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A FAST METHOD FOR COMPUTING CONVOLUTIONS WITH STRUCTURAL GREEN'S FUNCTION : APPLICATION TO TIRE/ROAD CONTACT PROBLEMS

R. Meftah[1] **and D. Duhamel**[1]

[1] Université Paris-Est, UR Navier (ENPC/IFSTTAR/CNRS),
Ecole des Ponts ParisTech, 6 et 8 Avenue Blaise Pascal,
Cité Descartes, Champs sur Marne, 77455 Marne la Vallée, cedex 2, France
Tel: 33 1 64 15 37 36
Fax: 33 1 64 15 37 41
e-mail: {rabie.meftah,denis.duhamel}@enpc.fr

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Abstract. *Tire/road contact represents the major source of traffic noise with driving speed above 50 km/h. One of the most important problems is to take into account the contact conditions and to calculate the contact forces in an accurate way. As a general approach, the dynamic response of the tire is calculated by convolving the contact forces with the Green function of the tire. The disadvantage of this method is that the computation can be time consuming. In this paper, an alternative which is a modal decomposition model is used. The developed method allows quicker calculations than the traditional convolution. It consists, at the first stage, on an approximation of the pre-calculated Green function on a series of modal contributions with the Least Square Complex Exponential (LSCE) algorithm then, on the calculation of the dynamic response in the time domain as a series of SDoF systems response. For verification, the approach is tested by using a Single Degree of Freedom (SDoF) oscillator where the system moves through a sinusoidal road profile with a constant speed. Then, it is applied to the Ring on Elastic Foundation (REF) Model.*

1 Introduction

In many advanced structures, the contact impact problems present an important interest. In numerical modeling of the effect of contact on structures, two classic approaches can be used. The Lagrange multiplier method [1] and the penalty method [2] are most commonly used in dynamic contact problems.

The dynamic contact problems can also be treated by using the Green's functions. In this approach the dynamic response is calculated by convolving the Green's functions with the contact forces. The convolution technique for contact problems is used by many authors : M. McIntyre al. [3] have applied the approach to the string/bow contact to study large-amplitude oscillations of musical instruments. C. Wang and J. Kim. [4],[5] have used the same approach for a thin beam impacting against a stop, A. Nordborq [6] for the wheel/rail contact problem and many other authors have used this technique in the tyre/road contact [7, 8, 9, 10].

The convolution technique for contact problems presents the advantage of simple implementation and is relatively less time consuming than classical methods. However, the time calculation for the convolution can be improved. G. Beylkin has developed [11] a fast convolution with free space Helmholtz Green's function. The convolution combines the spatial and Fourier domains. In the space domain, the Green's function is approximated by a sum of decaying Gaussians with positive coefficients and in the Fourier domain by a multiplication by a band-limited kernel.

In this paper, we present a fast convolution in which the Green's function in the frequency domain is approximated by a sum of modal contributions. The modal parameters are identified by LSCE algorithm and used in the time domain to construct a fast and accurate algorithm for computing convolutions with Green's functions. Two examples illustrate the efficiency of the method : first a SDoF system moving on a sinusoidal profile and in the second a MDoF on a random profile.

2 Structural Green's function

2.1 Standard convolution

A linear discretized dynamic problem can be generally expressed by a second order differential equation in the time domain:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{q}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass matrix, the damping matrix and the stiffness matrix, respectively. In the frequency domain, the problem can be written:

$$\left[-\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}\right]\mathbf{U}(\omega) = \mathbf{Q}(\omega) \tag{2}$$

The resolution of equation (1) first needs to search a general solution $\mathbf{u}_g(t)$ of the associated homogeneous equation and then to find a particular solution of the full equation. The computation of the Green's function $\mathbf{G}(\omega)$ is a systematic tool to get this particular solution. The traditional method when the time Green's function $\mathbf{g}(t)$ is known, is to calculate the dynamic response of the system by convolving the contact forces with the Green's function:

$$\mathbf{u}(t) = \mathbf{u}_g(t) + \int_0^t \mathbf{g}(t-\tau)\mathbf{q}(\tau)\mathrm{d}\tau \tag{3}$$

When the system is at rest until a certain time taken as origin $t = 0$, the solution of the homogeneous problem is null ($\mathbf{u}_g(t) = \mathbf{0}$), and the solution is reduced to the relation:

$$\mathbf{u}(t) = \int_0^t \mathbf{g}(t-\tau)\mathbf{q}(\tau)\mathrm{d}\tau \tag{4}$$

Equation (4) can be discretized as follows:

$$\mathbf{u}_k = \sum_{m=0}^{k} \mathbf{g}_{k-m}\mathbf{q}_m \tag{5}$$

Where $\mathbf{u}_k$ is the displacement at the time $k\Delta t$.

Let's note N_t the number of time steps used to calculate the displacement, and N_g the number of time steps for the influencing Green's function. The effect of the Green's function is neglected when the amplitudes of oscillation at time greater than $N_g\Delta t$ are hundred times smaller than the maximum of the Green's function $g(t)$.

Equation (5) is reduced to :

$$u_k = \sum_{m=0}^{\min(k,N_g)} g_{k-m}q_m \tag{6}$$

The number of arithmetic operations necessary to obtain the displacement until the time $N_t\Delta t$ is of the same order as N_tN_g.

2.2 Modal decomposition

The idea is to express the Green's function as a sum of modal contributions, and through numerical manipulations, the modal parameters can be identified. The Green's function $G(\omega)$ is supposed to be known, and $g(t)$ denotes its inverse Fourier transform. For the sake of simplicity, we restrict to one component for the displacement, in this case the Green's function $G(\omega)$ can be approximated as follows.

$$G(\omega) \simeq \sum_{k=1}^{k=N} \left[\frac{R_k}{j\omega - \lambda_k} + \frac{R_k^*}{j\omega - \lambda_k^*}\right] \tag{7}$$

One seeks to identify the modal parameters (residues A_k, damping ξ_k and frequencies ω_k). There exist several methods to deal with this kind of problems. The LSCE method

(Least Square Complex Exponential) remains the reference in the applications of experimental modal analysis. Equation (7) can be written in the equivalent form

$$G(\omega) \simeq \sum_{k=1}^{k=N} \frac{A_k}{-\omega^2 + 2j\xi_k\omega\omega_k + \omega_k^2} \tag{8}$$

The LSCE algorithm is used for the decomposition.

To check the accuracy of estimated modal data, the Green's function is regenerated. This method aims to find the best estimates of modal data that minimizes the error defined in equation (9)

$$E = \frac{\int_0^{\omega_{max}} |G(\omega) - \sum_{k=1}^{k=N} \frac{A_k}{-\omega^2 + 2j\xi_k\omega\omega_k + \omega_k^2}|\mathrm{d}\omega}{\int_0^{\omega_{max}} |G(\omega)|\mathrm{d}\omega} \tag{9}$$

3 Fast convolution

After finding the modal parameters A_k, ω_k and ξ_k, then truncating the decomposition to an order N, the Green's function in the time domain can be written in the form of a sum of the contribution of each mode:

$$g(t) = \sum_{k=1}^{k=N} \frac{A_k}{\omega_k^d} e^{-\xi_k\omega_k t} \sin(\omega_k^d\, t) \tag{10}$$

with

$$\omega_k^d = \omega_k\sqrt{1 - \xi_k^2}$$

The displacement can be calculated by a convolution product

$$u(t) = \int_0^t g(\tau)q(t-\tau)\mathrm{d}\tau = \int_0^t g(t-\tau)q(\tau)\mathrm{d}\tau \tag{11}$$

Replacing $g(t)$ by its decomposition in equation (11) yields

$$u(t) = \int_0^t \sum_{k=1}^{k=N} \frac{A_k}{\omega_k^d}\, e^{-\xi_k\omega_k\,(t-\tau)} \sin\left(\omega_k^d\,(t-\tau)\right) q(\tau)\mathrm{d}\tau \tag{12}$$

By separating the variables t and τ, then rearranging the terms, we can write the displacement in the form

$$u(t) = \sum_{k=1}^{k=N} \frac{A_k}{\omega_k^d} e^{-\xi_k\omega_k t} \left[\sin\left(\omega_k^d\, t\right)\alpha^k(t) - \cos\left(\omega_k^d\, t\right)\beta^k(t)\right] \tag{13}$$

where $\alpha^k(t)$ and $\beta^k(t)$ are given as

$$\alpha^k(t) = \int_0^t \quad e^{\xi_k \omega_k \tau} \cos(\omega_k^d\, \tau) q(\tau) \mathrm{d}\tau \tag{14}$$

$$\beta^k(t) = \int_0^t \quad e^{\xi_k \omega_k \tau} \sin(\omega_k^d\, \tau) q(\tau) \mathrm{d}\tau \tag{15}$$

$$\tag{16}$$

At the time $n\Delta t$ the discrete displacement is written

$$u(n\Delta t) = \sum_{k=1}^{k=N} \frac{A_k}{\omega_k^d} e^{-\xi_k \omega_k n\Delta t} \left[\sin\left(\omega_k^d\, n\Delta t\right) \alpha^k(n\Delta t) - \cos\left(\omega_k^d\, n\Delta t\right) \beta^k(n\Delta t)\right] \tag{17}$$

The coefficients $\alpha^k(n\Delta t)$ and $\beta^k(n\Delta t)$ are calculated by the recurrence equations

$$\alpha^k((n+1)\Delta t) = \alpha^k(n\Delta t) + e^{\xi_k \omega_k n\Delta t} \cos(\omega_k^d\, n\Delta t) q(n\Delta t)\Delta t \tag{18}$$

$$\beta^k((n+1)\Delta t) = \beta^k(n\Delta t) + e^{\xi_k \omega_k n\Delta t} \sin(\omega_k^d\, n\Delta t) q(n\Delta t)\Delta t \tag{19}$$

The number of arithmetic operations necessary to obtain the displacement until the time $N_t\Delta t$ is of the same order than $N_t N_m$, where N_m is the number of modes used to represent the Green's function.

4 Contact models using Green's functions

4.1 General procedure

To illustrate the approach presented above, let us consider a simple dynamic contact problem. The purpose of this example is to test the fast convolution method and to compare it with the traditional convolution. So we consider a mechanical system represented by its Green's function. The system is moving with a constant speed on a surface without slipping.

The displacement $u(t)$ at time t, depends on the contact forces history $f_c(t)$ imposed by the texture of the surface. Two situations arise: either, there is a contact between the system and the surface and the displacement of the system equals the height of the surface $u_r(t)$, or there is no contact and in this case the contact force is null and the displacement of the system is strictly higher than that of the surface. Only vertical displacements are considered here.

The ideal conditions of unilateral contact are given by:

$$u(t) = u_r(t) \ ; \ f_c(t) > 0 \tag{20}$$

$$u(t) > u_r(t) \ ; \ f_c(t) = 0 \tag{21}$$

The procedure of the computation is described in the following.

1. First the displacement history $u^h(n\Delta t)$ is computed by assuming no contact $f_c(n\Delta t) = 0$.

$$u^h(n\Delta t) = \Delta t \sum_{k=0}^{n-1} f_c(k\Delta t) g((n-k)\Delta t) \tag{22}$$

2. The displacement history is compared with the profile altitude:

$$\Delta x(n\Delta t) = u_r(n\Delta t) - u^h(n\Delta t) \tag{23}$$

3. If the assumption of no contact ($\Delta x < 0$) is satisfied, the displacement is:

$$u(n\Delta t) = u^h(n\Delta t) \tag{24}$$

   Then the procedure is repeated by assuming no contact for the next time step ($f_c((n+1)\Delta t) = 0$).

4. If $\Delta x \geq 0$, the contact force is computed from the cinematic conditions described below.

5. The procedure is repeated by assuming no contact condition for the next time step ($f_c((n+1)\Delta t) = 0$).

The same procedure is used for the modal decomposition model, but the displacement history $u^h(n\Delta t)$ is computed by

$$\begin{aligned} u^h(n\Delta t) \quad &= \sum_{k=1}^{k=N} \frac{A_k}{\omega_k^d} e^{-\xi_k \omega_k n\Delta t} \\ &\left[\sin\left(\omega_k^d\, n\Delta t\right) \alpha^k((n-1)\Delta t) - \cos\left(\omega_k^d\, n\Delta t\right) \beta^k((n-1)\Delta t)\right] \end{aligned} \tag{25}$$

4.1.1 Cinematic condition

The standard convolution can be written as following

$$\begin{aligned} u(t) \quad &= \int_0^{t-\Delta t} g(t-\tau)q(\tau)d\tau + \int_{t-\Delta t}^{t} g(t-\tau)q(\tau)d\tau \\ &= \underbrace{\int_0^{t-\Delta t} g(t-\tau)q(\tau)d\tau}_{u_h(t)} + \int_0^{\Delta t} g(\tau)q(t-\tau)d\tau \\ &= u_h(t) + \int_0^{\Delta t} g(\tau)q(t-\tau)d\tau() \end{aligned} \tag{26}$$

We derive the equation (26)

$$\begin{aligned}
v(t) = u'(t) \quad &= \int_0^{t-\Delta t} g'(t-\tau)q(\tau)d\tau + \int_{t-\Delta t}^{t} g'(t-\tau)q(\tau)d\tau \\
&= \underbrace{\int_0^{t-\Delta t} g'(t-\tau)q(\tau)d\tau}_{v_h(t)} + \int_0^{\Delta t} g'(\tau)q(t-\tau)d\tau \\
&= v_h(t) + \int_0^{\Delta t} g'(\tau)q(t-\tau)d\tau \qquad (27)
\end{aligned}$$

let's note $\mathbf{Y} = [u(t) \;\; v(t)]^T$ and $\mathbf{Y}_h = [u_h(t) \;\; v_h(t)]^T$, we can write

$$\mathbf{Y} = \mathbf{Y}_h + \Psi(q) \qquad (28)$$

where Ψ is an integral operator which reflects the influence of efforts at the present moment.

Where there is contact, the displacement is imposed by the rigid profile. We suppose that the contact point follows the profile. We can write the following contact conditions

$$u(t) \quad = u_r(t) \qquad (29)$$

$$v(t) \quad = \frac{du_r(t)}{dt} \qquad (30)$$

$$(31)$$

In the vectorial form

$$\mathbf{Y} = \mathbf{Y}_r = [u_r(t) \;\; \frac{du_r(t)}{dt}]^T \qquad (32)$$

Using the modal decomposition convolution we have

$$u(t) = \sum_{k=1}^{k=N} \frac{A_k}{\omega_k^d} e^{-\xi_k \omega_k t} \left[sin\left(\omega_k^d\, t\right) \alpha^k(t) - cos\left(\omega_k^d\, t\right) \beta^k(t) \right] \qquad (33)$$

and

$$\begin{aligned}
v(t) = \quad & -\sum_{k=1}^{k=N} \frac{A_k \xi_k \omega_k}{\omega_k^d} e^{-\xi_k \omega_k t} \left[sin\left(\omega_k^d\, t\right) \alpha^k(t) - cos\left(\omega_k^d\, t\right) \beta^k(t) \right] \\
& + \sum_{k=1}^{k=N} A_k e^{-\xi_k \omega_k t} \left[cos\left(\omega_k^d\, t\right) \alpha^k(t) + sin\left(\omega_k^d\, t\right) \beta^k(t) \right] \qquad (34)
\end{aligned}$$

Equations (33) and (34), can be written in the matrix form

$$\mathbf{Y} = \sum_{k=1}^{k=N} \mathbf{B}_k \mathbf{x}_k \qquad (35)$$

where

$$\mathbf{x}_k = [\alpha^k(t) \ \ \beta^k(t)]^T \tag{36}$$

and

$$\mathbf{B}_k = A_k e^{-\xi_k \omega_k t} \left[\begin{array}{cc} \frac{sin\left(\omega_k^d \, t\right)}{\omega_k^d} & -\frac{cos\left(\omega_k^d \, t\right)}{\omega_k^d} \\ \\ cos\left(\omega_k^d \, t\right) - \frac{\xi_k \omega_k}{\omega_k^d} sin\left(\omega_k^d \, t\right) & sin\left(\omega_k^d \, t\right) + \frac{\xi_k \omega_k}{\omega_k^d} cos\left(\omega_k^d \, t\right) \end{array} \right] \tag{37}$$

The contact conditions are

$$\mathbf{Y}_r = \mathbf{Y}^h + \Psi(\mathbf{q}) \tag{38}$$

The purpose is to find the contact force that verify simultaneously the conditions (32). Contact forces are related to displacements and velocities by the operator Ψ.

$$\begin{aligned} \Delta \mathbf{Y} &= \mathbf{Y}_r - \mathbf{Y}^h \\ &= \left[\begin{array}{c} \int_0^{\Delta t} g(t-\tau) q(\tau) d\tau \\ \int_0^{\Delta t} g'(t-\tau) q(\tau) d\tau \end{array} \right] \end{aligned} \tag{39}$$

Integrals can be computed using two Gauss points. The values of the contact force at the two Gauss points are then calculated by inverting the operator Ψ

$$\mathbf{q} = \left[\begin{array}{c} q_1 \\ q_2 \end{array} \right] \tag{40}$$

$$= \Psi^{-1}(t_1, t_2) \Delta \mathbf{Y} \tag{41}$$

The operator Ψ is given by

$$\Psi = \left[\begin{array}{cc} g(t-t_1) & g(t-t_2) \\ \\ g'(t-t_1) & g'(t-t_2) \end{array} \right] \Delta t \tag{42}$$

where,

$$g(t) = \sum_{k=1}^{k=N} \frac{A_k}{\omega_k^d} e^{-\xi_k \omega_k t} sin\left(\omega_k^d \, t\right) \tag{43}$$

On a time interval $[t \ \ t + \Delta t]$, the contact force is the medium of its two values at the two Gauss points t_1 and t_2.

where

$$t_1 = t + \left(1 - \frac{1}{\sqrt(3)}\right) \frac{\Delta t}{2} \tag{44}$$

$$t_2 = t + \left(1 + \frac{1}{\sqrt(3)}\right) \frac{\Delta t}{2} \tag{45}$$

Knowing the values of the contact force at the Gauss points t_1 and t_2 we can compute the parameters $\alpha_k(t+\Delta t)$ and $\beta_k(t+\Delta t)$ by the recursion equations

$$\alpha^k(t+\Delta t) = \alpha_h^k(t) + \frac{f^c(t_1)q_1 + f^c(t_2)q_2}{2}\Delta t \tag{46}$$

$$\beta^k(t+\Delta t) = \beta_h^k(t) + \frac{f^s(t_1)q_1 + f^s(t_2)q_2}{2}\Delta t \tag{47}$$

where

$$f^s(t) \qquad = e^{\xi_k \omega_k t} sin(\omega_k^d t) \tag{48}$$

$$f^c(t) \qquad = e^{\xi_k \omega_k t} cos(\omega_k^d t) \tag{49}$$

5 Examples

5.1 Case of a single DoF system

The simplest dynamic system considered in vibration problems is the Single Degree of Freedom (SDoF) oscillator as shown in Figure (1). The analysis of this system is used to compare the standard convolution with the developed approach.

In this example, the system moves through a profile $u_r(x)$ with a constant speed $V_0 = 0.1$ m.s^{-1}. It is supposed that the displacement is done without slipping as shown in figure (1).

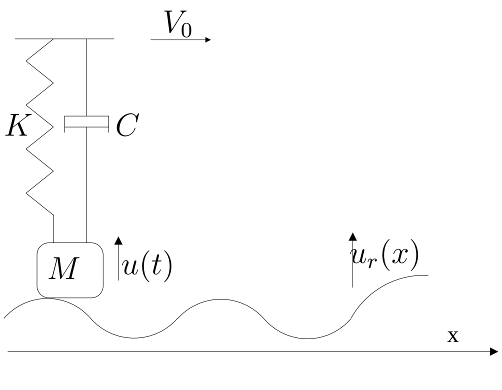

Figure 1: SDOF mass-spring system on a sinusoidal surface

Consider a sinusoidal profile

$$u_r = A_0 \sin(\omega_r x) = A_0 \sin(\frac{2\pi}{\lambda_r} V_0 t) \tag{50}$$

where $\lambda_r = 25\ mm$ is the wavelength of the profile and $A_0 = 5\ mm$ its amplitude.

The system verifies the equations:

$$M\ddot{u} + C\dot{u} + Ku = -Mg + F_c \tag{51}$$
$$u(t) \geq u_r(t) \tag{52}$$
$$F_c \geq 0 \tag{53}$$

with the initial conditions:

$$u_0 = u(0) = u_r(0) \tag{54}$$
$$v_0 = \frac{du(t)}{dt}|_{t=0} = 0 \tag{55}$$

If the mass is above the surface, there is no contact. The displacement and contact forces are given by

$$u(t) = e^{-\xi\omega_0(t-t_c)}\left[u_c\cos(\omega_d(t-t_c)) + \frac{v_c + \xi\omega_0 u_c}{\omega_d}\sin(\omega_d(t-t_c))\right] \tag{56}$$
$$F_c(t) = 0 \tag{57}$$

where u_c and v_c are respectively the displacement and velocity at the last contact moment t_c.

If the mass is below the surface, there is contact and the contact force is computed from the cinematic conditions described previously.

The parameters used in the model for the simulations are given in Table (1).

M[Kg]	K[N/m]	ξ
1	$4\ 10^5$	0.02

Table 1: SDoF parameters used in the simulations

Figures (2) and (3) show respectively the displacements $u(t)$ and the contact forces $F_c(t)$ calculated by the standard convolution method and the modal decomposition method. Both methods give the same result. In the part where there is contact, we notice that the displacement and contact force curves are fitting the shape of the surface. When this contact force is null, the system enters on a free vibration regime.

Using standard convolution is costly in terms of computing time, especially with a small time step. Indeed, from equations (5) and (17) we can see that in the case of a classical convolution, the number of calculation operations is proportional to the number of time steps N and to the size of the Green's functions N_g while in the modal decomposition it is proportional to N and to the approximation order. Table (2) shows a comparison of computing times between both methods.

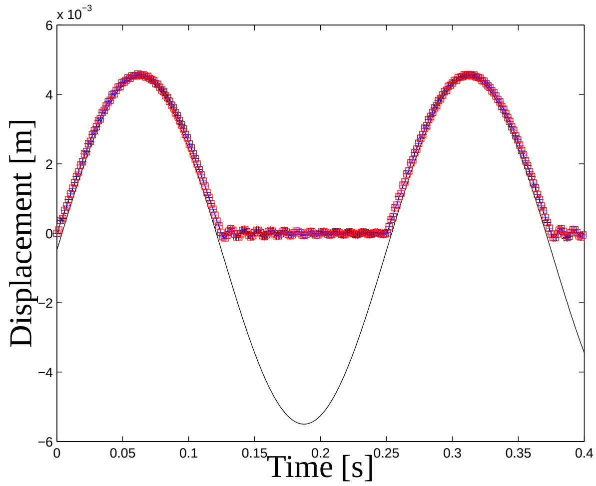

Figure 2: Displacement of a SDOF system on a sinusoidal surface : — profile, $-+-$ standard convolution, -□- modal decomposition

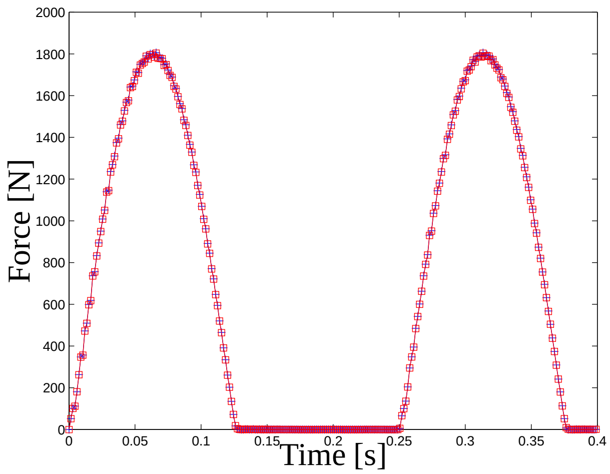

Figure 3: Contact force of a SDOF system on a sinusoidal surface :$-+-$ standard convolution, -□- modal decomposition

Time step [ms]	N_t	Standard convolution		Modal decomposition	
		N_g	Computing time [s]	N_m	Computing time [s]
0.1	2000	1900	0.06	1	0.03
0.1	20000	1900	1.01	1	0.17
0.01	20000	19000	5.50	1	0.26
0.01	200000	19000	105.14	1	2.80

Table 2: Comparison of the computing time: SDof system

5.2 Ring on Elastic Foundation Model

Modelling complex tire structures in details is a hard task. In the literature the ring on elastic foundation model was frequently used. In this model, the main dynamical properties of the tire are taken into account. The tread is modelled by a circular Euler-Bernoulli beam, the elastic properties of the sidewalls and the rim are modelled by distributed springs as shown in Figure (4).

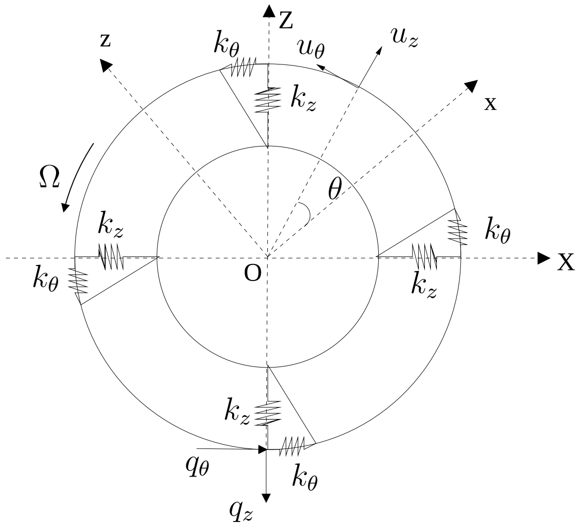

Figure 4: Ring on Elastic Foundation Model

The Green functions of the system is given by [12]:

$$\mathbf{G}(\omega) = \begin{bmatrix} G_{\theta\theta} & G_{\theta z} \\ G_{z\theta} & G_{zz} \end{bmatrix} = \sum_{n=-\infty}^{+\infty} \left[-\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{L}_n\right]^{-1} e^{jn\theta} \tag{58}$$

where $\mathbf{M}$ the mass matrix, $\mathbf{C}$ the gyroscopic matrix and $\mathbf{L}_n$ the matrix defined by equation (60):

$$\mathbf{M} = \rho S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\; ; \;\; \mathbf{C} = 2\rho S\Omega \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{59}$$

$$\mathbf{L}_n = \begin{pmatrix} \left(\frac{EI}{R^4} + \frac{K}{R^2} + \frac{pb}{R} + \rho S\Omega^2\right) n^2 + \frac{pb}{R} + k_\theta & -j\left[\frac{EI}{R^4}n^3 + \left(\frac{K}{R^2} + \frac{2pb}{R} + 2\rho S\Omega^2\right) n\right] \\ \\ j\left[\frac{EI}{R^4}n^3 + \left(\frac{K}{R^2} + \frac{2pb}{R} + 2\rho S\Omega^2\right) n\right] & \frac{EI}{R^4}n^4 + \left(\frac{pb}{R} + \rho S\Omega^2\right) n^2 + \frac{K}{R^2} + \frac{pb}{R} + k_z \end{pmatrix} \tag{60}$$

Only the normal component G_{zz} of the Green function is computed for the contact problem. The parameters used for the simulation are given in Table (3).

We assume that the contact line contains three points $A(\theta = -\pi/100)$, $B(\theta = 0)$ and $C(\theta = \pi/100)$. The matrix of Green functions is calculated at these three points. Figure (5) shows the Green functions in the frequency rang $[0\ 1000Hz]$. The figures (6) and (7) show the displacements and the contact forces at the three contact points.

Parameters	Values	Unit
Young modulus(E)	10^8	Pa
Density (ρ)	2280	Kgm^{-3}
Mean radius(R)	0.285	m
Thickness(h)	0.01	m
Width(b)	0.16	m
Membrane stiffness(k_z)	1.6410^6	Nm^{-2}
Circumferential stiffness(k_θ)	2.1910^5	Nm^{-2}

Table 3: Parameters used for the numerical simulations

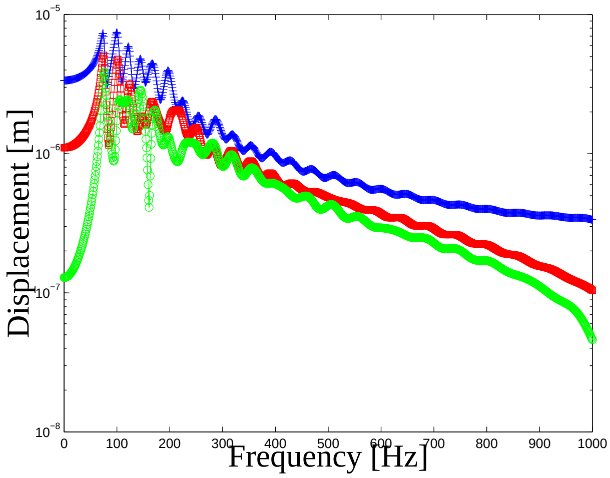

Figure 5: Green functions B : $-+-$ $G_{AA} = G_{BB} = G_{CC}$, -□- $G_{AB} = G_{BC}$,$-o-$ G_{AC}

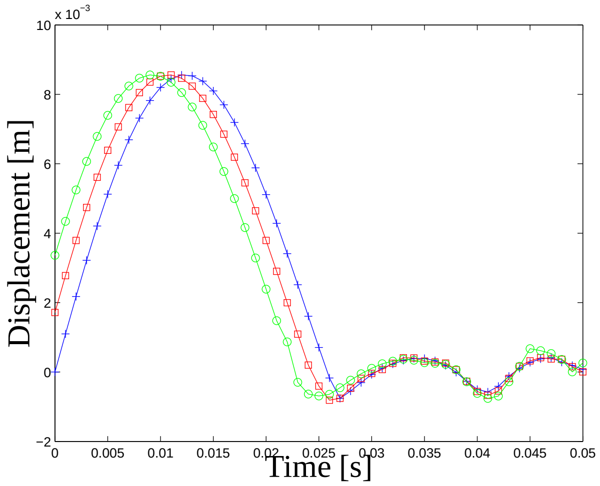

Figure 6: Displacements on the three contact points : — profile, $-+-$ point C, -□- point B , $-o-$ point A

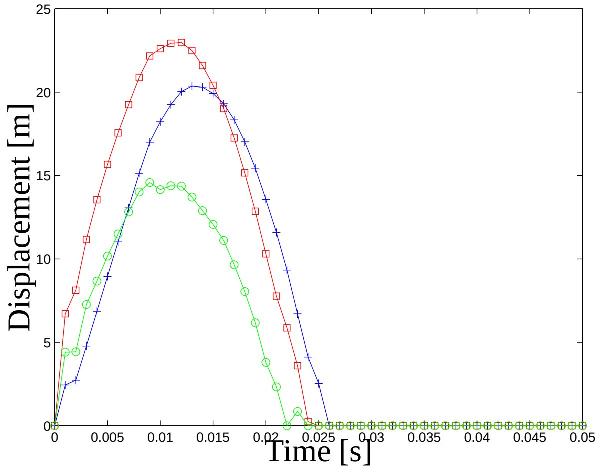

Figure 7: Contact forces on the three contact points : $-+-$ point C, -□- point B , $-o-$ point A

6 Conclusion

A unilateral contact dynamic model has been presented in this contribution. The model is based on a fast convolution using a modal decomposition of the Green functions. First, the Green function is approximated by a sum of a SDoF Green functions using the LSCE algorithm, then the identified modal parameters are used to constract a fast convolution. By exploiting the cinematic condition, the fast convolution can be inverted to calculate the contact forces when the contact occurs. Two examples are presented to prove the efficiency of the model.

REFERENCES

[1] K.J. Bathe and A.B. Chaudhary. A solution method for planar and axisymmetric contact problems. *International Journal for Numerical Methods in Engineering*, 21:65–88, 1985.

[2] Y. Kanto and G. Yakwa. A dynamic contact buckling analysis by the penalty finite element method. *International Journal for Numerical Methods in Engineering*, 29:755–774, 1990.

[3] M. McIntyre, R. Schumacher, and J. Woodhouse. On the oscillations of musical instruments. *Journal of the Acoustical Society of America*, 74 (1983):1325–1345, 1983.

[4] C. Wang and J. Kim. New analysis method for a thin beam impacting against a stop based on the full continuous mode. *Journal of Sound and Vibation*, 1996:809–823, 191.

[5] C. Wang and J. Kim. The dynamic analysis of a thin beam impacting against a stop of general three-dimensional geometry. *Journal of Sound and Vibation*, 203:237–249, 1997.

[6] A. Nordborg. Wheel/rail noise generation due to nonlinear effects and parametric excitation. *Journal of the Acoustical Society of America*, 111:1772–1781, 2002.

[7] W. Kropp. *Ein Modell zur Beschreibung des Rollgerusches eines unprofilierten Grtelreifens auf rauher Strassenoberflche (A model to describe the rolling noise of a smooth radial tire on a rough road surface)*. PhD thesis, T. U. Berlin, 1992.

[8] K. Larsson. *Modelling of dynamic contact-exemplified on tire/road interaction.* PhD thesis, Chalmers University of Technology, Goteborg, Sweden, 2002.

[9] F. Wullens. *Excitation of tyre vibration due to tyre/road interaction*. PhD thesis, Chalmers University of Technology, Goteborg, Sweden, 2004.

[10] P.B.U. Andersson and W. Kropp. Time domain contact model for tyre/road interaction including nonlinear contact stiffness due to small-scale roughness. *Journal of Sound and Vibration*, pages 296–312, 2008.

[11] G. Beylkin, C. Kurcza, and L. Monzona. Fast convolution with the free space Helmholtz Green's function. *Journal of Computational Physics*, 228:2770–2791, 2009.

[12] R. Meftah, D. Duhamel, J. Cesbron, F. Anfosso-Lde, H. Yin, and P. Argoul. Efficient computation of tire/road contact forces using an ARMA model of the Green function. In *XVIIth Symposium Vibrations, Chocs and Bruit, Ecole Centrale de Lyon*, 2010.