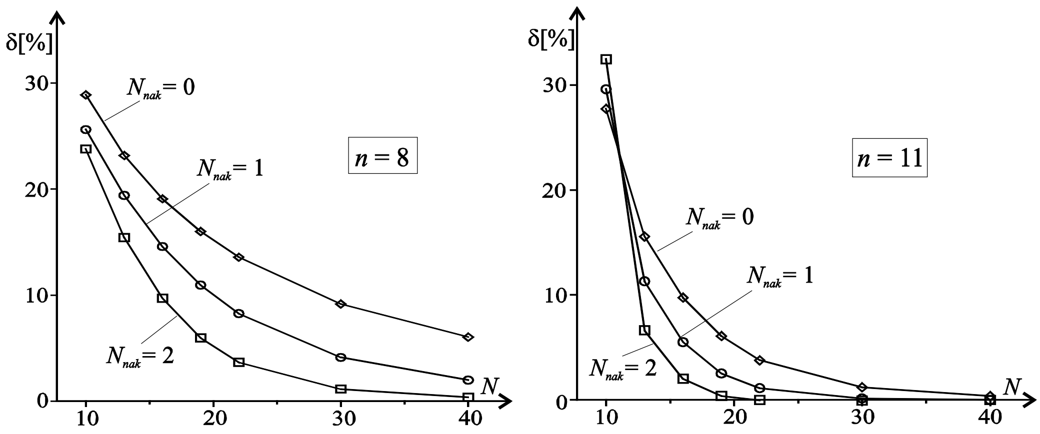

Figure 3: Error of fundamental frequency parameter λ for antisymmetric cross-plied laminated conical shell ($N_l = 2$, $h/R_2 = 0.01$, $\alpha = 30°$, $L\sin\alpha/R_2 = 0.25$, $m = 0$) – uniform grid distribution.

Presented results indicate that the use of the combined end conditions significantly improves the rate of convergence. When one spline segment covers more intervals at each end of the domain ($N_{nak} > 0$) then more accurate results are obtained. The results can be improved also using higher spline degree. It should be noted that the SSDQM converges, when the uniform node distribution is applied, unlike the conventional DQM.

Table 1 shows the fundamental frequency parameters (33) for different number of layers and various shell thickness. These results are obtained with the SSDQM based on the eleventh spline degree, imposing $N = 16$ nodes according to the Chebyshev-Gauss-Lobatto pattern and using the modification ($N_{nak} = 2$) discussed in section 3. As Fig. 2 reports, these computational parameters ensure fast convergence and high accuracy.

In the theoretical case, when the number of layers N_l approaches to infinity, coupling stiffnesses B_{11}, B_{22}, contained in Eq. (32), approach to zeros, unlike the case, when the number $N_l = 2$. Then the mentioned stiffnesses achieve maximum value.

h/R_2	Number of layers N_l					
	2	4	6	10	20	∞
0.01	0.1779 (0.1769)	0.1941	0.1962	0.1972	0.1976	0.1978 (0.1978)
0.02	0.2116 (0.2119)	0.2305	0.2332	0.2346	0.2352	0.2355 (0.2355)
0.03	0.2341 (0.2360)	0.2596	0.2637	0.2658	0.2667	0.2671 (0.2671)
0.04	0.2546 (0.2578)	0.2891	0.2947	0.2975	0.2987	0.2992 (0.2992)
0.05	0.2755 (0.2794)	0.3188	0.3254	0.3288	0.3302	0.3308 (0.3308)
0.06	0.2967 (0.3010)	0.3473	0.3548	0.3585	0.3600	0.3606 (0.3606)
0.07	0.3179 (0.3222)	0.3738	0.3817	0.3855	0.3871	0.3877 (0.3877)
0.08	0.3386 (0.3426)	0.3978	0.4057	0.4096	0.4111	0.4117 (0.4117)
0.09	0.3584 (0.3620)	0.4190	0.4268	0.4305	0.4320	0.4325 (0.4325)
0.10	0.3770 (0.3801)	0.4376	0.4450	0.4485	0.4500	0.4504 (0.4504)

Table 1: Fundamental frequency parameters λ for antisymmetric cross-plied laminated conical shells ($\alpha = 30°$, $L \sin\alpha / R_2 = 0.25$, $m = 0$).

In Tab. 1, for two extreme cases, the results obtained in [15] (values in parenthesis) are also presented for comparison. As one can notice, very good agreement is achieved, when the coupling stiffnesses vanish ($N_l = \infty$), while some differences occur, when these stiffnesses take maximum value ($N_l = 2$). The reason may lie in the use of various shell theories. In present paper Love's shell theory is taken under consideration, while in [15] Donnell's theory is used.

5 CONCLUSION

In the paper, the method based on the differential quadrature is presented and applied to vibration analysis of shells. These type methods belong to high order approximation techniques and provide high convergence and accuracy. In order to ensures the computational stability, the weighting coefficients in present method are computed using spline functions and the modification proposed in the paper improves the rate of convergence of the method under this conditions.

The effectiveness of this modification is shown by the example of the free vibration of the composite, conical shell. The results indicate that the combined end conditions, where the small number of not-a-knot end conditions are used, can significantly improve the results.

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