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NONLINEAR EFFECTS IN ELASTIC FLEXURAL – TORSIONAL VIBRATIONS OF BEAMS OF ARBITRARY CROSS SECTION

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Abstract. *In this paper a boundary element method is developed for the nonlinear flexural-torsional dynamic analysis of beams of arbitrary, simply or multiply connected, constant cross section, undergoing moderate large deflections and rotations under general boundary conditions, taking into account the effects of rotary and torsional warping inertia. Four boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to the angle of twist and solved using the Analog Equation Method, a BEM based method. The geometric, inertia, torsion and warping constants are evaluated employing the Boundary Element Method. The proposed model takes into account, both the Wagner's coefficients and the shortening effect. Numerical examples are worked out to illustrate the efficiency, wherever possible the accuracy, the range of applications of the developed method as well as the influence of the nonlinear effects to the response of the beam.*

1 INTRODUCTION

In engineering practice, we often come across the analysis of beam structures subjected to vibratory loading. This problem becomes much more complicated in the case the cross section's centroid does not coincide with its shear center (asymmetric beams), leading to the formulation of the flexural-torsional vibration problem. Besides, when arbitrary torsional boundary conditions are applied either at the edges or at any other interior point of the bar due to construction requirements, this bar under the action of general twisting loading is leaded to nonuniform torsion. Moreover, since weight saving is of paramount importance, frequently used thin-walled open section beams have low flexural and/or torsional stiffness and their deformations can be of such magnitude that it is not adequate to treat the cross section displacements and its angle of rotation as small. In these cases, the study of nonlinear effects on these members becomes essential, where this non-linearity results from retaining the nonlinear terms in the strain–displacement relations (finite displacement – small strain theory). When finite displacements are considered, the flexural-torsional dynamic analysis of bars becomes much more complicated, leading to the formulation of coupled and nonlinear flexural, torsional and axial equilibrium equations.

When the displacement components of a member are small, a wide range of linear analysis tools, such as modal analysis, can be used and some analytical results are possible. As these components become larger, the induced geometric nonlinearities result in effects that are not observed in linear systems. In such situations the possibility of an analytical solution method is significantly reduced and is restricted to special cases of beam boundary conditions or loading.

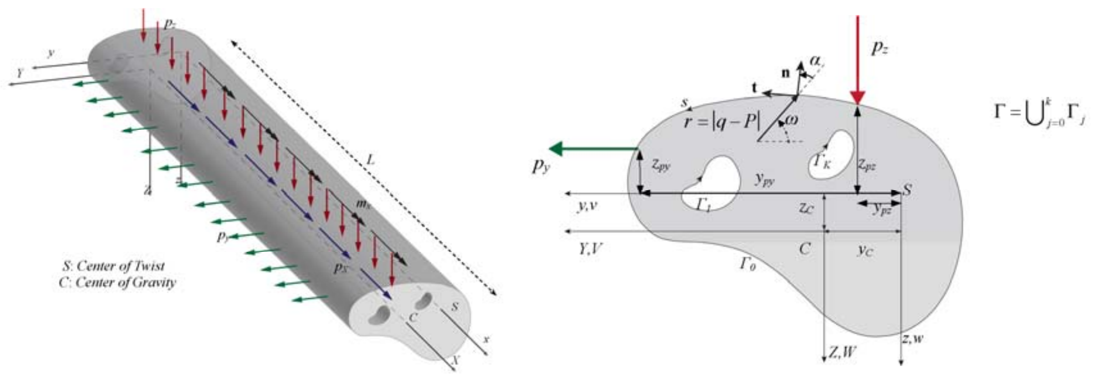
During the past few years, the nonlinear dynamic analysis of beams undergoing large deflections has received a good amount of attention in the literature. More specifically, Rozmarynowski and Szymczak in [1] studied the nonlinear free torsional vibrations of axially immovable thin-walled beams with doubly symmetric open cross section, employing the Finite Element Method. In this research effort only free vibrations are examined, the solution is provided only at points of reversal of motion (not in the time domain), no general axial, torsional or warping boundary conditions (elastic support case) are studied, while some nonlinear terms related to the finite twisting rotations as well as the axial inertia term are ignored. Crespo Da Silva in [2-3] presented the nonlinear flexural-torsional-extensional vibrations of Euler-Bernoulli doubly symmetric thin-walled closed cross section beams, primarily focusing to flexural vibrations and neglecting the effect of torsional warping. Pai and Nayfeh in [4-6] studied also the nonlinear flexural-torsional-extensional vibrations of metallic and composite slewing or rotating closed cross section beams, primarily focusing to flexural vibrations and neglecting again the effect of torsional warping. Simo and Vu-Quoc in [7] presented a FEM solution to a fully nonlinear (small or large strains, hyperelastic material) three dimensional rod model including the effects of transverse shear and torsion-warping deformation based on a geometrically exact description of the kinematics of deformation. Qaisi in [8] obtained nonlinear normal modes of free vibrating geometrically nonlinear beams of various edge conditions employing the harmonic balance analytical method. Moreover, Pai and Nayfeh in [9] studied a geometrically exact nonlinear curved beam model for solid composite rotor blades using the concept of local engineering stress and strain measures and taking into account the in-plane and out-of-plane warpings. Di Egidio et al. in [10-11] presented also a FEM solution to the nonlinear flexural-torsional vibrations of shear undeformable thin-walled open beams taking into account in-plane and out-of-plane warpings and neglecting warping inertia. In this paper, the torsional-extensional coupling is taken into account but the inextensionality assumption leads to the fact that the axial boundary conditions are not general. Mohri et. al. in [12] proposed a FEM solution to the linear vibration analysis of pre- and post- buckled thin-walled open cross section beams, neglecting warping and axial inertia, considering geometrical nonlinearity only for the static loading and presenting examples of bars subjected to free vibrations and special boundary conditions. Machado and Cortinez in [13] presented also a FEM solution to the linear free vibration analysis of bisymmetric thin-walled composite beams of open shaped cross section, taking into account static initial stresses and deformations considering geometrical nonlinearity only for the static loading and presenting examples of bars subjected to special boundary conditions. Avramov et. al. [14-15] studied the free flexural-torsional vibrations of beams and obtained nonlinear normal modes by expansion of the equations of motion employing the Galerkin technique and neglecting the cross-section warping. Lopes and Ribeiro [16] studied also the nonlinear flexural-torsional free vibrations of beams employing a FEM solution and neglecting the longitudinal and rotary inertia as well as the cross-section warping. Duan [17] presented a FEM formulation for the nonlinear free vibration problem of thin-walled curved beams of asymmetric cross-section based on a simplified displacement field. Finally, the boundary element method has also been used for the nonlinear flexural [18-20] and torsional [21] dynamic analysis of only doubly symmetric beams. To the authors' knowledge the general problem of coupled nonlinear flexural – torsional free or forced vibrations of asymmetric beams has not yet been presented.

In this paper, a boundary element method is developed for the nonlinear flexural-torsional dynamic analysis of beams of arbitrary, simply or multiply connected, constant cross section, undergoing moderate large deflections and twisting rotations under general boundary conditions, taking into account the effects of rotary and warping inertia. The beam is subjected to the combined action of arbitrarily distributed or concentrated transverse loading in both directions as well as to twisting and/or axial loading. Four boundary value problems are formulated with respect to the transverse displacements, to the axial displacement and to the angle of twist and solved using the Analog Equation Method [22], a BEM based method. Application of the boundary element technique leads to a system of nonlinear coupled Differential – Algebraic Equations (DAE) of motion, which is solved iteratively using the Petzold-Gear Backward Differentiation Formula (BDF) [23], a linear multistep method for differential equations coupled to algebraic equations (DAE). The geometric, inertia, torsion and warping constants are evaluated employing the Boundary Element Method. The essential features and novel aspects of the present formulation compared with previous ones are summarized as follows.

i. The cross section is an arbitrarily shaped thin or thick walled one. The formulation does not stand on the assumption of a thin-walled structure and therefore the cross section's torsional and warping rigidities are evaluated "exactly" in a numerical sense.
ii. The beam is subjected to an arbitrarily distributed or concentrated transverse loading and bending moments in both directions as well as to axial loading.
iii. The beam is supported by the most general boundary conditions including elastic support or restraint.
iv. For the first time in the literature, the effects of rotary and warping inertia are taken into account on the nonlinear flexural-torsional dynamic analysis of asymmetric beams subjected to arbitrary loading and boundary conditions.

v. The transverse loading can be applied at any arbitrary point of the beam cross section. The eccentricity change of the transverse loading during the torsional beam motion, resulting in additional torsional moment is taken into account.
vi. The proposed model takes into account the coupling effects of bending, axial and torsional response of the beam as well as the Wagner's coefficients and the shortening effect.
vii. The proposed method employs a BEM approach (requiring boundary discretization for the cross sectional analysis) resulting in line or parabolic elements instead of area elements of the FEM solutions (requiring the whole cross section to be discretized into triangular or quadrilateral area elements), while a small number of line elements are required to achieve high accuracy.

Numerical examples are worked out to illustrate the efficiency, wherever possible the accuracy, the range of applications of the developed method as well as the influence of the nonlinear effects to the response of the beam.



(a) (b)

Figure1: Prismatic beam in axial - flexural - torsional loading (a) of an arbitrary cross-section occupying the two dimensional region Ω (b).

3 STATEMENT OF THE PROBLEM

Let us consider a prismatic beam of length L (Fig.1), of constant arbitrary cross section of area A. The homogeneous isotropic and linearly elastic material of the beam's cross section, with modulus of elasticity E, shear modulus G and Poisson's ratio v occupies the two dimensional multiply connected region Ω of the y,z plane and is bounded by the Γ_j ($j=1,2,...,K$) boundary curves, which are piecewise smooth, i.e. they may have a finite number of corners. In Fig. 1 CYZ is the principal bending coordinate system through the cross section's centroid C, while y_C, z_C are its coordinates with respect to the Syz shear system of axes through the cross section's shear center S, with axes parallel to those of the CYZ system. The beam is subjected to the combined action of the arbitrarily distributed or concentrated, time dependent and conservative axial loading $p_X = p_X(X,t)$ along X direction, twisting moment $m_x = m_x(x,t)$ along x direction and transverse loading $p_y = p_y(x,t)$, $p_z = p_z(x,t)$ acting along the y and z directions, respectively, applied at distances y_{p_y}, z_{p_y} and y_{p_z}, z_{p_z}, with respect to the Syz shear system of axes (Fig. 1b). It is worth here noting that the aforementioned transverse loading can be applied at any arbitrary point of the beam's cross section and not necessarily at its centroid or at its shear center. In the case of a linear analysis, where the beam deflections and rotations are considered to be small, in order to overcome the fact that the external transverse loading is usually applied as tractions upon the surface of the beam, the superposition principle is adopted and the actual applied system of forces is replaced by a statically equivalent one, acting on the centroid or on the shear center of the beam (Fig. 2a). However in nonlinear analysis, where twisting rotations are considered to be large, the occurring change of eccentricity (Fig. 2b) may have substantial influence on the beam response and must be taken into account.

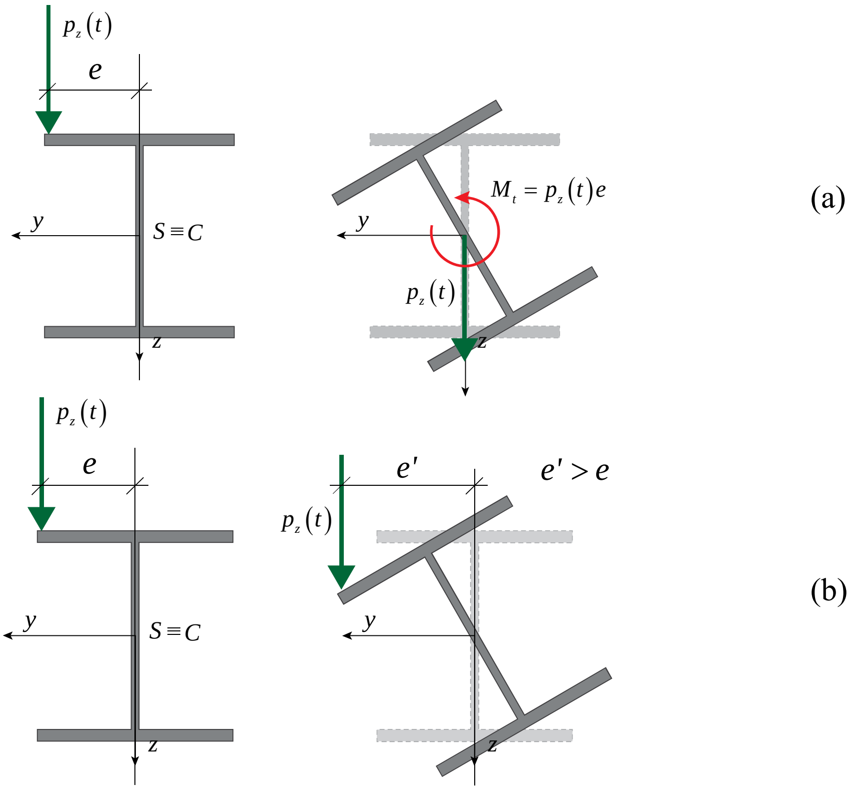


Figure 2: Superposition principle in linear analysis (a) and change of eccentricity of the transverse force p_z during the twisting motion of the cross section in nonlinear analysis (b).

Under the action of the aforementioned loading and employing the Euler-Bernoulli assumption, the displacement field of an arbitrary point of the cross section can be derived with respect to those of the shear center as [12]

$$\bar{u}(x,y,z,t) = u(x,t) - (y - y_C)\theta_Z(x,t) + (z - z_C)\theta_Y(x,t) + \theta_x'(x,t)\varphi_S^P(y,z) \tag{1a}$$

$$\bar{v}(x,y,z,t) = v(x,t) - z\sin(\theta_x(x,t)) - y\left[1 - \cos(\theta_x(x,t))\right] \tag{1b}$$

$$\bar{w}(x,y,z,t) = w(x,t) + y\sin(\theta_x(x,t)) - z\left[1 - \cos(\theta_x(x,t))\right] \tag{1c}$$

$$\theta_Y(x,t) = v'(x,t)\sin(\theta_x(x,t)) - w'(x,t)\cos(\theta_x(x,t)) \tag{1d}$$

$$\theta_Z(x,t) = v'(x,t)\cos(\theta_x(x,t)) + w'(x,t)\sin(\theta_x(x,t)) \tag{1e}$$

where $\bar{u}$, $\bar{v}$, $\bar{w}$ are the axial and transverse beam displacement components with respect to the Syz shear system of axes; $u(x,t) = \frac{1}{A}\int_A \bar{u}(x,y,z,t)\,dA$ denotes the average axial displacement of the cross section [24] and $v = v(x,t)$, $w = w(x,t)$ are the corresponding components of the shear center S ; $\theta_Y(x,t)$, $\theta_Z(x,t)$ are the angles of rotation of the cross section due to bending, with respect to its centroid; $\theta_x'(x,t)$ denotes the rate of change of the angle of twist $\theta_x(x,t)$ regarded as the torsional curvature and φ_S^P is the primary warping function with respect to the shear center S [25]. Employing the strain-displacement relations of the three-dimensional elasticity for moderate displacements [26, 27], the strain components can be written as

$$\begin{aligned}\varepsilon_{xx} = u' + (z - z_C)(v''\sin\theta_x - w''\cos\theta_x) - (y - y_C)(v''\cos\theta_x + w''\sin\theta_x) + \\ + \theta_x''\phi_S^P - \theta_x'\left(z_C(v'\cos\theta_x + w'\sin\theta_x) + y_C(v'\sin\theta_x - w'\cos\theta_x)\right) + \\ + \frac{1}{2}\left(v'^2 + w'^2 + (y^2 + z^2)(\theta_x')^2\right)\end{aligned} \tag{2a}$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \left(\frac{\partial\phi_S^P}{\partial y} - z\right)\theta_x' \tag{2b}$$

$$\gamma_{xz} = 2\varepsilon_{xz} = \left(\frac{\partial\phi_S^P}{\partial z} + y\right)\theta_x' \tag{2c}$$

while considering strains to be small, the non vanishing stress components of the second Piola – Kirchhoff stress tensor are obtained as

$$\begin{aligned}S_{xx} = E\Big[u' + (z - z_C)(v''\sin\theta_x - w''\cos\theta_x) - (y - y_C)(v''\cos\theta_x + w''\sin\theta_x) + \\ + \theta_x''\phi_S^P - \theta_x'\left(z_C(v'\cos\theta_x + w'\sin\theta_x) + y_C(v'\sin\theta_x - w'\cos\theta_x)\right) + \\ + \frac{1}{2}\left(v'^2 + w'^2 + (y^2 + z^2)(\theta_x')^2\right)\Big]\end{aligned} \tag{3a}$$

$$S_{xy} = G \cdot \theta_x' \cdot \left(\frac{\partial\phi_S^P}{\partial y} - z\right) \tag{3b}$$

$$S_{xz} = G \cdot \theta_x' \cdot \left(\frac{\partial\phi_S^P}{\partial z} + y\right) \tag{3c}$$

In order to establish the nonlinear equations of motion, the principle of virtual work

$$\delta W_{\text{int}} + \delta W_{mass} = \delta W_{ext} \tag{4}$$

where

$$\delta W_{\text{int}} = \int_V \left(S_{xx}\delta\varepsilon_{xx} + S_{xy}\delta\gamma_{xy} + S_{xz}\delta\gamma_{xz}\right) dV \tag{5a}$$

$$\delta W_{mass} = \int_V \rho\left(\ddot{\bar{u}}\delta\bar{u} + \ddot{\bar{v}}\delta\bar{v} + \ddot{\bar{w}}\delta\bar{w}\right) dV \tag{5b}$$

$$\delta W_{ext} = \int_L \left(p_X\delta\bar{u}_C + p_y\delta\bar{v}_{p_y} + p_z\delta\bar{w}_{p_z} + m_x\delta\theta_x\right) dx \tag{5c}$$

under a Total Lagrangian formulation, is employed. In the above equations, $\delta(\cdot)$ denotes virtual quantities, $(\dot{\ })$ denotes differentiation with respect to time, V is the volume of the beam, $\bar{u}_C$ is the axial displacement of the centroid and $\bar{v}_{p_y}$, $\bar{w}_{p_z}$ are the transverse displacements of the points where the loads p_y, p_z, respectively, are applied. It is worth here noting that the aforementioned relation of the external work δW_{ext} (eqn. (5c)) is expressed in terms of the applied external forces and virtual quantities of the kinematical components in the deformed configuration of the beam. This expression takes into account the change of the eccentricity of the external conservative transverse loading, arising from the cross section torsional rotation, inducing additional (positive or negative) torsional moment (Fig. 2b). Substituting the expressions of virtual quantities in eqn. (5c), the external work can be written as

$$\begin{aligned}\delta W_{ext} = \int_L \Big[& p_X \delta u + p_y \delta v + p_z \delta w + \Big(p_y \left(-z_{p_y} \cos\theta_x - y_{p_y} \sin\theta_x \right) + \\ & + p_z \left(y_{p_z} \cos\theta_x - z_{p_z} \sin\theta_x \right) + m_x \Big) \delta\theta_x \Big] dx\end{aligned} \tag{6}$$

Expanding the trigonometric functions in terms of Taylor series and keeping the first two terms [12], the following approximate expressions are obtained

$$\cos\theta_x = 1 - \frac{\theta_x^{\ 2}}{2!} = 1 - \frac{\theta_x^{\ 2}}{2} \tag{7a}$$

$$\sin\theta_x = \theta_x - \frac{\theta_x^{\ 3}}{3!} = \theta_x - \frac{\theta_x^{\ 3}}{6} \tag{7b}$$

Using equations (7), equation (6) can be written as

$$\begin{aligned}\delta W_{ext} = \int_L \Bigg[& p_X \delta u + p_y \delta v + p_z \delta w + \Bigg(p_y \left(-z_{p_y} - y_{p_y}\theta_x + \frac{1}{2}\theta_x^{\ 2} z_{p_y} + \frac{1}{6}\theta_x^{\ 3} y_{p_y} \right) + \\ & + p_z \left(y_{p_z} - z_{p_z}\theta_x - \frac{1}{2}\theta_x^{\ 2} y_{p_z} + \frac{1}{6}\theta_x^{\ 3} z_{p_z} \right) + m_x \Bigg) \delta\theta_x \Bigg] dx\end{aligned} \tag{8}$$

As it can be observed from equation (8), neglecting all the nonlinear terms leads to the well known linear expression of the external twisting moment, arising from the superposition principle. In the present study, these terms are retained, so that the change of eccentricity is taken into account. Moreover, the stress resultants of the beam can be defined as

$$N = \int_\Omega S_{xx} d\Omega \tag{9a}$$

$$M_Y = \int_\Omega S_{xx} Z d\Omega \tag{9b}$$

$$M_Z = -\int_\Omega S_{xx} Y d\Omega \tag{9c}$$

$$M_t^P = \int_\Omega \left[S_{xy} \left(\frac{\partial \varphi_S^P}{\partial y} - z \right) + S_{xz} \left(\frac{\partial \varphi_S^P}{\partial z} + y \right) \right] d\Omega \tag{9d}$$

$$M_w = -\int_\Omega S_{xx} \varphi_S^P d\Omega \tag{9e}$$

$$M_R = \int_\Omega S_{xx} \left(y^2 + z^2 \right) d\Omega \tag{9f}$$

where M_t^P is the primary twisting moment [25, 28] resulting from the primary shear stress distribution S_{xy}, S_{xz}, M_w is the warping moment due to torsional curvature and M_R is a higher order stress resultant. Substituting the expressions of the stress components (3) into equations (9a-9f), the stress resultants are obtained as

$$\begin{aligned} N = EA\Bigg[& u' + \frac{1}{2}\left(v'^2 + w'^2 + \frac{I_S}{A}\theta_x'^2 \right) - \\ & -\theta_x' \Big(z_C \left(v' \cos\theta_x + w' \sin\theta_x \right) + y_C \left(v' \sin\theta_x - w' \cos\theta_x \right) \Big) \Bigg] \end{aligned} \tag{10a}$$

$$M_Y = -EI_Y \left(w'' \cos\theta_x - v'' \sin\theta_x - \beta_Z \theta_x'^2 \right) \tag{10b}$$

$$M_Z = EI_Z \left(v'' \cos\theta_x + w'' \sin\theta_x - \beta_Y \theta_x'^2 \right) \tag{10c}$$

$$M_t^P = GI_t \theta_x' \tag{10d}$$

$$M_w = -EC_S \left(\theta_x'' + \beta_\omega \theta_x'^2 \right) \tag{10e}$$

$$\begin{aligned} M_R = {} & N\frac{I_S}{A} - 2EI_Z\beta_Y \left(v'' \cos\theta_x + w'' \sin\theta_x \right) - 2EI_Y\beta_Z \left(w'' \cos\theta_x - v'' \sin\theta_x \right) + \\ & + 2EC_S\beta_\omega\theta_x'' + \frac{1}{2}E\left(I_R - \frac{I_S^2}{A} \right)\theta_x'^2 \end{aligned} \tag{10f}$$

where the area *A*, the polar moment of inertia I_S with respect to the shear center *S*, the principal moments of inertia I_Y, I_Z with respect to the cross section's centroid, the fourth moment of inertia I_R with respect to the shear center *S*, the torsion constant I_t and the warping constant C_S with respect to the shear center S, are given as

$$A = \int_\Omega d\Omega \tag{11a}$$

$$I_S = \int_\Omega \left(y^2 + z^2 \right) d\Omega \tag{11b}$$

$$I_Y = \int_\Omega Z^2 d\Omega \tag{11c}$$

$$I_Z = \int_\Omega Y^2 d\Omega \tag{11d}$$

$$I_R = \int_\Omega \left(y^2 + z^2 \right)^2 d\Omega \tag{11e}$$

$$C_S = \int_\Omega \left(\varphi_S^P \right)^2 d\Omega \tag{11f}$$

$$I_t = \int_\Omega \left(y^2 + z^2 + y\frac{\partial \varphi_S^P}{\partial z} - z\frac{\partial \varphi_S^P}{\partial y} \right) d\Omega \tag{11g}$$

while the Wagner's coefficients β_Z, β_Y and β_ω are given as

$$\beta_Z = \frac{1}{2I_Y}\int_\Omega (z - z_C)(y^2 + z^2)d\Omega \tag{12a}$$

$$\beta_Y = \frac{1}{2I_Z}\int_\Omega (y - y_C)(y^2 + z^2)d\Omega \tag{12b}$$

$$\beta_\omega = \frac{1}{2C_S}\int_\Omega (y^2 + z^2)\varphi_S^P d\Omega \tag{12c}$$

Employing the expressions of strain obtained in equations (2), the expressions of stress resultants given in equations (10) and applying the principle of virtual work (eqn. (4)), the equations of motion of the beam can be derived. The arising set of equations is coupled and highly complicated. Simplification can be achieved by neglecting the axial inertia of the beam, denoted by the term $\rho A\ddot{u}$, and employing the approximate expressions given in equations (7). Thus, using the aforementioned approximations and ignoring the nonlinear terms of the fourth or greater order [12], the governing partial differential equations of motion for the beam at hand can be written as

$$\begin{aligned}-EA\Big[u'' + w'w'' + v'v'' + \frac{I_S}{A}\theta_x'\theta_x'' - z_C\left(\theta_x\theta_x'w'' + \theta_x\theta_x''w' + \theta_x''v' + \theta_x'v'' + \theta_x'^2 w'\right) - \\ -y_C\left(\theta_x\theta_x''v' + \theta_x\theta_x'v'' - \theta_x''w' - \theta_x'w'' + \theta_x'^2 v'\right)\Big] = p_X\end{aligned} \tag{13a}$$

$$\begin{aligned}&EI_Z v'''' - N\left[-z_C\theta_x'' - y_C\left(\theta_x'^2 + \theta_x''\theta_x\right) + v''\right] + \\ &+\left(EI_Z - EI_Y\right)\left(w''''\theta_x + 2w'''\theta_x' + w''\theta_x'' - v''''\theta_x^2 - 4v'''\theta_x\theta_x' - 2v''\theta_x\theta_x'' - 2v''\theta_x'^2\right) + \\ &+EI_Z\beta_Y\left(-2\theta_x'\theta_x''' - 2\theta_x''^2\right) + EI_Y\beta_Z\left(2\theta_x'\theta_x'''\theta_x + 2\theta_x''^2\theta_x + 5\theta_x'^2\theta_x''\right) + \rho A\ddot{v} + \\ &+\rho\left[\left(I_Z - I_Y\right)\left(\theta_x v'' + \theta_x'v'\right) - I_Z w'' - A\left(y_C\theta_x + z_C - \frac{1}{2}z_C\theta_x^2\right)\right]\ddot{\theta}_x + \rho\left(I_Z - I_Y\right)\cdot \\ &\cdot\left[2\theta_x\theta_x'\ddot{v}' + \theta_x^2\ddot{v}'' + 2\left(\theta_x'\dot{\theta}_x + \theta_x\dot{\theta}_x'\right)\dot{v}' + 2\theta_x\dot{\theta}_x\dot{v}'' - \theta_x'\ddot{w}' - \theta_x\ddot{w}'' + v'\theta_x\ddot{\theta}_x'\right] + \rho I_Z\cdot \\ &\cdot\left[-w'\ddot{\theta}_x' - \ddot{v}'' - 2\dot{\theta}_x\dot{w}'' + v''\dot{\theta}_x^2 + 2v'\dot{\theta}_x\dot{\theta}_x' - 2\dot{\theta}_x'\dot{w}'\right] - \rho A\left(y_C - z_C\theta_x\right)\dot{\theta}_x^2 = p_y - \\ &-p_X\left(v' - y_C\theta_x'\theta_x - z_C\theta_x'\right)\end{aligned} \tag{13b}$$

$$\begin{aligned}&EI_Y w'''' - N\left[w'' + y_C\theta_x'' - z_C\left(\theta_x'^2 + \theta_x\theta_x''\right)\right] + \\ &+\left(EI_Z - EI_Y\right)\left(v''''\theta_x + 2v'''\theta_x' + v''\theta_x'' + w''''\theta_x^2 + 4w'''\theta_x\theta_x' + 2w''\theta_x\theta_x'' + 2w''\theta_x'^2\right) +\end{aligned}$$

$$+EI_Z\beta_Y\left(-2\theta_x{}'\theta_x{}'''\theta_x-5\theta_x{}'^2\theta_x{}''-2\theta_x{}''^2\theta_x\right)+EI_Y\beta_Z\left(-2\theta_x{}'\theta_x{}'''-2\theta_x{}''^2\right)+\rho A\ddot{w}+$$
$$+\rho\left[\left(I_Z-I_Y\right)\left(-\theta_x w''-\theta_x{}'w'\right)+I_Y v''+A\left(-z_C\theta_x+y_C-\frac{1}{2}y_C\theta_x^2\right)\right]\ddot{\theta}_x-\rho\left(I_Z-I_Y\right)\cdot$$
$$\cdot\left[2\theta_x\theta_x{}'\ddot{w}'+\theta_x^2\ddot{w}''+2\left(\theta_x{}'\dot{\theta}_x+\theta_x\dot{\theta}_x{}'\right)\dot{w}'+2\theta_x\dot{\theta}_x\dot{w}''+\theta_x{}'\ddot{v}'+\theta_x\ddot{v}''+w'\theta_x\ddot{\theta}_x{}'\right]-\rho I_Y\cdot$$
$$\cdot\left[-v'\ddot{\theta}_x{}'+\ddot{w}''-2\dot{\theta}_x\dot{v}''-w''\dot{\theta}_x^2-2w'\dot{\theta}_x\dot{\theta}_x{}'-2\dot{\theta}_x{}'\dot{v}'\right]-\rho A\left(z_C+y_C\theta_x\right)\dot{\theta}_x^2=p_z-$$
$$-p_X\left(w'+y_C\theta_x{}'-z_C\theta_x{}'\theta_x\right) \tag{13c}$$

$$EC_S\theta_x{}''''-GI_t\theta_x{}''-\frac{3}{2}E\left(I_R-\frac{I_S{}^2}{A}\right)\theta_x{}'^2\theta_x{}''-N\left[\frac{I_S}{A}\theta_x{}''+y_C\left(w''-v''\theta_x\right)-z_C\left(v''+w''\theta_x\right)\right]+$$
$$+\left(EI_Z-EI_Y\right)\left(v''w''-v''^2\theta_x+w''^2\theta_x\right)+$$
$$+EI_Z\beta_Y\left(2\theta_x{}''w''\theta_x+w''\theta_x{}'^2+2\theta_x{}'v'''+2\theta_x{}''v''+2\theta_x{}'w'''\theta_x\right)+$$
$$+EI_Y\beta_Z\left(-2\theta_x{}''v''\theta_x+2\theta_x{}''w''+2\theta_x{}'w'''-v''\theta_x{}'^2-2\theta_x{}'v'''\theta_x\right)+\rho\left(v'^2I_Y+w'^2I_Z+I_S\right)\ddot{\theta}_x-$$
$$-\rho A\left(z_C+y_C\theta_x-\frac{1}{2}z_C\theta_x^2\right)\ddot{v}+\rho A\left(y_C-z_C\theta_x-\frac{1}{2}y_C\theta_x^2\right)\ddot{w}+\rho\left(I_Z-I_Y\right)\theta_x\left(w'\ddot{w}'-v'\ddot{v}'\right)+$$
$$+\rho I_Z\left(w'\ddot{v}'+2\dot{\theta}_x\dot{w}'w'\right)+\rho I_Y\left(-v'\ddot{w}'+2\dot{\theta}_x\dot{v}'v'\right)-\rho C_S\ddot{\theta}_x{}''=m_x+p_zy_{p_z}-p_yz_{p_z}+$$
$$+\left(\frac{1}{2}\theta_x^2z_{p_y}+\frac{1}{6}\theta_x^3y_{p_y}-\theta_xy_{p_y}\right)p_y+\left(\frac{1}{6}\theta_x^3z_{p_z}-\frac{1}{2}\theta_x^2y_{p_z}-z_{p_z}\theta_x\right)p_z-$$
$$-p_X\left(\frac{I_S}{A}\theta_x{}'-y_Cv'\theta_x-z_Cw'\theta_x-z_Cv'+y_Cw'\right) \tag{13d}$$

while the expression of N is given as

$$N=EA\left[u'+\frac{1}{2}\left(v'^2+w'^2+\frac{I_S}{A}\theta_x{}'^2\right)-\theta_x{}'\left(z_C\left(w'\theta_x+v'\right)+y_C\left(v'\theta_x-w'\right)\right)\right] \tag{14}$$

The above governing differential equations (eqns. (13)) are also subjected to the initial conditions ($x\in\left(0,l\right)$)

$$u\left(x,0\right)=u_0\left(x\right) \qquad \dot{u}\left(x,0\right)=\dot{u}_0\left(x\right) \tag{15a,b}$$
$$v\left(x,0\right)=v_0\left(x\right) \qquad \dot{v}\left(x,0\right)=\dot{v}_0\left(x\right) \tag{16a,b}$$
$$w\left(x,0\right)=w_0\left(x\right) \qquad \dot{w}\left(x,0\right)=\dot{w}_0\left(x\right) \tag{17a,b}$$
$$\theta_x\left(x,0\right)=\theta_{x0}\left(x\right) \qquad \dot{\theta}_x\left(x,0\right)=\dot{\theta}_{x0}\left(x\right) \tag{18a,b}$$

together with the corresponding boundary conditions of the problem at hand, which are given as

$$a_1 u(x,t) + \alpha_2 N(x,t) = \alpha_3 \tag{19}$$

$$\beta_1 v(x,t) + \beta_2 V_y(x,t) = \beta_3 \qquad \bar{\beta}_1 \theta_Z(x,t) + \bar{\beta}_2 \mathrm{M}_Z(x,t) = \bar{\beta}_3 \tag{20a,b}$$

$$\gamma_1 w(x,t) + \gamma_2 V_z(x,t) = \gamma_3 \qquad \bar{\gamma}_1 \theta_Y(x,t) + \bar{\gamma}_2 \mathrm{M}_Y(x,t) = \bar{\gamma}_3 \tag{21a,b}$$

$$\delta_1 \theta_x(x,t) + \delta_2 M_t(x,t) = \delta_3 \qquad \bar{\delta}_1 \theta_x'(x,t) + \bar{\delta}_2 M_w(x,t) = \bar{\delta}_3 \tag{22a,b}$$

at the beam ends $x = 0, l$, where V_y, V_z and M_Z, M_Y are the reactions and bending moments with respect to y, z or to Y, Z axes, respectively, given by the following relations (ignoring again the nonlinear terms of the fourth or greater order)

$$\begin{aligned} V_y = {} & N\left(v' - z_C\theta_x' - y_C\theta_x\theta_x'\right) + \\ & + EI_Z\left(v'''\theta_x^2 - w''\theta_x' - w'''\theta_x - v''' + 2v''\theta_x\theta_x' + 2\beta_Y\theta_x'\theta_x''\right) + \\ & + EI_Y\left(w'''\theta_x + w''\theta_x' - 2v''\theta_x\theta_x' - v'''\theta_x^2 - \beta_Z\theta_x'^3 - 2\beta_Z\theta_x\theta_x'\theta_x''\right) \end{aligned} \tag{23a}$$

$$\begin{aligned} V_z = {} & N\left(w' + y_C\theta_x' - z_C\theta_x'\theta_x\right) + \\ & + EI_Y\left(w'''\theta_x^2 + v'''\theta_x - w''' + v''\theta_x' + 2w''\theta_x\theta_x' + 2\beta_Z\theta_x'\theta_x''\right) - \\ & - EI_Z\left(v''\theta_x' + w'''\theta_x^2 + v'''\theta_x + 2w''\theta_x\theta_x' - 2\beta_Y\theta_x\theta_x'\theta_x'' - \beta_Y\theta_x'^3\right) \end{aligned} \tag{23b}$$

$$M_Z = EI_Z\left(w''\theta_x - \beta_Y\theta_x'^2 + v'' - v''\theta_x^2\right) + EI_Y\left(-w''\theta_x + v''\theta_x^2 + \beta_Z\theta_x'^2\theta_x\right) \tag{23c}$$

$$M_Y = EI_Z\left(-w''\theta_x^2 + \beta_Y\theta_x'^2\theta_x - v''\theta_x\right) + EI_Y\left(\beta_Z\theta_x'^2 - w'' + w''\theta_x^2 + v''\theta_x\right) \tag{23d}$$

while M_t and M_w are the torsional and warping moments at the boundary of the bar, respectively, given as

$$\begin{aligned} M_t = {} & GI_t\theta_x' - EC_S\theta_x''' + N\left(-z_C w'\theta_x + y_C w' - y_C v'\theta_x - z_C v' + \frac{I_S}{A}\theta_x'\right) + \\ & + EI_Z\beta_Y\left(-2\theta_x' v'' - 2\theta_x' w''\theta_x\right) + EI_Y\beta_Z\left(-2\theta_x' w'' + 2\theta_x' v''\theta_x\right) + \frac{1}{2}E\left(I_R - \frac{I_S^2}{A}\right)\theta_x'^3 \end{aligned} \tag{24a}$$

$$M_w = -EC_S\left(\theta_x'' + \beta_\omega\theta_x'^2\right) \tag{24b}$$

Finally, α_k, β_k, $\bar{\beta}_k$, γ_k, $\bar{\gamma}_k$, δ_k, $\bar{\delta}_k$ $(k = 1,2,3)$ are time dependent functions specified at the boundaries of the bar ($x = 0, l$). The boundary conditions eqns. (19)-(22) are the most general boundary conditions for the problem at hand, including also the elastic support. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from these equations by specifying appropriately these functions (e.g. for a clamped edge it is $\alpha_1 = \beta_1 = \gamma_1 = \delta_1 = 1$, $\bar{\beta}_1 = \bar{\gamma}_1 = \bar{\delta}_1 = 1$, $\alpha_2 = \alpha_3 = \beta_2 = \beta_3 = \gamma_2 = \gamma_3 = \delta_2 = \delta_3 = \bar{\beta}_2 = \bar{\beta}_3 = \bar{\gamma}_2 = \bar{\gamma}_3 = \bar{\delta}_2 = \bar{\delta}_3 = 0$). In a case of eccentric axial loading or additional concentrated or distributed bending or warping moments, additional terms in governing equations (13) would arise. These terms could be taken into account without any special difficulty, by modifying appropriately equations (5c), (6), (8).

4 INTEGRAL REPRESENTATIONS-NUMERICAL SOLUTION

4.1 Evaluation of the axial displacement $u(x,t)$, the transverse displacements $v(x,t)$, $w(x,t)$ and the angle of twist $\theta_x(x,t)$

According to the precedent analysis, the nonlinear flexural-torsional vibration problem of a beam reduces in establishing the axial displacement component $u(x,t)$ having continuous partial derivatives up to the second order and the transverse displacement components $v(x,t)$, $w(x,t)$ and the angle of rotation $\theta_x(x,t)$ having continuous partial derivatives up to the fourth order with respect to x and up to the second order with respect to t, satisfying the nonlinear initial boundary value problem described by the coupled governing differential equations of motion (eqns. (13)) along the beam, the initial conditions (eqns. (15)-(18)) and the boundary conditions (eqns. (19)-(22)) at the beam ends $x = 0, l$. Eqns. (13) and (15)-(22) are solved using the Analog Equation Method [22] as it is developed for hyperbolic differential equations in [29, 30].

5 NUMERICAL EXAMPLES

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the validation, the efficiency, wherever possible the accuracy and the range of applications of the developed method. The numerical results have been obtained employing 21 nodal points (longitudinal discretization) and 400 boundary elements (cross section discretization).

Example 1

In the first example, for comparison reasons, the forced vibrations of a steel-I beam (Fig. 3), ($E = 2{,}1\times10^8\, kN/m^2$, $G = 8{,}0769\times10^7\, kN/m^2$, $\rho = 7{,}85\, tn/m^3$, flange width $t_f = 0{,}03m$, web width $t_w = 0{,}012m$, total height $h = 0{,}56m$, total width $b = 0{,}30m$) having geometric constants presented in Table 1, under three load cases, are examined. In the first load case (case (a), Fig. 3a), a beam with hinged-hinged ends, of length $l = 10\, m$ is considered. The beam is subjected to a uniformly distributed 'static' loading $p_z(t) = p_{z0}(t/t_1)$ for $0 \le t \le t_1$ and $p_z(t) = p_{z0}$ for $t \ge t_1$ ($t_1 = 0{,}02\sec$, $p_{z0} = 3000\, kN/m$), as this is shown in Fig. 3a. In Fig. 4 the time histories of the displacements $u(x_0,t)$, $w(x_0,t)$ at $x_0 = 6{,}9048\, m$ are presented, respectively, as compared with those obtained from a BEM solution [20], noting the accuracy of the proposed method.

$A = 2,4\times10^{-2}\,m^2$	$I_R = 1,14831\times10^{-4}\,m^6$
$I_Y = 1,39035\times10^{-3}\,m^4$	$I_t = 5,38871\times10^{-6}\,m^4$
$I_Z = 1,35072\times10^{-4}\,m^4$	$C_S = 9,48415\times10^{-6}\,m^6$
$I_S = 1,5254\times10^{-3}\,m^4$	

Table 1: Geometric constants of the beam of example 1.

	Linear Analysis	Nonlinear Analysis					
		Load case (b)			Load case (c)		
		Present study	FEM 960 solid elements [31]	FEM 2080 shell elements [31]	Present study	FEM 960 solid elements [31]	FEM 2080 shell elements [31]
$u(l)_{\max}$	0,0000	-0,0033	-0,0034	-0,0037	-0,0031	-0,0032	-0,0032
$v(l)_{\max}$	0,0000	0,0615	0,0603	0,0675	0,0331	0,0329	0,0328
$w(l)_{\max}$	0,2040	0,2045	0,2108	0,2114	0,2040	0,2108	0,2078

Table 2: Maximum values of the kinematical components $u(l,t)$ (m), $v(l,t)$ (m) and $w(l,t)$ (m) of the cantilever beam of example 1 for load cases (b), (c).

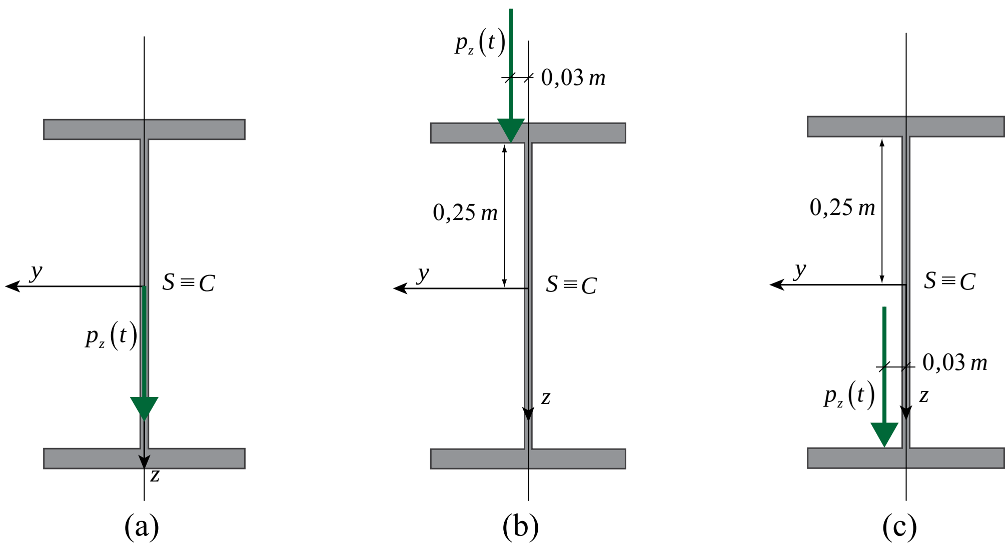


Figure 3: Transverse load applied on the centroid concerning the first load case (a) and on the flange concerning the second load case (b) and the third load case (c) of example 1.

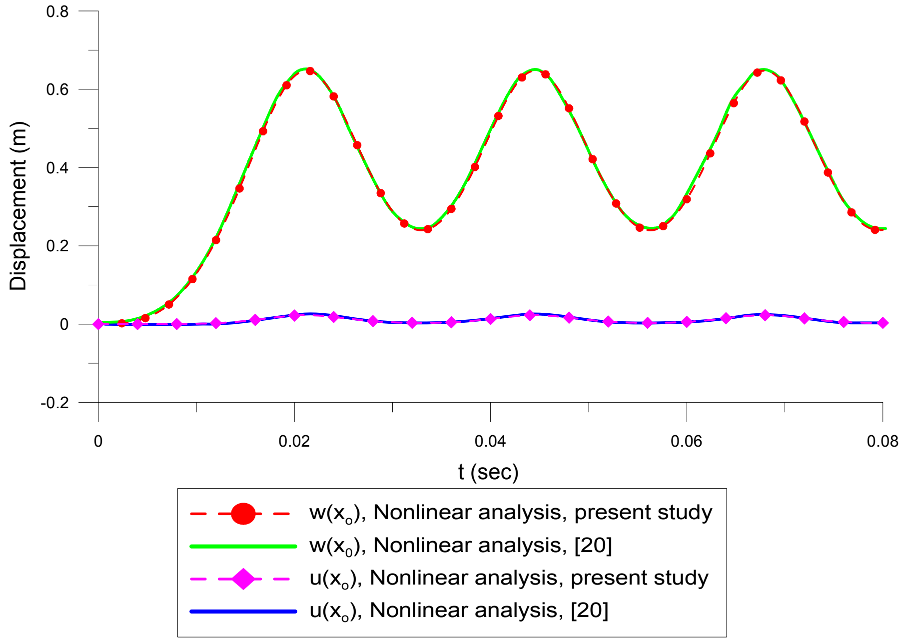


Figure 4: Time history of the displacements u and w at $x_0 = 6,9048\,m$ of the hinged-hinged beam of example 1 for load case (a).

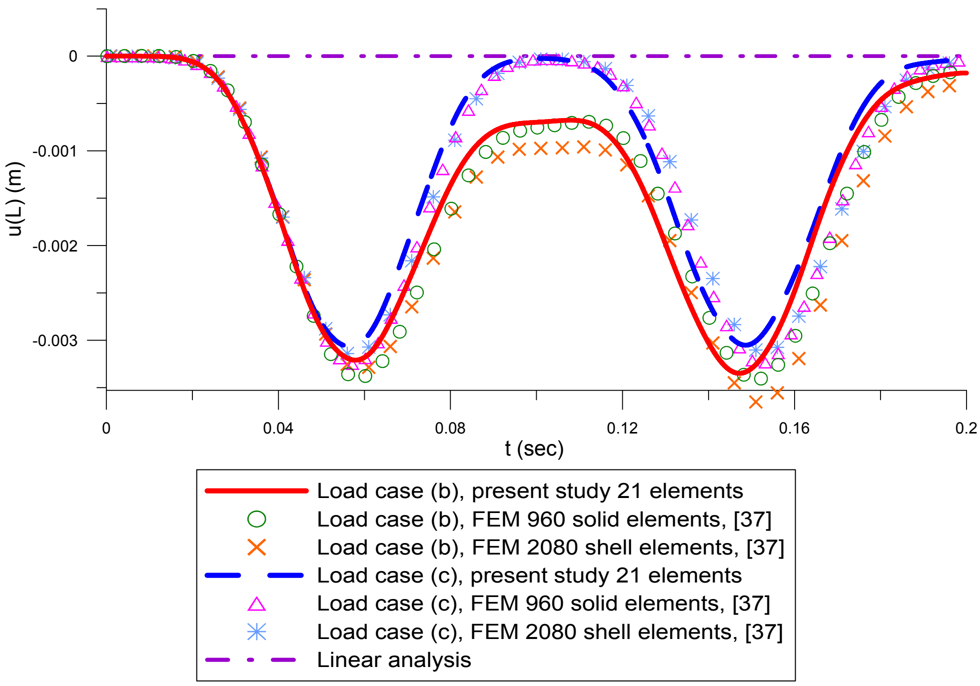


Figure 5: Time history of the displacement u at the tip of the cantilever beam of example 1 for load cases (b), (c).

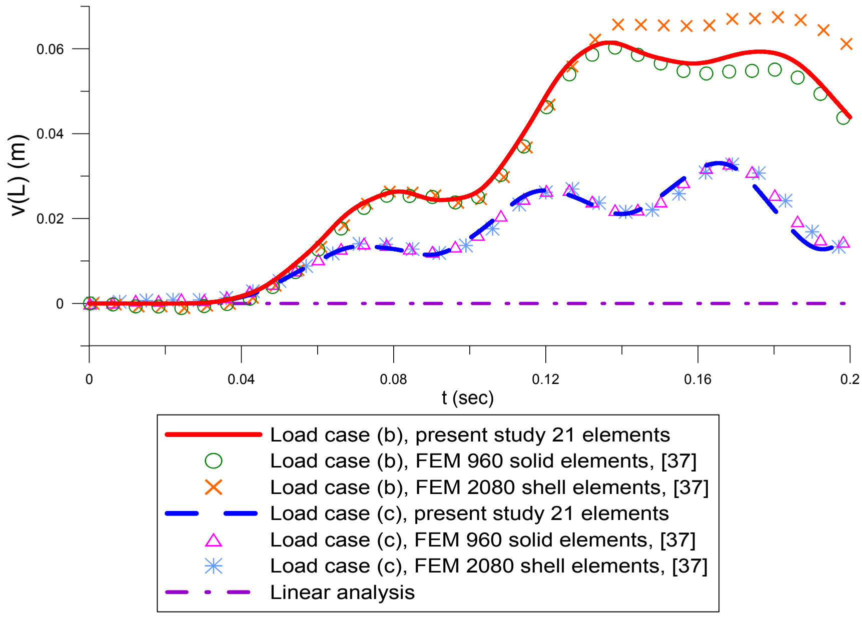


Figure 6: Time history of the displacement v at the tip of the cantilever beam of example 1 for load cases (b), (c).

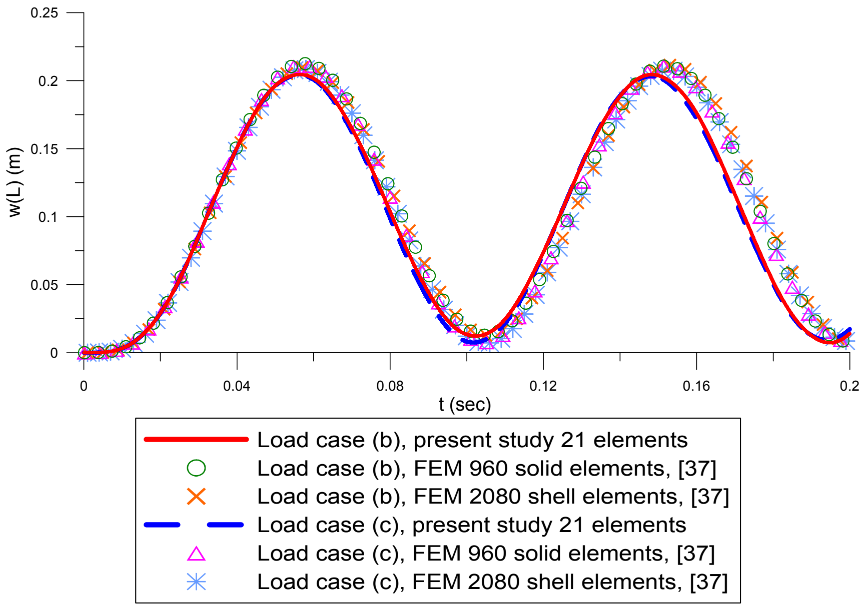
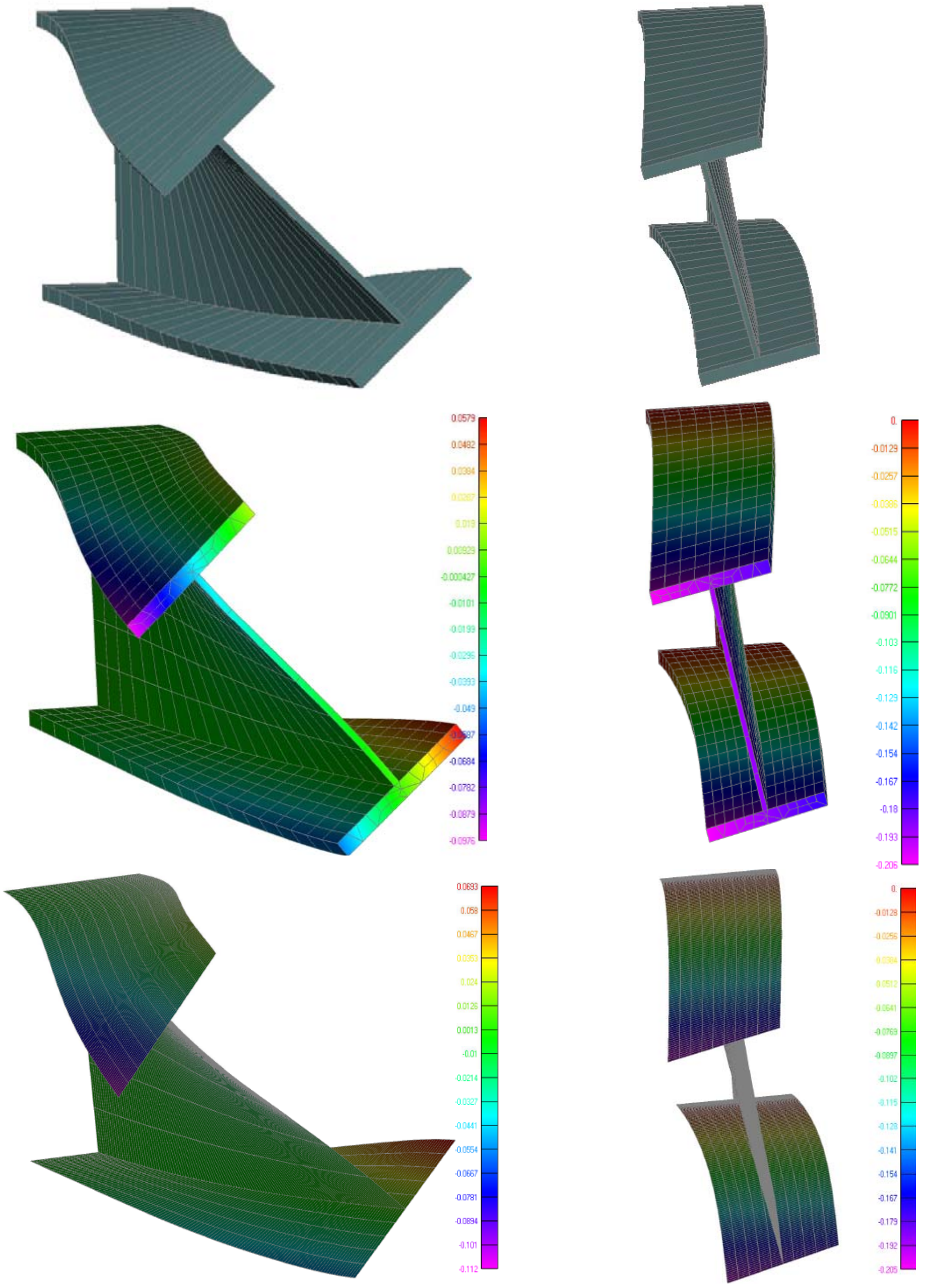


Figure 7: Time history of the displacement w at the tip of the cantilever beam of example 1 for load cases (b), (c).

Moreover, in order to demonstrate the influence of the loading position upon the cross section, a cantilever beam of the same cross section of length $l = 8\,m$, undergoing a uniformly distributed 'static' loading ($t_1 = 0,02\,\text{sec}$, $p_{z0} = 60\,kN/m$) is examined for the load cases (b), (c) (Figs. 3b,c). In Figs. 5-7 the time histories of the kinematical components $u(l,t), v(l,t)$ and $w(l,t)$, respectively, at the tip cross section of the cantilever beam are presented as compared with those obtained from a FEM solution [31, 32], employing either shell or solid finite elements. In Fig. 8 the deformed configurations of the beam at the time instant $t = 0,1\,\text{sec}$ for the load case (b) and at $t = 0,067\,\text{sec}$ for the load case (c) are presented as compared with those obtained from a FEM solution [31, 32]. Finally, in Table 2 the maximum values of the kinematical components $u(l,t), v(l,t)$ and $w(l,t)$ of the present study are presented together with those obtained from the aforementioned FEM solutions [31, 32]. From these figures and table the accuracy of the proposed method is demonstrated. Moreover, as it can easily be observed, the loading position may have important influence on the dynamic response of the beam.

Load case (b) at $t = 0,1\,\text{sec}$ Load case (c) at $t = 0,067\,\text{sec}$



(a)

(b)

(c)

Figure 8: Deformed configurations of the cantilever beam of example 1 from the present study (a) and a FEM solution using solid elements (b) or shell elements (c).

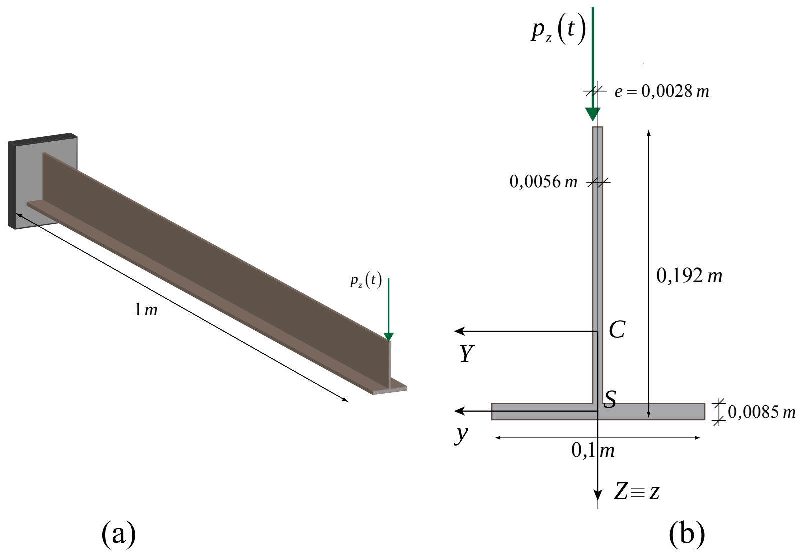


Figure 9: Cantilever beam of example 2 (a) and load with eccentricity with respect to the shear center of its cross section (b).

$A = 1,878\times10^{-3}\,m^2$	$I_t = 3,049\times10^{-8}\,m^4$
$I_Y = 7,176\times10^{-6}\,m^4$	$C_S = 3,495\times10^{-11}\,m^6$
$I_Z = 7,110\times10^{-7}\,m^4$	$\beta_z = -7,276\times10^{-2}\,m$
$I_S = 1,295\times10^{-5}\,m^4$	$z_c = -5,2\times10^{-2}\,m$
$I_R = 2,581\times10^{-7}\,m^6$	

Table 3: Geometric constants of the beam of example 2.

	Linear Analysis	Nonlinear Analysis	
		Ignoring eccentricity change	Taking into account eccentricity change
$v(l)_{\max}$	0,0009	−0,0017	0,0021
$w(l)_{\max}$	0,0068	0,0068	0,0068
$\theta_x(l)_{\max}$	0,0304	0,0264	0,1090

Table 4: Maximum values of the kinematical components $v(l,t)$ (m), $w(l,t)$ (m) and $\theta_x(l,t)$ (rad) of the cantilever beam of example 2.

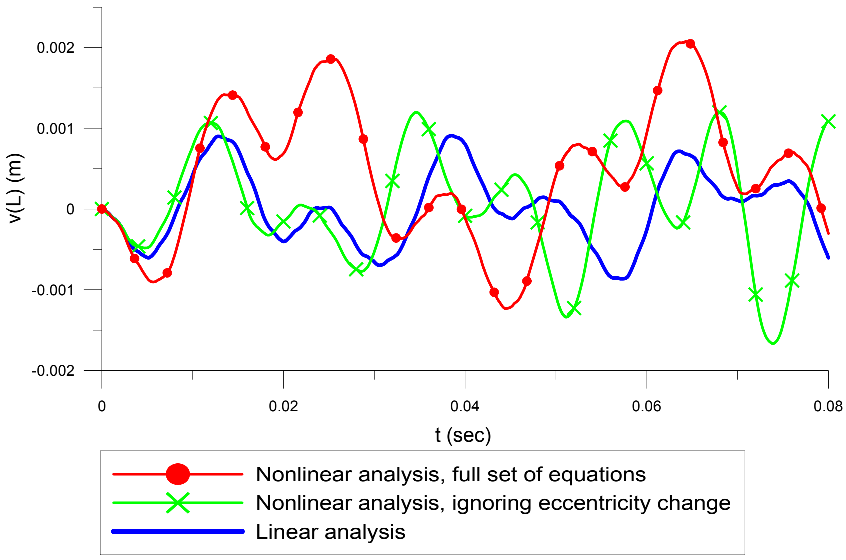


Figure 10: Time history of the displacement v at the tip of the cantilever beam of example 2.

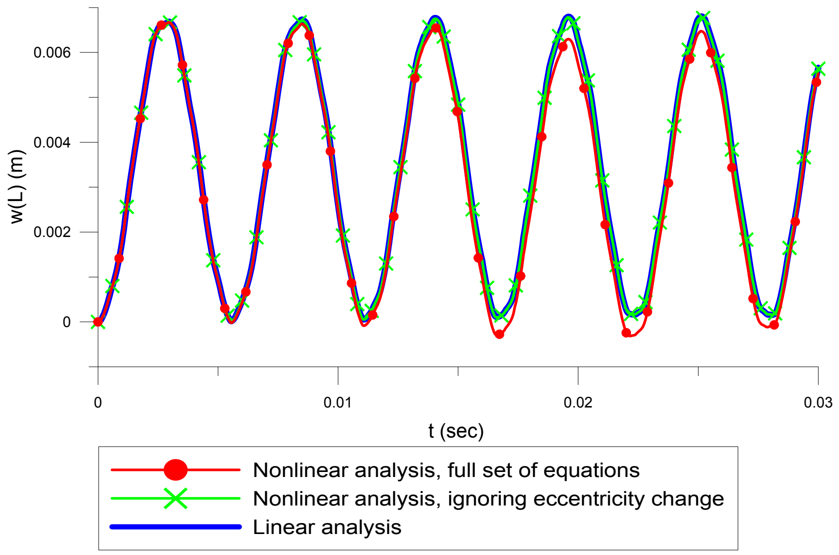


Figure 11: Time history of the displacement w at the tip of the cantilever beam of example 2.

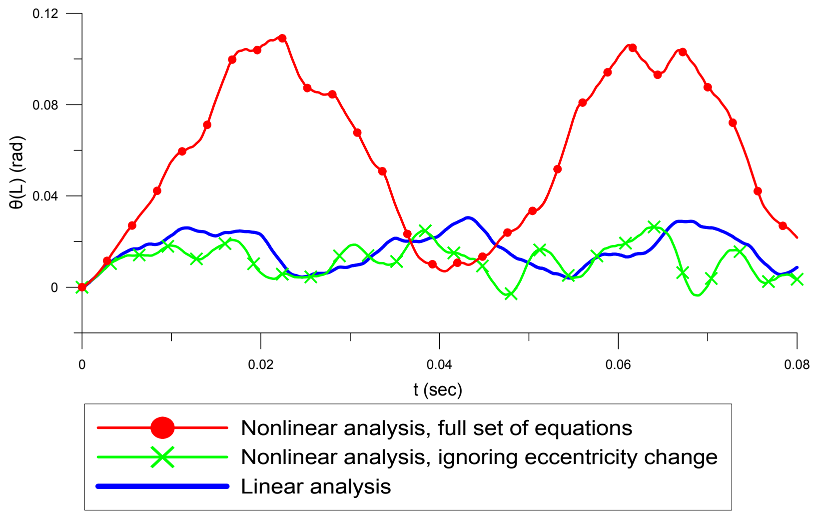


Figure 12: Time history of the angle of twist θ_x at the tip of the cantilever beam of example 2.

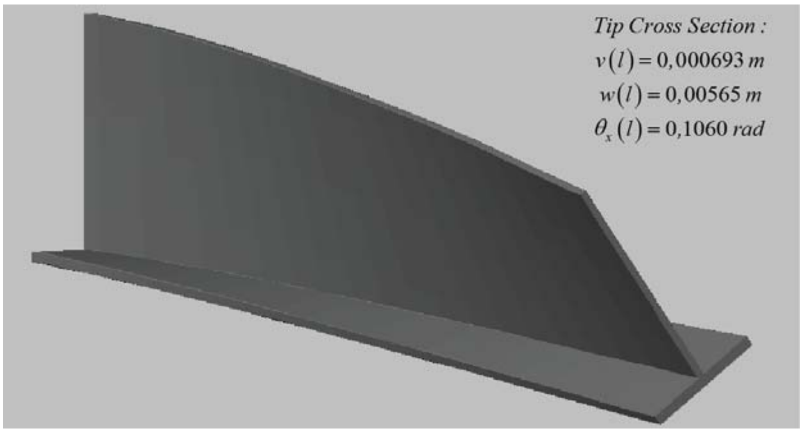


Figure 13. Deformation of the cantilever beam of example 2 at $t = 0,02\sec$.

Example 2

In order to investigate the influence of eccentricity change of the transverse loading in nonlinear flexural-torsional vibrations, the forced vibration of a cantilever beam ($E = 2,1\times10^8\ kN/m^2$, $G = 8,0769\times10^7\ kN/m^2$, $\rho = 7,85\,tn/m^3$, $l = 1\,m$) of a monosymmetric thin-walled open shaped cross section (its geometric constants are presented in Table 3), as this is shown in Fig. 9, has been studied. More specifically, the beam is subjected to a suddenly applied concentrated force $P_z(t) = 16\,kN$ having a slight eccentricity $e = 0,0028\,m$ with respect to the shear center of the tip cross section (Fig. 9b). In Figs. 10-12 the time histories of the transverse displacements $v(l,t)$, $w(l,t)$ and the angle of twist $\theta_x(l,t)$ of the cantilever beam, respectively, in Table 4 the maximum values of its kinematical components and in Fig. 13 the deformation of the cantilever beam at the time instant $t = 0,02\sec$, are presented. From these figures and table, the influence of the relatively small 'imperfection' of the application point of the transverse loading to the beam response is remarked.

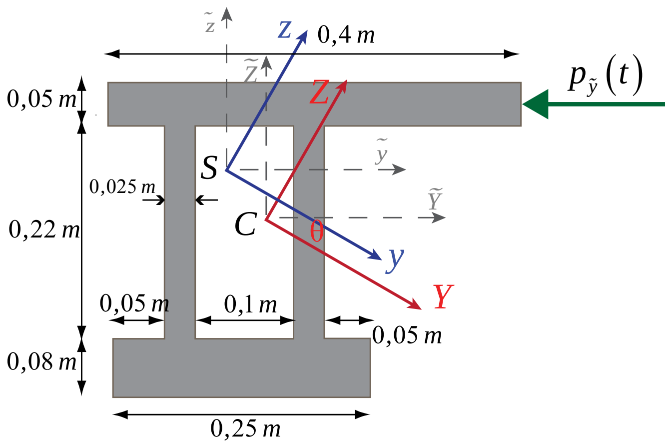


Figure 14: Cross section of the cantilever beam of example 3.

$A = 5{,}1\times10^{-2}\,m^2$	$\theta = -0{,}41330\,rad$
$I_Y = 9{,}64614\times10^{-4}\,m^4$	$I_t = 2{,}6965\times10^{-4}\,m^4$
$I_Z = 3{,}90079\times10^{-4}\,m^4$	$C_S = 4{,}6911\times10^{-6}\,m^6$
$I_S = 1{,}58177\times10^{-3}\,m^4$	$\beta_y = 8{,}56633\times10^{-2}\,m$
$I_R = 7{,}14223\times10^{-5}\,m^6$	$\beta_z = -8{,}89913\times10^{-3}\,m$
$y_c = 6{,}20527\times10^{-2}\,m$	$\beta_\omega = 0{,}39404$
$z_c = -2{,}45329\times10^{-2}\,m$	

Table 5: Geometric constants of the beam of example 3.

Example 3

In order to demonstrate the range of applications of the developed method, in this final example the forced vibration of a hinged-hinged (axially immovable ends) beam of length $l = 3{,}5\,m$ ($E = 3{,}0\times10^7\,kN/m^2$, $G = 1{,}25\times10^7\,kN/m^2$, $\rho = 2{,}5\,tn/m^3$) having an asymmetric closed shaped cross section (its geometric constants are presented in Table 5), subjected to a uniformly distributed suddenly applied loading $p_{\tilde{y}} = 1000\,kN/m$, as this is shown in Fig. 14, is examined. In Figs. 15-17 the time histories of the transverse displacements $\tilde{v}(l/2,t)$, $\tilde{w}(l/2,t)$ and the angle of twist $\theta_x(l/2,t)$, respectively, in Table 6 the maximum values of its kinematical components and in Fig. 18 the deformation of the hinged-hinged beam at the time instant $t = 0{,}017\,\sec$, are presented. As it can be observed, in this case, the eccentricity change did not affect significantly the dynamic response of the beam, while geometrical nonlinearity due to axially immovable ends had important influence, especially on the transverse displacements $\tilde{v}$, $\tilde{w}$.

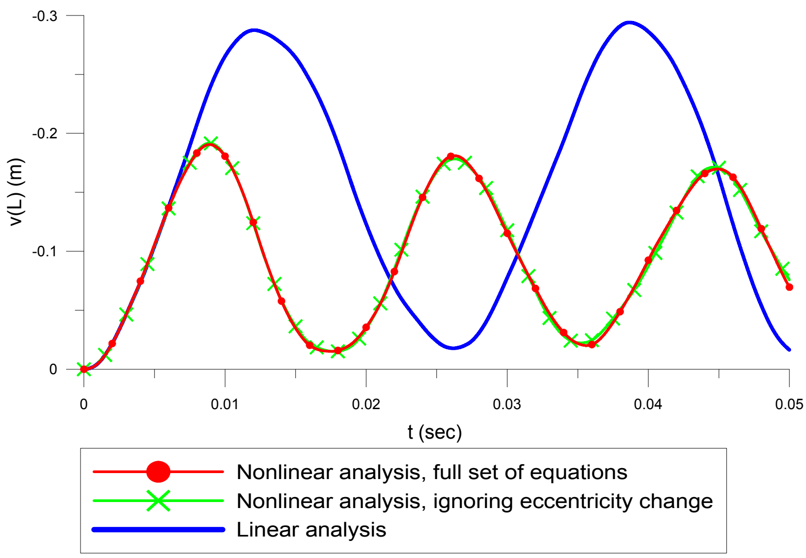


Figure 15: Time history of the displacement $\tilde{v}$ at the midpoint of the hinged-hinged beam of example 3.

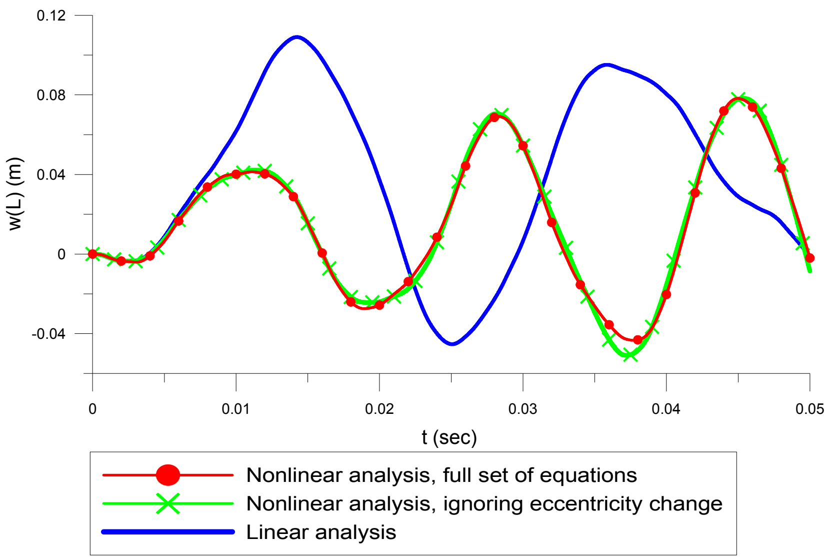


Figure 16: Time history of the displacement $\tilde{w}$ at the midpoint of the hinged-hinged beam of example 3.

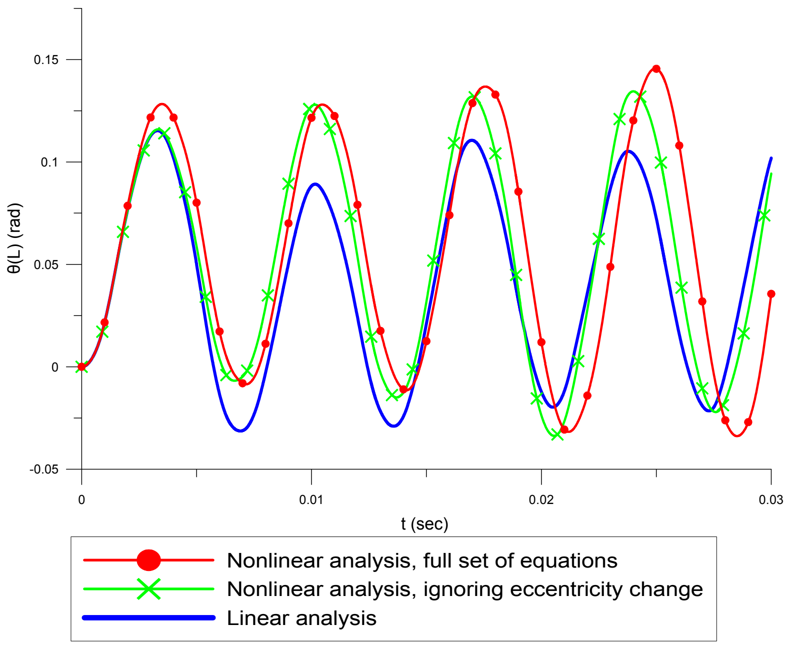


Figure 17: Time history of the angle of rotation θ_x at the midpoint of the hinged-hinged beam of example 3.

	Linear Analysis	Nonlinear Analysis	
		Ignoring eccentricity change	Taking into account eccentricity change
$\tilde{v}(l/2)_{\max}$	$-0{,}2940$	$-0{,}1920$	$-0{,}1910$
$\tilde{w}(l/2)_{\max}$	0,1090	0,0782	0,0782
$\theta_x(l/2)_{\max}$	0,1150	0,1340	0,1460

Table 6: Maximum values of the kinematical components $\tilde{v}(l/2,t)$ (m), $\tilde{w}(l/2,t)$ (m) and $\theta_x(l/2,t)$ (rad) of the fixed-fixed beam of example 3.

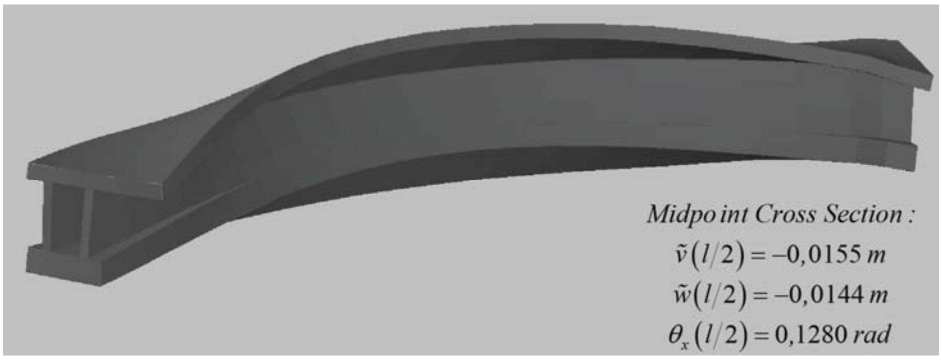


Figure 18: Deformation of the hinged-hinged beam of example 3 at $t = 0,017\,\text{sec}$.

6 CONCLUSIONS

In this paper a boundary element method is developed for the nonlinear elastic flexural-torsional dynamic analysis of beams of arbitrary cross section, undergoing moderately large displacements and angles of twist, taking into account the effects of rotary and warping inertia and the change of eccentricity of transverse loads during torsional rotation. The main conclusions that can be drawn from this investigation are:

- The numerical technique presented in this investigation is well suited for computer aided analysis of beams of arbitrary simply or multiply connected cross section supported by the most general boundary conditions and subjected to the combined action of arbitrarily distributed or concentrated time dependent loading.
- The geometrical nonlinearity leads to strong coupling between the axial, torsional and bending equilibrium equations, resulting a significantly different response of the beam compared with this obtained from a linear analysis.
- The change of eccentricity of the transverse loading during the torsional rotation of the cross section affects significantly the torsional response of the beam
- Different loading positions upon the cross section may alter significantly the torsional response of the beam.
- Accurate results are obtained using a relatively small number of nodal points along the beam.
- The developed procedure retains most of the advantages of a BEM solution over a FEM approach, although it requires longitudinal domain discretization.

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