COMPDYN 2011
III ECCOMAS Thematic Conference on
Computational Methods in Structural Dynamics and Earthquake Engineering
M. Papadrakakis, M. Fragiadakis, V. Plevris (eds.)
Corfu, Greece, 25–28 May 2011

THE DECOUPLING OF LINEAR DYNAMICAL SYSTEMS

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Keywords: Modal Analysis, Quadratic Eigenvalue Problem, Coordinate Coupling.

Abstract. *The purpose of this paper is report on a recently developed methodology to decouple any linear dynamical system. The decoupling transformation is time-varying and reduces to the well known time-invariant modal transformation for linear systems that are undamped or classically damped.*

1 INTRODUCTION

It has long been recognized that coordinate coupling in damped linear systems is a considerable barrier to analysis and design. In the absence of damping, a linear system possesses classical normal modes, which constitute a linear time-invariant coordinate transformation that decouples the system. This process of decoupling the equation of motion of a dynamical system is a time-honored procedure termed modal analysis. A damped linear system cannot be decoupled by modal analysis unless it also possesses a full set of classical normal modes, in which case the system is said to be classically damped. Rayleigh [1] showed that a system is classically damped if its damping matrix is a linear combination of its inertia and stiffness matrices. Practically speaking, classical damping means that energy dissipation is almost uniformly distributed throughout the system. In general, there is no reason why this condition should be satisfied and thus damped linear systems cannot be decoupled by modal analysis [2-6]. In addition, it has been shown [7] that no time-invariant linear transformations in the configuration space can decouple all damped systems. Even partial decoupling, i.e. simultaneous transformation of the coefficient matrices of the equation of motion to upper triangular forms, cannot be ensured with time-invariant linear transformations [8].

It was shown in [9, 10] that a non-classically damped system in free vibration can be transformed into one with classical damping by synchronizing the phase angles in its non-classically damped modes. This process, termed phase synchronization, generates a real time-varying transformation that decouples the system in configuration space. Furthermore, the decoupling procedure reduces to modal analysis under classical damping. When additional time shifts induced by external forces are accounted for, the decoupling transformation for free vibration can be extended to decouple forced vibration [10]. The purpose of this paper is to summarize the decoupling of all linear systems concisely.

2 PROBLEM STATEMENT

The equation of motion of an n-degree-of-freedom viscously damped linear system can be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real symmetric and positive definite square matrices of order n. The generalized coordinate $\mathbf{q}$ and the excitation $\mathbf{f}(t)$ are real n-dimensional vectors. Unless the three coefficient matrices are diagonal, Eq. (1) is coupled. Coupling is not an inherent property of a system but depends on the generalized coordinates used. The "classical decoupling problem" is concerned with the transformation of Eq. (1) into

$$\ddot{\mathbf{p}} + \mathbf{D}_1\dot{\mathbf{p}} + \mathbf{\Omega}_1\mathbf{p} = \mathbf{g}(t) \tag{2}$$

where $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are real diagonal matrices of order n, and $\mathbf{p}$ and $\mathbf{g}(t)$ are also real. Over the years, various types of decoupling approximation were employed in the analysis of damped systems [11-20]. Different indices of coupling were also introduced to quantify coordinate coupling [21-29]. However, a solution to the "classical decoupling problem" has not been reported in the open literature.

2.1 The quadratic eigenvalue problem

Associated with system (1) is the quadratic eigenvalue problem [30-33]

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0} \tag{3}$$

There are $2n$ eigenvalues λ_j but there cannot be more than n linearly independent eigenvectors $\mathbf{v}_j$, where $j = 1,\ldots,2n$. Because the matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real, any complex eigen-

values must occur in complex conjugate pairs. The quadratic eigenvalue problem plays a key role in decoupling system (1). In fact, all required parameters for the decoupling transformation can be obtained from the solution of the quadratic eigenvalue problem (3).

2.2 Assumptions and possible relaxation

It will be assumed that the $2n$ eigenvalues λ_j of Eq. (3) are distinct. This assumption is made to streamline the introduction of new concepts and it can be readily relaxed. There will be little change to the exposition of decoupling under the less restrictive condition that eigenvectors associated with repeated eigenvalues are independent. When an eigenvalue is repeated m times and a full complement of m independent eigenvectors cannot be found, the eigenvalue problem (3) is said to be defective [30]. Decoupling of systems with defective eigenvalue problems is of a purely theoretical nature but is still relatively straightforward [see 9, 10 for more details].

2.3 Classical modal analysis

To set up the necessary terminology and notation, the decoupling of classically damped systems will be concisely surveyed. Since **M**, **K** are positive definite, solution of the symmetric eigenvalue problem

$$\mathbf{Ku} = \lambda\mathbf{Mu} \tag{5}$$

generates n positive eigenvalues λ_i and n real eigenvectors $\mathbf{u}_i$ that are orthogonal with respect to either **M** or **K**. Upon mass normalization, the eigenvectors constitute the columns of a modal matrix $\mathbf{U} = [\mathbf{u}_1 \,|\, \mathbf{u}_2 \,|\, \cdots \,|\, \mathbf{u}_n]$. Define a modal transformation by

$$\mathbf{q} = \mathbf{Up} \tag{6}$$

In terms of the principal coordinate **p**, Eq. (1) takes the canonical form

$$\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{\Omega p} = \mathbf{U}^T\mathbf{f}(t) \tag{7}$$

where $\mathbf{\Omega} = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$ is the spectral matrix and $\mathbf{D} = \mathbf{U}^T\mathbf{CU}$ is the modal damping matrix. A system is classically damped if it can be decoupled by classical modal analysis, whereby **D** is diagonal. A necessary and sufficient condition [34] for classical damping is $\mathbf{CM}^{-1}\mathbf{K} = \mathbf{KM}^{-1}\mathbf{C}$. There is, of course, no particular reason why this condition should be satisfied. In general, a damped linear system cannot be decoupled by classical modal analysis.

Classical modal analysis utilizes real coordinate transformations and is amenable to physical interpretation. Foss and others [35-37] extended classical modal analysis to a process of complex modal analysis in the state space to treat non-classically damped systems. However, complex modal analysis still cannot decouple all damped linear systems. A condition of non-defective eigenvectors in the state space must be satisfied in order for complex modal analysis to achieve complete decoupling. Furthermore, upon state-space transformation it is generally not possible to classify the $2n$ state variables as displacements and velocities. Physical insight is thus greatly diminished. Perhaps it is even fair to say that decoupling in configuration space renders decoupling in state space unnecessary; the state of a system can always be obtained from displacements and their time derivatives.

3 THE DECOUPLING OF FREE VIBRATION

Suppose all eigenvalues of Eq. (3) are complex and distinct. The eigenvalues λ_j of Eq. (3) and the corresponding eigenvectors $\mathbf{v}_j$ occur in n pairs of complex conjugates. Let

$$\lambda_j = \alpha_j + i\omega_j \tag{8}$$

$$\mathbf{v}_j = [r_{j1}e^{-i\varphi_{j1}} \quad r_{j2}e^{-i\varphi_{j2}} \quad \cdots \quad r_{jn}e^{-i\varphi_{jn}}]^T \tag{9}$$

where α_j, ω_j, r_{jk} and φ_{jk} are real parameters for $j,k = 1,\ldots,n$. The $2n$ complex conjugate eigensolutions $\mathbf{v}_j e^{\lambda_j t}$ are sometimes referred to as the complex modes in configuration space [30, 38, 39]. Two eigensolutions $\mathbf{v}_j e^{\lambda_j t}$ and $\overline{\mathbf{v}}_j e^{\overline{\lambda}_j t}$ combine to generate a non-classically damped mode of vibration [9]. A characteristic of a non-classically damped mode is a constant phase drift between system components in each mode. If suitable phase shifts are introduced into each mode, various components of the system vibrate in a synchronous manner, passing through their equilibrium positions at the same instant of time. In other words, the system is transformed into one with classical damping. A basic objective of a process termed phase synchronization is to do just that [9, 10]. Upon phase synchronization, the resulting classically damped system can be decoupled by modal analysis. When $\mathbf{f}(t) = \mathbf{0}$ and all eigenvalues λ_j of Eq. (3) are complex, Eq. (1) can always be converted into Eq. (2), for which

$$\mathbf{D}_1 = -\text{diag}[\lambda_j + \overline{\lambda}_j] = -\text{diag}[2\alpha_1, 2\alpha_2, ..., 2\alpha_n] \tag{11}$$

$$\mathbf{\Omega}_1 = \text{diag}[\lambda_j \overline{\lambda}_j] = \text{diag}[\alpha_1^2 + \omega_1^2, \alpha_2^2 + \omega_2^2, ..., \alpha_n^2 + \omega_n^2] \tag{12}$$

and $\mathbf{g}(t) = \mathbf{0}$. The free response $\mathbf{q}(t)$ of Eq. (1) can be recovered from the free response $\mathbf{p}(t)$ of Eq. (2) by

$$\mathbf{q}(t) = \sum_{j=1}^{n} \text{diag}[p_j(t - \varphi_{j1}/\omega_j), p_j(t - \varphi_{j2}/\omega_j), \ldots, p_j(t - \varphi_{jn}/\omega_j)]\mathbf{z}_j \tag{13}$$

where

$$\mathbf{z}_j = [r_{j1}e^{\alpha_j \varphi_{j1}/\omega_j} \quad r_{j2}e^{\alpha_j \varphi_{j2}/\omega_j} \quad \cdots \quad r_{jn}e^{\alpha_j \varphi_{jn}/\omega_j}]^T \tag{14}$$

If system (1) is non-oscillatory, both real and complex eigenvalues λ_j of Eq. (3) occur simultaneously. Free vibration with mixed eigenvalues can be decoupled by dividing its eigensolutions into two groups: one associated with the real eigenvalues and the other with complex eigenvalues. The complex eigensolutions are treated by complex conjugation and the real ones by real quadratic conjugation [10].

Specifically, let $2c$ eigenvalues be complex and $2r = 2(n - c)$ be real. Separate the distinct eigenvalues into two disjoint sets so that

$$\{\lambda\} = \{2c \text{ complex } \lambda\} \cup \{2r \text{ real } \lambda\} \tag{15}$$

The complex eigenvalues occur as c pairs of complex conjugates and the real eigenvalues can be classified into primary eigenvalues (the r smallest ones) and secondary eigenvalues (the r largest ones). Enumerate the eigenvalues in such a way that

$$\{2c \text{ complex } \lambda\} = \{\lambda_1, ..., \lambda_c, \lambda_{n+1} = \overline{\lambda}_1, \ldots, \lambda_{n+c} = \overline{\lambda}_c\} \tag{16}$$

$$\{2r \text{ real } \lambda\} = \{\lambda_{c+1} < ... < \lambda_n < \lambda_{n+c+1} = \widetilde{\lambda}_{c+1} < ... < \lambda_{2n} = \widetilde{\lambda}_n\} \tag{17}$$

The above indexing means that the first c eigenvalues are complex and the next $r = n - c$ are the real secondary eigenvalues. These are followed by c complex conjugates of the first set and finally by r real primary eigenvalues. Simultaneous phase synchronization of all damped modes [9, 10] produces a decoupled system (2) for which

$$\mathbf{D}_1 = -\text{diag}[\lambda_j + \overline{\lambda}_j, \lambda_k + \widetilde{\lambda}_k] = -\text{diag}[\lambda_1 + \lambda_{n+1}, \lambda_2 + \lambda_{n+2}, \ldots, \lambda_n + \lambda_{2n}] \tag{18}$$

$$\mathbf{\Omega}_1 = \text{diag}[\lambda_j \overline{\lambda}_j, \lambda_k \widetilde{\lambda}_k] = \text{diag}[\lambda_1 \lambda_{n+1}, \lambda_2 \lambda_{n+2}, \ldots, \lambda_n \lambda_{2n}] \tag{19}$$

The decoupling transformation is given by Eq. (13). See [10] for details on how to compute the parameters α_j, ω_j, r_{jk} and φ_{jk} using the concept of real quadratic conjugation.

4 DECOUPLING OF FORCED VIBRATION

It has been shown [9, 10] that a damped system in free vibration can be decoupled if suitable phase shifts are introduced into each damped mode of vibration so that all components are either in phase or out of phase. This process is termed phase synchronization, and its purpose is to compensate for the time drifts caused by viscous damping. The required phase drifts are constant; that is why the time-varying decoupling transformation for free vibration in Eq. (13) is merely time-shifting. In the presence of an external excitation $\mathbf{f}(t)$, the interplay between these time drifts and $\mathbf{f}(t)$ generates a genuinely time-varying decoupling transformation for forced vibration [10].

Postulate that system (1) is decoupled into the form (2), where the diagonal matrices $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are still given by Eqs. (18) and (19). What is the relationship between $\mathbf{f}(t)$ and $\mathbf{g}(t)$? How is the decoupling transformation (13) generalized? While it is possible to investigate these issues in the configuration space, with the theoretical framework that has been set up it is more efficient to perform additional manipulations in state space.

Let $2c$ eigenvalues be complex, $2r = 2(n-c)$ be real and enumerate the eigenvalues as in Eqs. (16) – (17). Without loss of generality, normalize the eigenvectors according to

$$2\lambda_j \mathbf{v}_j^T \mathbf{M}\mathbf{v}_j + \mathbf{v}_j^T \mathbf{C}\mathbf{v}_j = \lambda_j - \lambda_{n+j} \tag{20}$$

$$2\lambda_{n+j} \mathbf{v}_{n+j}^T \mathbf{M}\mathbf{v}_{n+j} + \mathbf{v}_{n+j}^T \mathbf{C}\mathbf{v}_{n+j} = \lambda_{n+j} - \lambda_j \tag{21}$$

for $j = 1,\ldots,n$. The above normalization reduces to mass normalization if system (1) is undamped or classically damped [6, 9, 10]. Cast Eq. (1) in state space in the symmetric form

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix} \tag{22}$$

Define a state transformation by

$$\begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} (\mathbf{V}\overline{\mathbf{\Lambda}} - \overline{\mathbf{V}}\mathbf{\Lambda})(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} & (\overline{\mathbf{V}} - \mathbf{V})(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} \\ (\mathbf{V} - \overline{\mathbf{V}})\mathbf{\Lambda}\overline{\mathbf{\Lambda}}(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} & (\overline{\mathbf{V}}\overline{\mathbf{\Lambda}} - \mathbf{V}\mathbf{\Lambda})(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} \tag{23}$$

where

$$\mathbf{V} = [\mathbf{v}_n \,|\, \cdots \,|\, \mathbf{v}_n],\ \mathbf{V}^* = [\mathbf{v}_{n+1} \,|\, \cdots \,|\, \mathbf{v}_{2n}] \tag{24}$$

$$\mathbf{\Lambda} = \mathrm{diag}[\lambda_1,\ldots,\lambda_n],\ \mathbf{\Lambda}^* = \mathrm{diag}[\lambda_{n+1},\ldots,\lambda_{2n}] \tag{25}$$

Equation (22) becomes

$$\begin{bmatrix} \mathbf{D}_1 & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{\Omega}_1 & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1^T \mathbf{f} \\ \mathbf{T}_2^T \mathbf{f} \end{bmatrix} \tag{26}$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ are given by

$$\mathbf{T}_1 = (\mathbf{V}\mathbf{\Lambda}^* - \mathbf{V}^*\mathbf{\Lambda})(\mathbf{\Lambda}^* - \mathbf{\Lambda})^{-1} \tag{27}$$

$$\mathbf{T}_2 = (\mathbf{V}^* - \mathbf{V})(\mathbf{\Lambda}^* - \mathbf{\Lambda})^{-1} \tag{28}$$

and where $\mathbf{D}_1$, $\mathbf{\Omega}_1$ are given by Eqs. (18)–(19) (see [10] for details of the calculation). The upper and lower halves of Eq. (26) are

$$\mathbf{q} = \mathbf{T}_1\mathbf{p} + \mathbf{T}_2\dot{\mathbf{p}} - \mathbf{T}_2\mathbf{T}_2^T\mathbf{f} \tag{29}$$

$$\mathbf{q} = \mathbf{T}_1\mathbf{p}_1 + \mathbf{T}_2\mathbf{p}_2 \tag{30}$$

Eliminate $\mathbf{p}_2$ from the above two equations to get

$$\ddot{\mathbf{p}}_1 + \mathbf{D}_1\dot{\mathbf{p}}_1 + \mathbf{\Omega}_1\mathbf{p}_1 = \mathbf{T}_1^T\mathbf{f} + \mathbf{T}_2^T\dot{\mathbf{f}} \tag{31}$$

When Eqs. (2) and (31) are compared, it becomes obvious that $\mathbf{p}_1$ can be identified with $\mathbf{p}$. Therefore, the relationship between $\mathbf{f}(t)$ and $\mathbf{g}(t)$ is

$$\mathbf{g}(t) = \mathbf{T}_1^T \mathbf{f}(t) + \mathbf{T}_2^T \dot{\mathbf{f}}(t) \tag{32}$$

Observe that $\mathbf{T}_1$, $\mathbf{T}_2$, and hence $\mathbf{g}(t)$ can be readily obtained from the solution of the quadratic eigenvalue problem (3). Upon substitution of Eq. (32) in (2), forced vibration with complex eigenvalues is decoupled in the configuration space.

The relationship between the solution $\mathbf{p}(t)$ of Eq. (2) and solution $\mathbf{q}(t)$ of the original system (1) can be deduced from Eqs. (23), (30) and (31)

$$\mathbf{q} = \mathbf{T}_1 \mathbf{p} + \mathbf{T}_2 \dot{\mathbf{p}} - \mathbf{T}_2 \mathbf{T}_2^T \mathbf{f} \tag{33}$$

The above expression represents a time-varying decoupling transformation in the configuration space. A closer examination of Eq. (33) reveals that it consists of two parts. The first part, $\mathbf{T}_1 \mathbf{p} + \mathbf{T}_2 \dot{\mathbf{p}}$, accounts for time drifts caused by viscous damping in free vibration. The second part, $\mathbf{T}_2 \mathbf{T}_2^T \mathbf{f}$, accounts for time drifts induced by the excitation $\mathbf{f}(t)$. When $\mathbf{f}(t) = \mathbf{0}$, Eq. (33) reduces to $\mathbf{q} = \mathbf{T}_1 \mathbf{p} + \mathbf{T}_2 \dot{\mathbf{p}}$ which, by direct manipulations, is the same as Eq. (13). Thus, the decoupling transformation (33) generalizes the decoupling transformation (13) for free vibration and, therefore, represents **a general decoupling transformation for all non-defective linear dynamical systems**.

To cast Eq. (33) in state space, simply rewrite Eq. (23) in the form

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V\Lambda} & \overline{\mathbf{V\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{T}_2^T \mathbf{f}(t) \end{bmatrix} \tag{34}$$

Initial conditions of $\mathbf{p}(t)$ in Eq. (2) and $\mathbf{q}(t)$ in the original system (1) can be connected by putting $t = 0$ in Eq. (34). A flowchart depicting the decoupling method is shown in Fig. 1.

5 REDUCTION TO CLASSICAL MODAL ANALYSIS

The decoupling methodology developed herein is a direct generalization of classical modal analysis. Without loss of generality, assume that all eigenvectors are normalized in accordance with either Eqs. (20)-(21). When real and complex eigenvalues of Eq. (3) occur simultaneously, and the system is classically damped, $\mathbf{V} = \mathbf{V}^* = \mathbf{U}$ [10]. Using Eqs. (27) and (28), it can now be shown that the transformation of excitation in Eq. (32) reduces to $\mathbf{g}(t) = \mathbf{U}^T \mathbf{f}(t)$, while the decoupling transformation in Eq. (33) simplifies to the classical modal transformation $\mathbf{q} = \mathbf{U}\mathbf{p}$. Thus, the decoupling method presented reduces to classical modal analysis for systems that are classically damped.

6 EFFICIENCY OF SOLUTION BY DECOUPLING

Response calculation is probably not the most important reason for decoupling. It is the possibility, for example, of modal reduction and of an investigation of energy distribution among independent coordinates that would make decoupling worthwhile. Nevertheless, it may still be instructive to examine the efficiency of solution of Eq. (1) by decoupling. It will be assumed that the excitation $\mathbf{f}(t)$ and response $\mathbf{q}(t)$ are sufficiently smooth (adequate if twice differentiable).

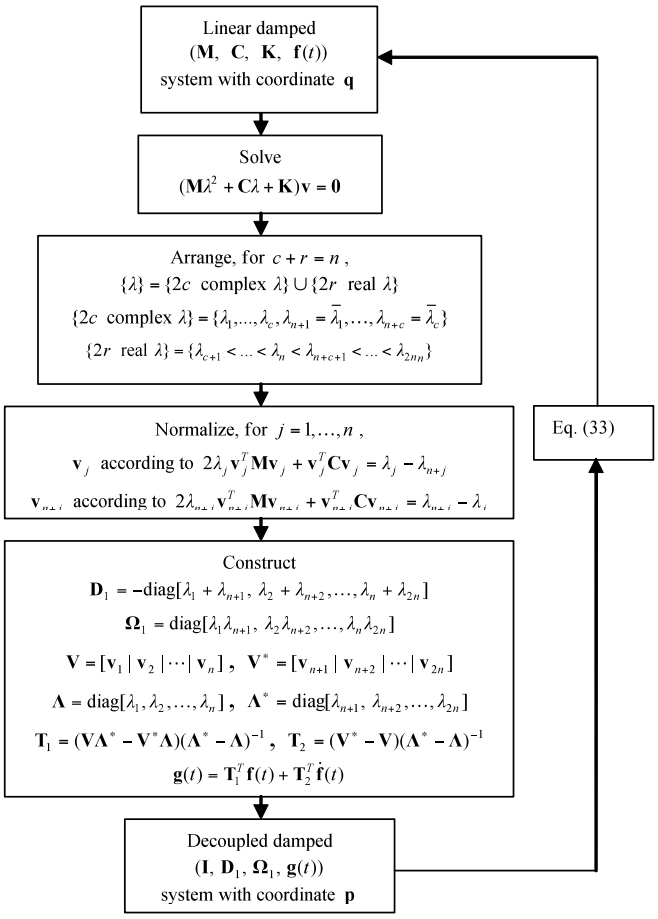

Figure 1: Algorithm for decoupling all linear systems.

One measure of the performance of an algorithm is the number of flops (floating point operations) required to evaluate the response at m points within a given time window. The flops associated with two procedures are compared. (a) In direct numerical integration, a standard procedure is to recast the second-order Eq. (1) in state space as a first-order system of dimension $2n$. The state equation is then discretized, and the resulting difference equation is solved by matrix computations [40]. The estimate of flops of this procedure at m instants is [9, 10, 41-43] $N_1 = 160n^3 + 16mn^2$, where n is the number of degrees of freedom and $m >> n$ in general. (b) In solving Eq. (1) by decoupling, Eq. (2) is obtained through solution of Eq. (3) and evaluation of Eq. (32). Each independent decoupled system in Eq. (2) is then solved numerically at m instants with the same algorithm used in procedure (a). Subsequently, Eq. (33) is employed to compute the response $\mathbf{q}(t)$. The estimate of flops is [9, 10, 41-43] $N_2 = 10mn^2 + 16mn + 213n^3 + 4n^2$. The variations of N_1 and N_2 with n are illustrated in Fig. 2 for a window containing $m = 10^6$ instants.

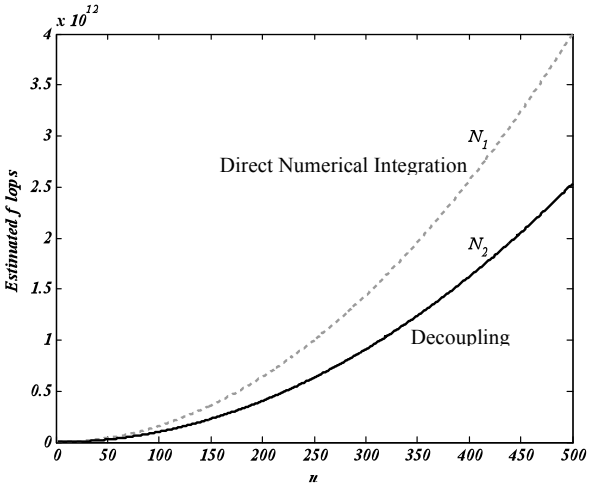

Figure 2: Comparison of efficiency in response calculation.

It is observed that response calculation by decoupling generally reduces the flops and therefore economizes on both core memory and computing time.

In fact, Fig. 2 is rather conservative because N_2 has been estimated by using the same time step in the integration of all decoupled equations. If an optimal time step is individually chosen for each decoupled equation, N_2 may decrease substantially. Moreover, each decoupled equation may be solved exactly in many applications. On the other hand, the efficiency of response calculation by decoupling depends on the size of the time window. In addition, validity of the flop estimates requires that the excitation is $\mathbf{f}(t)$ sufficiently smooth. Distributional excitation such as an impulse leading to weak solutions ($\mathbf{q}(t)$ less than twice differentiable) are excluded [44, 45]. Thus Fig. 2 should be interpreted as indicative rather than absolute in the comparison of efficiency.

7 NUMERICAL EXAMPLE

A four-degree-of-freedom system possessing both real and complex eigenvalues is decoupled to illustrate the method. The example is taken from [10]. In Eq. (1), let $\mathbf{M} = \mathbf{I}$,

$$\mathbf{C} = \begin{bmatrix} 0.1 & -0.1 & 0 & 0 \\ -0.1 & 0.2 & -0.1 & 0 \\ 0 & -0.1 & 0.2 & -0.1 \\ 0 & 0 & -0.1 & 1.35 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1.1 \end{bmatrix} \tag{35}$$

and $\mathbf{f}(t) = [0,0,0,1]^T te^{-0.3t} \sin 2t$ and assume zero initial conditions $\mathbf{q}(0) = \mathbf{0}$, $\dot{\mathbf{q}}(0) = \mathbf{0}$. This system is non-classically damped and, therefore, can only be decoupled by the method presented herein. Solution of the quadratic eigenvalue problem (3) yields $c = 3$ pairs of complex conjugate and $r = 1$ pairs of real quadratic conjugate eigenvalues and eigenvectors [10]. From Eqs. (27) and (28),

$$\mathbf{T}_1 = \begin{bmatrix} -0.38 & 0.53 & 0.30 & 0.70 \\ -0.05 & -0.45 & -0.70 & 0.68 \\ 0.52 & -0.66 & 0.64 & 0.64 \\ 1.11 & 0.37 & -0.16 & 0.57 \end{bmatrix} \tag{36}$$

$$\mathbf{T}_2 = \begin{bmatrix} 0.49 & -0.07 & -0.02 & 0.53 \\ 0.71 & -0.07 & 0.03 & 0.31 \\ 0.90 & 0.15 & 0.02 & -0.17 \\ 0.70 & 0.27 & -0.09 & -0.93 \end{bmatrix} \tag{37}$$

It can be checked that the decoupled $(\mathbf{I},\ \mathbf{D}_1,\ \mathbf{\Omega}_1,\ \mathbf{g}(t))$ system is given by

$$\mathbf{D}_1 = -\text{diag}[\lambda_j + \bar{\lambda}_j\,(j=1,2,3), \lambda_4 + \tilde{\lambda}_4] = \text{diag}[0.65, 0.40, 0.39, 0.41] \tag{38}$$

$$\mathbf{\Omega}_1 = \text{diag}[\lambda_j \bar{\lambda}_j\,(j=1,2,3), \lambda_4 \tilde{\lambda}_4] = \text{diag}[0.46, 1.75, 3.32, 0.04] \tag{39}$$

$$\mathbf{g}(t) = e^{-0.3t} \begin{bmatrix} 1.4t\cos(2t) + (0.70 + 0.90t)\sin(2t) \\ 0.54t\cos(2t) + (0.27 + 0.29t)\sin(2t) \\ -0.18t\cos(2t) - (0.09 + 0.13t)\sin(2t) \\ -1.86t\cos(2t) + (-0.93 + 0.85t)\sin(2t) \end{bmatrix} \tag{40}$$

\with initial conditions $\mathbf{p}(0) = \mathbf{0}$, $\dot{\mathbf{p}}(0) = \mathbf{0}$. The decoupled system can be readily solved and solution $\mathbf{q}(t)$ of the original $(\mathbf{M}, \mathbf{C}, \mathbf{K}, \mathbf{f}(t))$ system can be recovered from $\mathbf{p}(t)$ by Eq. (33). As shown in Fig. 3, $\mathbf{g}(t)$, $\mathbf{p}(t)$, and $\mathbf{q}(t)$ are all oscillatory.

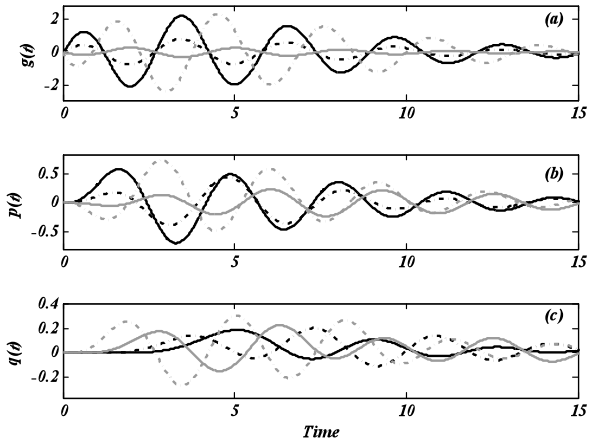

Figure 3: Simulation results.

It can be verified that $\mathbf{q}(t)$, whether generated by decoupling or by direct numerical integration, is the same.

8 CONLUSIONS

A method has been presented to decouple all linear dynamical systems under viscous damping. The decoupling methodology developed herein possesses ample physical insight and it also lends itself to numerical computations. If the linear system is in free vibration, the time varying decoupling transformation in Eq. (33) reduces to a time shifting transformation in Eq. (13). Under classical damping, the decoupling method reduces to modal analysis. Major findings of this paper are summarized in the following statements.

1. In free or forced vibration, all parameters required for the decoupling of a linear system are obtained through the solution of a quadratic eigenvalue problem.

2. Any viscously damped linear system (no restrictions) can be completely decoupled. A flowchart outlining the decoupling procedure is given in Fig. 1.
3. While damped linear systems are decoupled in the configuration space, transformations of initial conditions are prescribed in state space (otherwise initial values are connected at different time instants due to phase synchronization).

To streamline the introduction of new concepts, most formulas have been established with the assumption that eigenvalues of the quadratic eigenvalue problem (3) are distinct. These formulas remain unchanged when Eq. (3) is non-defective (each repeated eigenvalue possesses a full complement of independent eigenvectors). However, a system for which Eq. (3) is defective can still be decoupled by the method presented herein. Due to the lack of practical importance of defective systems (a system is non-defective with probability 1) and because the mathematics becomes quite involved, details on the decoupling of defective systems are deferred to a future paper on general mode-by-mode transformations

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