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HIGH ORDER ABSORBING BOUNDARY CONDITIONS FOR ELASTODYNAMICS

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Abstract. *A high-order Absorbing Boundary Condition (ABC) is devised on an artificial boundary for time-dependent elastic waves in unbounded domains. The configuration considered is that of a two-dimensional elastic waveguide. The proposed ABC is an extension of the Hagstrom-Warburton ABC which was originally designed for acoustic waves, and is applied directly to the displacement field. The order of the ABC determines its accuracy and can be chosen to be arbitrarily high. A special variational formulation is constructed which incorporates the ABC. A standard FE discretization is used in space, and a Newmark-type scheme is used for time-stepping. A long-time instability is observed, but simple means are shown to dramatically postpone its onset so as to make it harmless during the simulation time of interest.*

1 INTRODUCTION

A well-known computational technique for treating wave problems in unbounded domains is the use of Absorbing Boundary Conditions (ABCs), also known by other names such as Radiating Boundary Conditions and Non-reflecting Boundary Conditions; see the review papers [1–3]. An ABC is a condition imposed on an artificial boundary which truncates the unbounded domain, thus allowing the replacement of the original problem by another problem defined in a finite domain. The latter can then be solved using standard numerical techniques like finite element (FE) or finite difference schemes. Application areas include acoustics, oceanography, electromagnetic waves and Solid-Earth Geophysics (SEG).

In 1993, Collino [4] proposed the first genuinely *high-order* local ABC for the wave equation, which does not involve high derivatives and can be implemented up to any desired order. The key here is the use of *auxiliary variables*, which are introduced on the artificial boundary and enable the elimination of the high derivatives from the ABC equations. Various authors followed Collino in proposing different high-order ABCs; see the review in [5]. Here we concentrate on the Hagstrom-Warburton [6] high-order ABC.

The use of ABCs in SEG, for the solution of direct or inverse elastic wave problems, goes back to the seminal 1969 paper of Lysmer and Kuhlemeyer [7], who proposed a dashpot-type ABC. Despite its crude accuracy, the dashpot ABC is still used today, along with various improved ABCs and absorbing layers which have been proposed in the last four decades; see, e.g., [8–10].

In contrast to the situation with PML, the only high-order ABC developed for elastic waves to date, to the best of our knowledge, is that of Tsogka and Joly [11, 12]. Their ABC is an extension of the Collino ABC [4] to elastodynamics, and is based on using special potential functions. The ABC was incorporated in a mixed FE formulation. Very recently [13], a long-time instability has been observed in solutions generated by this scheme. We shall see that the scheme proposed here suffers from the same malady, although the onset of the instability can be postponed dramatically by using various means.

In the present paper, we propose another high-order ABC for elastodynamics, which is an extension of the Hagstrom-Warburton ABC [6]. The latter is, in turn, a high-order form of the Higdon ABC [14], and a symmetrized modification of the Givoli-Neta high-order formulation [15, 16]. This ABC, developed originally for scalar time-dependent problems and hyperbolic conservation systems, has been shown to be extremely effective in a variety of situations [6, 17–21]. Here we extend the ABC for use in elastodynamics, basing it directly on the displacement field.

2 PROBLEM STATEMENT

We consider a two-dimensional semi-infinite elastic waveguide of width b, as shown in Fig. 1(a).

In the waveguide we consider the two-dimensional (plane strain) linear equations of elastodynamics. Some boundary conditions are specified on all three boundaries. Initial conditions are also prescribed. We assume that outside a compact region, denoted Ω_0, the following simplified conditions hold: (a) the medium is *homogeneous*, namely the material properties are constant; (b) the material is *isotropic*; (c) body forces are absent; and (d) the initial values vanish. As a result of these assumptions, in the semi-infinite region outside Ω_0 the governing equations are reduced to the Navier equations.

We now truncate the semi-infinite domain by introducing the artificial boundary Γ_E, located

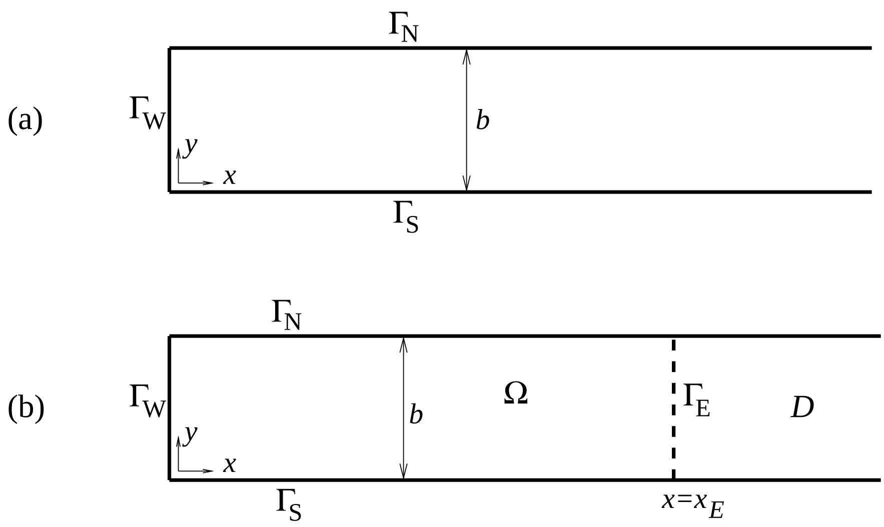

Figure 1: A semi-infinite waveguide: (a) the original setup, (b) the setup of the problem with truncated domain.

at $x = x_E$, $0 \leqslant y \leqslant b$; see Fig. 1(b). This boundary divides the original semi-infinite domain into two subdomains: the exterior domain D, and a finite computational domain Ω, which is bounded by $\Gamma_W, \Gamma_S, \Gamma_N$ and Γ_E. We choose the location of Γ_E such that Ω_0 is strictly contained in Ω.

The initial boundary value problem in Ω consists of the elastic equations in Ω, the given initial conditions, the boundary conditions on Γ_W, Γ_N and Γ_S, and an ABC imposed on Γ_E. We shall consider this ABC in the next sections.

3 THE HIGH-ORDER ABC: GENERAL APPROACH

The general approach devised in [15] and adopted in [6] to construct a high-order ABC consists of the following four steps (presented here in the context of a scalar problem, for simplicity):

Step 1. We start with a basic, low-order, ABC, $B_0 u = 0$ on Γ_E. The only essential requirement from this ABC is that it be associated with a reflection coefficient smaller than 1.

Step 2. Taking a product of operators B_j, each one of which has the same form as B_0, possibly with different coefficients, we obtain a Pth-order ABC,

$$\mathcal{B}^P u \equiv \left(\prod_{j=0}^{P} B_j \right) u = 0 \, .$$

Its reflection coefficient is the product of the reflection coefficients R_j associated with each B_j, and hence becomes exponentially small as P is increased.

Step 3. This high-order ABC is not computationally practical because it involves high-order derivatives. We introduce the auxiliary variables ϕ_j which are defined recursively; using them we eliminate all the high derivatives, and thus obtain a high-order ABC that involves only low derivatives.

Step 4. The latter high-order ABC is still not computationally practical, since it involves (in general) normal derivatives of the auxiliary variables ϕ_j on the boundary Γ_E. These derivatives do not allow us to discretize the ϕ_j on Γ_E alone. We therefore use the wave equation itself (which we prove to be satisfied by each ϕ_j) to eliminate all normal derivatives of all auxiliary variables. This finally results in a practical high-order ABC.

For details on the development of the elastic ABC based on this approach, see [22].

4 THE HIGH-ORDER ABC: FINAL FORM

The final form of the high-order ABC is as follows.
For $j = 2, \ldots, P$:

$$\begin{aligned}\left[\left(a_{j-1}a_j^2 + a_j a_{j-1}^2\right)(\lambda+2\mu) + \left(a_{j-1}+a_j\right)c^2\rho\right]\partial_{tt}\phi_j^x - \left(a_{j-1}+a_j\right)c^2\mu\partial_{yy}\phi_j^x \\ = \left[a_{j-1}a_j^2(\lambda+2\mu) - a_{j-1}c^2\rho\right]\partial_{tt}\phi_{j+1}^x + a_{j-1}a_j c(\lambda+\mu)\partial_{yt}\phi_{j+1}^y + a_{j-1}c^2\mu\partial_{yy}\phi_{j+1}^x \\ + \left[a_j a_{j-1}^2(\lambda+2\mu) - a_j c^2\rho\right]\partial_{tt}\phi_{j-1}^x - a_j a_{j-1}c(\lambda+\mu)\partial_{yt}\phi_{j-1}^y + a_j c^2\mu\partial_{yy}\phi_{j-1}^x,\end{aligned} \tag{1}$$

$$\begin{aligned}\left[\left(a_{j-1}a_j^2 + a_j a_{j-1}^2\right)\mu + \left(a_{j-1}+a_j\right)c^2\rho\right]\partial_{tt}\phi_j^y - \left(a_{j-1}+a_j\right)c^2(\lambda+2\mu)\partial_{yy}\phi_j^y \\ = \left[a_{j-1}a_j^2\mu - a_{j-1}c^2\rho\right]\partial_{tt}\phi_{j+1}^y + a_{j-1}a_j c(\lambda+\mu)\partial_{yt}\phi_{j+1}^x + a_{j-1}c^2(\lambda+2\mu)\partial_{yy}\phi_{j+1}^y \\ + \left[a_j a_{j-1}^2\mu - a_j c^2\rho\right]\partial_{tt}\phi_{j-1}^y - a_j a_{j-1}c(\lambda+\mu)\partial_{yt}\phi_{j-1}^x + a_j c^2(\lambda+2\mu)\partial_{yy}\phi_{j-1}^y,\end{aligned} \tag{2}$$

For $j = 1$:

$$\begin{aligned}\left[\left(a_0a_1^2 + 2a_1a_0^2\right)(\lambda+2\mu) + a_0c^2\rho\right]\partial_{tt}\phi_1^x - a_0a_1c(\lambda+\mu)\partial_{yt}\phi_1^y - a_0c^2\mu\partial_{yy}\phi_1^x \\ = \left[a_0a_1^2(\lambda+2\mu) - a_0c^2\rho\right]\partial_{tt}\phi_2^x + a_0a_1c(\lambda+\mu)\partial_{yt}\phi_2^y + a_0c^2\mu\partial_{yy}\phi_2^x \\ + \left[2a_1a_0^2(\lambda+2\mu) - 2a_1c^2\rho\right]\partial_{tt}\phi_0^x - 2a_1a_0c(\lambda+\mu)\partial_{yt}\phi_0^y + 2a_1c^2\mu\partial_{yy}\phi_0^x,\end{aligned} \tag{3}$$

$$\begin{aligned}\left[\left(a_0a_1^2 + 2a_1a_0^2\right)\mu + a_0c^2\rho\right]\partial_{tt}\phi_1^y - a_0a_1c(\lambda+\mu)\partial_{yt}\phi_1^x - a_0c^2(\lambda+2\mu)\partial_{yy}\phi_1^y \\ = \left[a_0a_1^2\mu - a_0c^2\rho\right]\partial_{tt}\phi_2^y + a_0a_1c(\lambda+\mu)\partial_{yt}\phi_2^x + a_0c^2(\lambda+2\mu)\partial_{yy}\phi_2^y \\ + \left[2a_1a_0^2\mu - 2a_1c^2\rho\right]\partial_{tt}\phi_0^y - 2a_1a_0c(\lambda+\mu)\partial_{yt}\phi_0^x + 2a_1c^2(\lambda+2\mu)\partial_{yy}\phi_0^y,\end{aligned} \tag{4}$$

These equations are accompanied by (6) and (9) which correspond to $j = 0$; namely, they connect u_i and ϕ_1^i:

$$\left(a_0\partial_t + c\partial_x\right)u_x = a_0\partial_t\phi_1^x \quad , \quad \left(a_0\partial_t + c\partial_x\right)u_y = a_0\partial_t\phi_1^y \, . \tag{5}$$

In addition we have the 'closure' conditions

$$\left(a_0\partial_t + c\partial_x\right)u_x = a_0\partial_t\phi_1^x, \tag{6}$$

$$\left(a_j\partial_t + c\partial_x\right)\phi_j^x = \left(a_j\partial_t - c\partial_x\right)\phi_{j+1}^x, \quad j = 1, \ldots, P, \tag{7}$$

$$\phi_{P+1}^x = 0 \quad \text{on} \quad \Gamma_E, \tag{8}$$

$$\left(a_0\partial_t + c\partial_x\right)u_y = a_0\partial_t\phi_1^y, \tag{9}$$

$$\left(a_j\partial_t + c\partial_x\right)\phi_j^y = \left(a_j\partial_t - c\partial_x\right)\phi_{j+1}^y, \quad j = 1, \ldots, P, \tag{10}$$

$$\phi_{P+1}^y = 0 \quad \text{on} \quad \Gamma_E \, , \tag{11}$$

as well as

$$\phi_0^x = u_x, \quad , \quad \phi_0^y = u_y \, . \tag{12}$$

Eqs. (1)–(12) constitute together the desired ABC on Γ_E.

5 NUMERICAL EXAMPLE

We take zero initial conditions, and we 'drive' the problem through a persistent surface force applied on the north boundary Γ_N. On this boundary we take $u_y = 0$ and the nonzero tangential traction $T_x = \sigma_{xy} = T_x^N$, where

$$T_x^N(x,t) = \begin{cases} 1 & \text{for} \quad 0 \leqslant x \leqslant 8 \\ 0 & \text{for} \quad x > 8 \end{cases} \quad , \qquad t > 0 \; . \tag{13}$$

Snapshots of the reference solution (obtained in a long domain) and the truncated-domain solution using the ABC with $P = 20$ are shown in Fig. 2. Steady state is rapidly reached due to the persistence of the applied force. In later times the solution does not change significantly, and hence no snapshots are shown after $t = 3$. It is apparent that the agreement between the two solutions is excellent.

6 LONG-TIME INSTABILITY

Numerical experiments show that our scheme exhibits a long-time instability, which causes the solution to grow exponentially after a sufficiently long amount of time. A similar phenomenon was observed with the potential-based high-order ABC of Tsogka and Joly [11, 12]. The origin of this instability is not clear yet. Nevertheless, we have found that the onset of the instability can be dramatically *postponed* by injecting small *numerical damping* into the time-stepping scheme on the boundary Γ_E. See [22] for more details.

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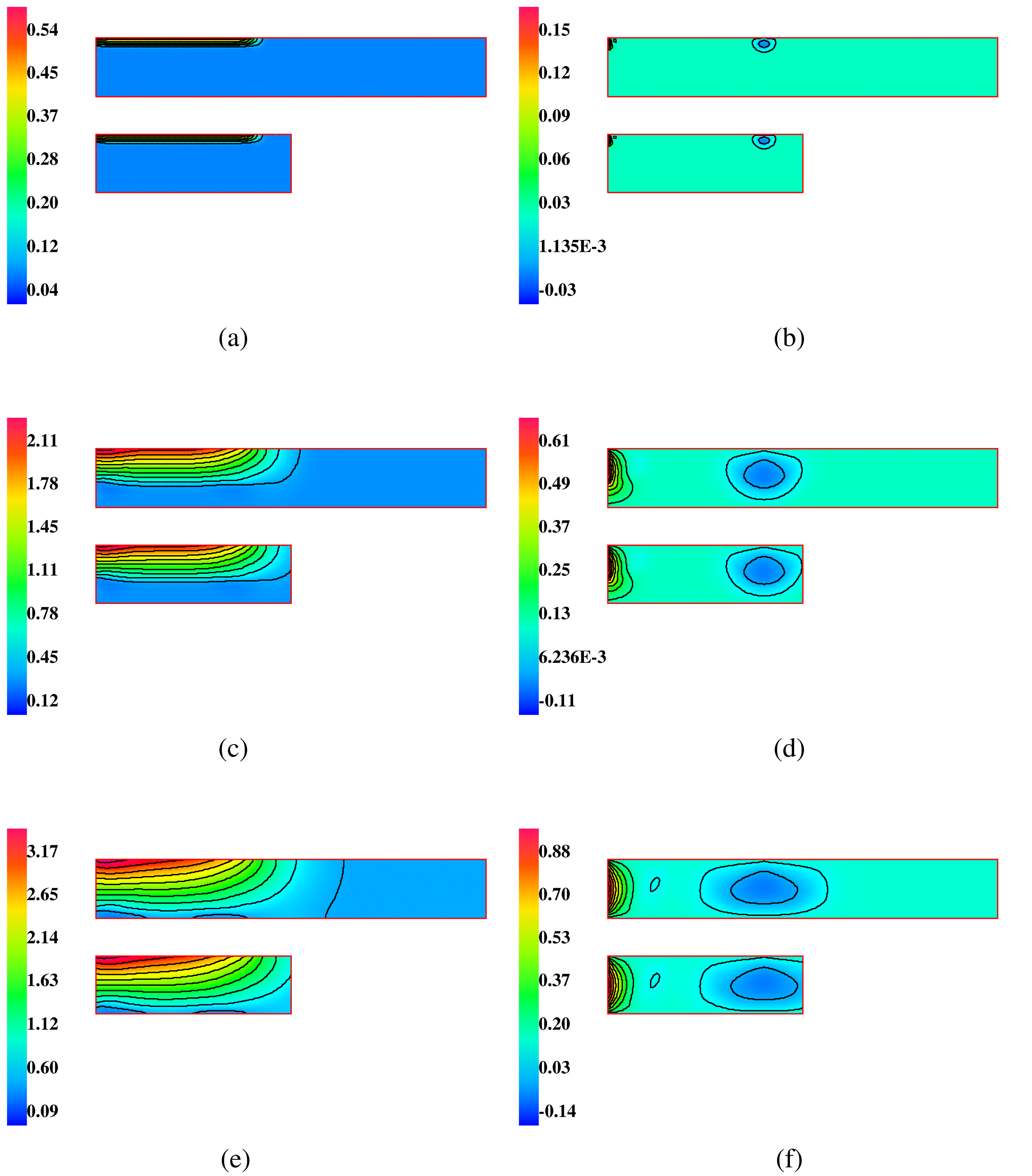

Figure 2: Surface loading problem: snapshots of solution. (a) u_x at $t = 0.5$; (b) u_y at $t = 0.5$; (c) u_x at $t = 2$; (d) u_y at $t = 2$; (e) u_x at $t = 3$, (f) u_y at $t = 3$.

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