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DYNAMIC CHARACTERISTICS OF STRUCTURES WITH VISCOELASTIC DAMPERS MODELED BY MEANS OF GENERALIZED RHEOLOGICAL MODELS

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Abstract. *Frame structures with viscoelastic (VE) dampers mounted on them are considered in this paper. Generalized rheological models are used to model the VE dampers. The finite element method is used to derive the equations of motion of a structure with dampers and such equations are written in terms of both physical and state-space variables. A solution to motion equations in the frequency domain is provided and the dynamic properties of the structure with VE dampers are determined as a solution to the appropriately defined eigenvalue problem. The dynamic characteristics of a relatively large structure with VE are determined and discussed.*

1 INTRODUCTION

Viscoelastic (VE) dampers are a passive type of energy dissipation devices, frequently used to mitigate excessive vibrations of structures due to winds or earthquakes. The properties of VE dampers, such as the possibility of energy dissipation and stiffness, are frequency and temperature dependent and are commonly defined in terms of experimentally obtained storage and loss modules. The frequency dependence of the properties of VE dampers can be accurately described by means of rheological models. Both the classical rheological models and the so-called rheological models with the fractional derivative are used [1, 2].

The dynamic analysis of frame or building structures with viscous and/or viscoelastic dampers are presented in a number of papers [3 - 14]. The most popular models of VE dampers are the simple rheological models, i.e., the Kelvin model and the Maxwell model. In papers [6 – 8] the simple Maxwell model was used. Moreover, in papers [7, 9, 13] the simple Kelvin model is used to describe the dynamic behavior of dampers. These simple models are used in [7, 12, 13] to solve the problem of optimum design of structures with VE dampers. However, as shown in [15], the simple Maxwell or Kelvin models cannot accurately approximate the frequency dependence of the storage modulus and the loss modulus of VE dampers. Recently, more advanced models of VE dampers have also been used when considering the dynamic analysis of structures with VE dampers. The dynamics of structures with VE dampers modeled by means of the generalized rheological models is very rarely discussed in the literature. These models are used in papers [4, 16] only. In papers [3, 5, 14] a three-parameter rheological model with the fractional-derivative model is used to model the VE dampers' behavior. Moreover, rational polynomial approximation modeling is used in paper [17] for an analysis of structures with VE dampers.

In this paper, frame structures equipped with VE dampers are considered. The frame is treated as a linear system while the generalized Kelvin model and the generalized Maxwell model are used to accurately describe the dynamic behaviour of VE dampers. The considered generalized models of VE dampers contain more parameters than the rheological models with the fractional derivative but lead to traditional differential equations of motion. The finite element method is adopted to write the equation of motion of the considered system in terms of both physical and state nodal variables. In particular, the solution of motion equation of a whole system in the frequency domain is considered and the dynamic properties of structure with VE dampers are determined. The frequencies of vibration, the non-dimensional damping ratios together with the corresponding eigenvectors are determined as a solution to the appropriately defined linear eigenvalue problem. Moreover, the frequency response functions are also determined. The results of typical calculations are presented and discussed.

Up to now, the dynamic characteristics of structures with VE dampers modeled by means of the generalized rheological models have not been considered or discussed in the available literature. Moreover, it was found that the VE damper model is not unique, i.e., both of the considered models can be used for modeling accurately the real damper for a sufficiently great number of model parameters.

2 THE GENERALIZED RHEOLOGICAL MODELS OF VE DAMPERS

The frequency dependence of the properties of VE dampers can be captured using generalized rheological models. The generalized Kelvin model and the generalized Maxwell model, as shown in Figs. 1 and 2, are used for modeling the VE dampers in this paper. The generalized Kelvin model is built of a spring and a set of the m Kelvin elements connected in series (see Fig.1). The Kelvin element is build of a spring and a dashpot connected in parallel. The generalized Maxwell model is built of the spring and a set of the m Maxwell elements connected in parallel (see Fig.2). The Maxwell element is the classical Maxwell rheological model, i.e., the spring and dashpot connected in series. As shown in [18], the frequency dependence of the properties of VE dampers can be accurately taken into account using the generalized rheological models.

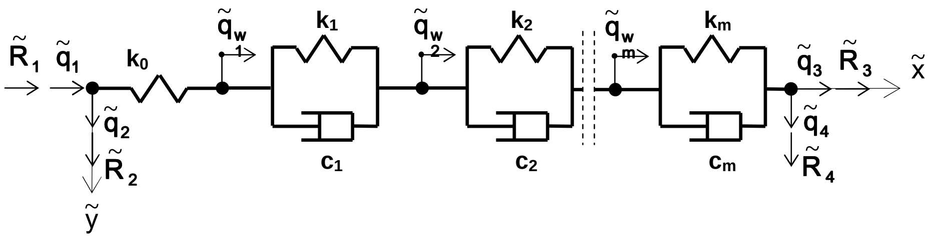

Figure 1: A schematic of the generalized Kelvin model

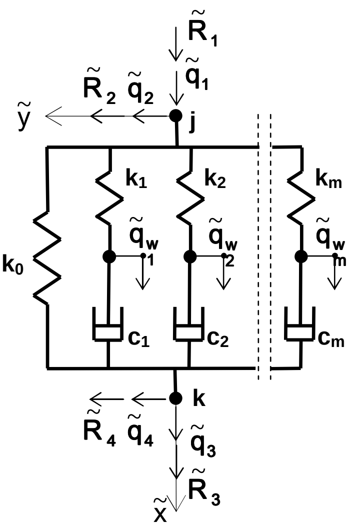

Figure 2: A schematic of the generalized Maxwell model

The concept of superelement and the concept of so-called internal variables have been used in describing both models. The dynamic behavior of the Kelvin damper can be described by means of the following equations:

$$u_0(t) = k_0(\tilde{q}_{w,1}(t) - \tilde{q}_1(t)) \ , \tag{1}$$

$$u_i(t) = k_i(\tilde{q}_{w,i+1}(t) - \tilde{q}_{w,i}(t)) + c_i(\dot{\tilde{q}}_{w,i+1}(t) - \dot{\tilde{q}}_{w,i}(t)) \ , \tag{2}$$

$$u_m(t) = k_m(\tilde{q}_3(t) - \tilde{q}_{w,m}) + c_m(\dot{\tilde{q}}_3(t) - \dot{\tilde{q}}_{w,m}) \ , \tag{3}$$

where $u_i(t)$ is the force in the *i-th* element of the model ($i = 0,1,..,m$), k_i and c_i are the spring stiffness and the damping factor of the dashpot of the *i-th* element of the model, respectively, and symbols $\tilde{q}_1(t)$ and $\tilde{q}_3(t)$ denote the external nodes displacements given in the local coordinate system (compare Fig.1). Moreover, the dot denotes differentiation with respect to time t and the symbol $\tilde{q}_{w,i}(t)$ denotes the internal variable ($i = 1,..,m$).

After introducing the vector of external reactions $\tilde{\mathbf{R}}_z(t) = col(\tilde{R}_1(t), \tilde{R}_2(t), \tilde{R}_3(t), \tilde{R}_4(t))$ (see Figure 1) and utilizing the equilibrium conditions of the external nodes: $\tilde{R}_1(t) = -u_0(t)$, $\tilde{R}_2(t) = 0$, $\tilde{R}_1(t) = u_m(t)$ and $\tilde{R}_4(t) = 0$ we can write the following matrix equation:

$$\tilde{\mathbf{R}}_z(t) = \tilde{\mathbf{K}}_{zz}\tilde{\mathbf{q}}_z(t) + \tilde{\mathbf{K}}_{zw}\tilde{\mathbf{q}}_w(t) + \tilde{\mathbf{C}}_{zz}\dot{\tilde{\mathbf{q}}}_z(t) + \tilde{\mathbf{C}}_{zw}\dot{\tilde{\mathbf{q}}}_w(t) \ , \tag{4}$$

where $\tilde{\mathbf{q}}_z(t) = col(\tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t), \tilde{q}_4(t))$, $\tilde{\mathbf{q}}_w(t) = col(\tilde{q}_{w,1}(t), ..., \tilde{q}_{w,m}(t))$. The matrices $\tilde{\mathbf{K}}_{zz}$, $\tilde{\mathbf{K}}_{zw}$, $\tilde{\mathbf{C}}_{zz}$ and $\tilde{\mathbf{C}}_{zw}$ are defined in Appendix A.

Moreover, the equilibrium conditions of the internal nodes, i.e., $u_{i-1}(t) - u_1(t) = 0$ for $i = 1,..,m$ lead to the following matrix equation:

$$\tilde{\mathbf{K}}_{wz}\tilde{\mathbf{q}}_z(t) + \tilde{\mathbf{K}}_{ww}\tilde{\mathbf{q}}_w(t) + \tilde{\mathbf{C}}_{wz}\dot{\tilde{\mathbf{q}}}_z(t) + \tilde{\mathbf{C}}_{ww}\dot{\tilde{\mathbf{q}}}_w(t) = \mathbf{0} \ , \tag{5}$$

where $\tilde{\mathbf{K}}_{wz} = \tilde{\mathbf{K}}_{zw}^T$, $\tilde{\mathbf{C}}_{wz} = \tilde{\mathbf{C}}_{zw}^T$ and the matrices $\tilde{\mathbf{K}}_{ww}$, $\tilde{\mathbf{C}}_{ww}$ are defined in Appendix A.

The equation of motion of the Kelvin model of a VE damper written in the local coordinate system can be finally presented in the form:

$$\tilde{\mathbf{R}}_d(t) = \tilde{\mathbf{K}}_d\tilde{\mathbf{q}}_d(t) + \tilde{\mathbf{C}}_d\dot{\tilde{\mathbf{q}}}_d(t) \ , \tag{6}$$

where $\tilde{\mathbf{R}}_d(t) = col(\tilde{\mathbf{R}}_z(t),\ \mathbf{0})$, $\tilde{\mathbf{q}}_d(t) = col(\tilde{\mathbf{q}}_z(t),\ \tilde{\mathbf{q}}_w(t))$,

$$\tilde{\mathbf{K}}_d = \begin{bmatrix} \tilde{\mathbf{K}}_{zz} & \tilde{\mathbf{K}}_{zw} \\ \tilde{\mathbf{K}}_{wz} & \tilde{\mathbf{K}}_{ww} \end{bmatrix}, \qquad \tilde{\mathbf{C}}_d = \begin{bmatrix} \tilde{\mathbf{C}}_{zz} & \tilde{\mathbf{C}}_{zw} \\ \tilde{\mathbf{C}}_{wz} & \tilde{\mathbf{C}}_{ww} \end{bmatrix}. \tag{7}$$

Transforming nodal parameters to the global coordinate system, the usual transformation of displacements of the external nodes of the damper is used while the internal variables of the damper are still defined in the local coordinate system. This means that the transformation matrix is:

$$\mathbf{T}_d = \begin{bmatrix} \tilde{\mathbf{T}}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{8}$$

where

$$\tilde{\mathbf{T}}_d = \begin{bmatrix} \tilde{\mathbf{T}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{T}} \end{bmatrix}, \qquad \tilde{\mathbf{T}} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}, \tag{9}$$

$c = \cos\alpha$, $s = \sin\alpha$, α is the angle between the global and local coordinate systems and $\mathbf{I}$ is the $(m \times m)$ identity matrix.

In the global coordinate system the generalized Kelvin model equation has the form:

$$\mathbf{R}_d(t) = \mathbf{K}_d\mathbf{q}_d(t) + \mathbf{C}_d\dot{\mathbf{q}}_d(t) \ , \tag{10}$$

where

$$\mathbf{R}_d(t) = col(\mathbf{R}_z(t),\ \mathbf{0}) = \mathbf{T}_d^T\tilde{\mathbf{R}}_d\mathbf{T}_d \ , \qquad \mathbf{R}_z(t) = col(R_1(t), R_2(t), R_3(t), R_4(t)) \ ,$$

$$\mathbf{q}_d(t) = col(\mathbf{q}_z(t),\ \mathbf{q}_w(t) = \tilde{\mathbf{q}}_w(t)) = \mathbf{T}_d^T\tilde{\mathbf{q}}_d\mathbf{T}_d \ , \quad \mathbf{q}_z(t) = col(q_1(t), q_2(t), q_3(t), q_4(t)) \ , \tag{11}$$

are the vector of nodal reactions and the vector of nodal parameters, respectively, in the global coordinate system. The explicit forms of matrices $\mathbf{K}_d$ and $\mathbf{C}_d$ are given in Appendix A.

The dynamic behavior of the Maxwell damper could be described in a similar way. Using the internal variables defined in Fig. 2, the following equations can be written:

$$u_0(t) = k_0(\tilde{q}_3(t) - \tilde{q}_1(t)) \ , \tag{12}$$

$$u_{is}(t) = k_i(\tilde{q}_{w,i}(t) - \tilde{q}_1(t)) \ , \tag{13}$$

$$u_{id}(t) = c_i(\dot{\tilde{q}}_3(t) - \dot{\tilde{q}}_{w,i}(t)) \ , \tag{14}$$

for the spring element and for the *i-th* Maxwell element ($i = 1,..,m$), respectively. The symbols $u_0(t)$, $u_{is}(t)$ and $u_{id}(t)$ in the above relationships denote the force in the spring element, the force in the spring of the *i-th* Maxwell element, and the force in the dashpot of the *i-th* Maxwell element, respectively.

The nodal reactions in the local coordinate system are:

$$\tilde{R}_1(t) = -u_0(t) - \sum_{i=1}^{m} u_{is}(t) \ , \quad \tilde{R}_2(t) = 0 \ , \quad \tilde{R}_3(t) = u_0(t) + \sum_{i=1}^{m} u_{id}(t) \ , \quad \tilde{R}_4(t) = 0 \ . \tag{15}$$

After introducing relationships (12) – (14) into Eqns (15) we obtain again Eqn (4) though its matrices are defined in Appendix B. Moreover, in the global coordinate system, Eqn (10) is also valid with the matrices $\mathbf{K}_d$ and $\mathbf{C}_d$ given in Appendix B.

Many particular rheological models described in the literature can be obtained by varying the number of elements in the generalized models mentioned above.

However, the simple Kelvin and Maxwell models, which contain only one Kelvin or Maxwell element, respectively, are not particular instances of the generalized models being discussed because the spring element with stiffness k_0 is not present. For the reader's convenience, a brief description of the above-mentioned simple models is also presented. In the matrix notation the equation of both models could be written in the form of Eqns (6) and (10). The matrices and vectors appearing in these equations are defined in Appendix C.

3 EQUATION OF MOTION AND DYNAMIC CHARACTERISTICS OF STRUCTURES WITH VE DAMPERS

Planar frame structures with VE dampers are modeled using the finite element method. A typical two-node bar element with six nodal parameters is used to describe the structure treated as the elastic system. The mass and stiffness matrices together with the vector of nodal forces of the element can be found in many books. The equation of motion of the structure with VE dampers modeled using the generalized rheological models can be written in the following form:

$$\mathbf{M}_{ss}\ddot{\mathbf{q}}_s(t) + \mathbf{C}_{ss}\dot{\mathbf{q}}_s(t) + \mathbf{C}_{sd}\dot{\mathbf{q}}_d(t) + \mathbf{K}_{ss}\mathbf{q}_s(t) + \mathbf{K}_{sd}\mathbf{q}_d(t) = \mathbf{p}_s(t) \ , \tag{16}$$

$$\mathbf{C}_{ds}\dot{\mathbf{q}}_s(t) + \mathbf{C}_{dd}\dot{\mathbf{q}}_d(t) + \mathbf{K}_{ds}\mathbf{q}_s(t) + \mathbf{K}_{dd}\mathbf{q}_d(t) = \mathbf{0} \ , \tag{17}$$

where the symbols $\mathbf{M}_{ss}$, $\mathbf{C}_{ss}$, $\mathbf{C}_{sd} = \mathbf{C}_{ds}^T$, $\mathbf{C}_{dd}$, $\mathbf{K}_{ss}$, $\mathbf{K}_{sd} = \mathbf{K}_{ds}^T$ and $\mathbf{K}_{dd}$ denote the mass, damping, and stiffness matrices of the system (i.e., structure with dampers), respectively, written in the global coordinate system. The dimension of matrices $\mathbf{M}_{ss}$, $\mathbf{C}_{ss} = \mathbf{C}_{ss}^{(s)} + \mathbf{C}_{ss}^{(d)}$

and $\mathbf{K}_{ss} = \mathbf{K}_{ss}^{(s)} + \mathbf{K}_{ss}^{(d)}$ is ($n \times n$). The matrices $\mathbf{M}_{ss}$, $\mathbf{C}_{ss}^{(s)}$ and $\mathbf{K}_{ss}^{(s)}$ describe the inertia, damping, and elastic properties of the structure without dampers, while the matrices $\mathbf{C}_{ss}^{(d)}$, $\mathbf{K}_{ss}^{(d)}$ and the ($n \times r$) matrices $\mathbf{C}_{sd} = \mathbf{C}_{ds}^T$, $\mathbf{K}_{sd} = \mathbf{K}_{ds}^T$ represent the effect of the coupling of dampers with the structure. The ($r \times r$) matrices $\mathbf{C}_{dd}$ and $\mathbf{K}_{dd}$ describe the damping and stiffness properties of dampers with braces, respectively. Moreover, $\mathbf{q}_s(t)$, $\mathbf{q}_d(t)$ and $\mathbf{p}_s(t)$ are the global vectors of nodal generalized displacements, internal variables and nodal excitation forces, respectively. The concept of proportional damping is used to model the damping properties of the structure, i.e.: $\mathbf{C}_{ss}^{(s)} = \alpha \mathbf{M}_{ss} + \kappa \mathbf{K}_{ss}^{(s)}$ where α and κ are proportionality factors.

The equation of motion written in terms of state variables will also be useful. After introducing the following state vector $\mathbf{x}(t) = col(\mathbf{q}_s(t),\ \dot{\mathbf{q}}_s(t),\ \mathbf{q}_d(t))$ and adding the equation:

$$\mathbf{M}_{ss}\dot{\mathbf{q}}_s(t) - \mathbf{M}_{ss}\dot{\mathbf{q}}_s(t) = \mathbf{0}, \tag{18}$$

to the system of Eqns (16) and (17) the following state equation could be written

$$\mathbf{A}\dot{\mathbf{x}}(t) + \mathbf{B}\mathbf{x}(t) = \mathbf{s}(t), \tag{19}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{M}_{ss} & \mathbf{C}_{sd} \\ \mathbf{M}_{ss} & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_{ds} & \mathbf{0} & \mathbf{C}_{dd} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{0} & \mathbf{K}_{sd} \\ \mathbf{0} & -\mathbf{M}_{ss} & \mathbf{0} \\ \mathbf{K}_{ds} & \mathbf{0} & \mathbf{K}_{dd} \end{bmatrix}, \quad \mathbf{s}(t) = \begin{Bmatrix} \mathbf{p}(t) \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}. \tag{20}$$

Please note that the matrices $\mathbf{A}$ and $\mathbf{B}$ are symmetrical and the matrix $\mathbf{B}$ is non-singular.

The solution to the homogenous state equation, i.e., when $\mathbf{s}(t) = \mathbf{0}$, is assumed in the form:

$$\mathbf{x}(t) = \mathbf{a} \exp(st). \tag{21}$$

Introducing (21) into the state equation (19) we obtain the following eigenvalue problem:

$$(s\mathbf{A} + \mathbf{B})\,\mathbf{a} = \mathbf{0}. \tag{22}$$

The linear eigenvalue problem (22) must be solved to determine the ($2n + r$) eigenvalues s_i and eigenvectors $\mathbf{a}_i$. In the case of an undercritically damped structure the $2n$ eigenvalues (eigenvectors) are complex and conjugate numbers (vectors) while the remaining r eigenvalues (eigenvectors) are real numbers (vectors).

The dynamic behavior of a frame with VE dampers is characterized by the natural frequencies ω_i and the non-dimensional damping parameters γ_i. The above-mentioned quantities are defined as:

$$\omega_i^2 = \mu_i^2 + \eta_i^2, \qquad \gamma_i = -\mu_i / \omega_i, \tag{23}$$

where $\mu_i = \mathrm{Re}(s_i)$, $\eta_i = \mathrm{Im}(s_i)$. The formulae (23) refer to complex eigenvalues only.

The third dynamic characteristic of the considered system are the frequency response functions. Before determination of these functions it is useful to rewrite equations of motion (16) and (17) in the form of the following one matrix equation:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \overline{\mathbf{p}}(t), \tag{24}$$

where

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{ss} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{ss} & \mathbf{C}_{sd} \\ \mathbf{C}_{ds} & \mathbf{C}_{dd} \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sd} \\ \mathbf{K}_{ds} & \mathbf{K}_{dd} \end{bmatrix}, \quad \mathbf{q}(t) = \begin{Bmatrix} \mathbf{q}_s(t) \\ \mathbf{q}_d(t) \end{Bmatrix}, \quad \overline{\mathbf{p}}(t) = \begin{Bmatrix} \mathbf{p}(t) \\ \mathbf{0} \end{Bmatrix} . \quad (25)$$

To determine the frequency response functions the steady state harmonic responses of the system are considered. If the excitation forces vary harmonically in time, i.e., when

$$\overline{\mathbf{p}}(t) = \overline{\mathbf{P}} \exp(\mathrm{i}\lambda t) , \quad (26)$$

then the steady state response of the system under consideration can be described by:

$$\mathbf{q}(t) = \mathbf{Q} \exp(\mathrm{i}\lambda t) , \quad (27)$$

where $\mathrm{i} = \sqrt{-1}$ is an imaginary unit.

After substituting Eqns (26) and (26) into Eqn (24) the following standard formula describes the matrix of the frequency response functions:

$$\mathbf{H}(\lambda) = (-\lambda^2 \mathbf{M} + \mathrm{i}\lambda \mathbf{C} + \mathbf{K})^{-1} . \quad (28)$$

If the structure with dampers modeled by the simple Maxwell model is considered, then all of the relationships presented above in this section are valid provided that the matrices given in Appendix C are used to generate the global matrices appearing in Eqns (16) and (17).

The vector of internal variables $\mathbf{q}_d(t)$ does not exist in the case of a structure with dampers modeled by the simple Kelvin model, and the motion equation (16) takes the form:

$$\mathbf{M}_{ss}\ddot{\mathbf{q}}_s(t) + (\mathbf{C}_{ss} + \mathbf{C}_{dd})\,\dot{\mathbf{q}}_s(t) + (\mathbf{K}_{ss} + \mathbf{K}_{dd})\,\mathbf{q}_s(t) = \mathbf{p}_s(t) . \quad (29)$$

The matrices $\mathbf{C}_{dd}$ and $\mathbf{K}_{dd}$ appearing in (29) are built from the matrices $\mathbf{C}_d$ and $\mathbf{K}_d$, respectively, given by formulae (C.4).

The state equation has the form of Equation (19) where, now

$$\mathbf{x}(t) = \begin{Bmatrix} \mathbf{q}_s(t) \\ \dot{\mathbf{q}}_s(t) \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{C}_{ss} + \mathbf{C}_{dd} & \mathbf{M}_{ss} \\ \mathbf{M}_{ss} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{K}_{ss} + \mathbf{K}_{dd} & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ss} \end{bmatrix}, \quad \mathbf{s}(t) = \begin{Bmatrix} \mathbf{p}(t) \\ \mathbf{0} \end{Bmatrix} . \quad (30)$$

Moreover, the matrix of frequency response functions must be defined as

$$\mathbf{H}(\lambda) = \left[-\lambda^2 \mathbf{M}_{ss} + \mathrm{i}\lambda(\mathbf{C}_{ss} + \mathbf{C}_{dd}) + \mathbf{K}_{ss} + \mathbf{K}_{dd} \right]^{-1} . \quad (31)$$

4 RESULTS OF TYPICAL CALCULATIONS

4.1 Description of a representative structure and VE dampers

An eight-storey RC frame with three-bays is selected to determine the dynamic characteristics of a structure with dampers. The frame is designed according to the EC8 Part 1 for class B soils. The height of the columns is 3.0 m, the span of the beams is 5.0 m and Young's modulus for concrete is 31.0 GPa. The dimensions of the cross-section of the structural elements are presented in Table 1 while the unit masses of the frame elements are given in Table 2.

Table 3 shows the natural frequencies of vibration of the frame without dampers. The dynamic properties of the structure are obtained by way of a two-dimensional analysis of the frame. Any axial deformations and internal damping of the structure are neglected.

The dampers are attached in the middle bay on all floors of the structure. The dampers are modeled using the following rheological models: i) the simple Kelvin model; ii) the simple

Maxwell model; iii) the generalized Kelvin model with seven parameters, and iv) the generalized Maxwell model also with seven parameters.

Storey level	Lateral column [cm]	Central column [cm]	Beams [cm]
7, 8	35×35	40×40	30×40
5, 6	40×40	45×45	30×45
3, 4	45×45	53×53	30×50
1, 2	50×50	60×60	30×50

Table 1: Dimensions of eight-storey frame elements

Storey level	Unit lateral column mass [kg/m]	Unit central column mass [kg/m]	Unit beam mass [kg/m]
7, 8	306.2	400.0	15000.0
5, 6	400.0	506.2	15000.0
3, 4	506.2	702.2	15000.0
1, 2	625.0	900.0	15000.0

Table 2: Unit mass of eight-storey frame elements

Natural frequencies [rad/sec]			
3.1311	8.6582	15.4268	23.7804
31.2647	40.1148	42.1251	51.1550
52.3598	57.6067	65.6532	69.9862

Table 3: Natural frequencies of frame without dampers

Data from the real experiment are not adopted in this paper. Instead, the storage and loss modulus of dampers are calculated from the formulae:

$$K'(\lambda) = k + c\lambda^{\alpha} \cos(\alpha\pi / 2) \ , \qquad K''(\lambda) = c\lambda^{\alpha} \sin(\alpha\pi / 2) \ , \tag{32}$$

which are the analytical formulae of the fractional-derivative Kelvin model of dampers. The chosen parameters of the fractional-derivative Kelvin model are: $\alpha = 0.63$, $k = 0.4 \times 10^6 \, N/m$ and $c = 3.6 \times 10^6 \, N \sec^{\alpha}/m$. The value of the parameter α is similar to the one used in papers [1, 4], but the original values of k and c are divided by 2.0.

In paper [1] the parameters of generalized models are obtained by minimizing the mean square norm of the differences between the targeted modules and the analytical modules of the considered model. The parameters of the generalized Kelvin model and the generalized Maxwell model, both with seven parameters and used in this paper, are given in Table 4.

	Stiffness $(\times 10^6)\,[N/m]$			Damping factor $(\times 10^6)\,[N\sec/m]$	
	Kelvin model	Maxwell model		Kelvin model	Maxwell model
k_0	57.650	0.1065		–	–
k_1	18.350	33.385	c_1	2.729	1.478
k_2	6.160	3.310	c_2	6.190	1.732
k_3	0.5545	1.443	c_3	8.675	8.305

Table 4: Parameters of generalized Kelvin and Maxwell models

The energy dissipated by the damper was calculated assuming that the amplitude of a harmonically varying vibration of the damper is equal to $0.01\,m$ in all of the considered cases. From this calculation, it can be concluded that the loss modulus and dissipation energy of the fractional-derivative Kelvin model and both generalized models are approximately equal in the range 0 – 15.0 rad/sec of excitation frequency. This range of frequency covers the range of the first three natural frequencies of vibration of the structure considered.

The values of the parameters of the simple Kelvin model are: $k = 0.74637 \times 10^7 \; N/m$, $c = 0.134420 \times 10^7 \; N\sec/m$ and the values of the parameters of the simple Maxwell model are: $k = 1.90392 \times 10^7 \; N/m$ and $c = 0.338669 \times 10^7 \; N\sec/m$. These parameters are calculated by minimizing the mean square norm of difference between the target modules, given by (32), and the analytical modules of the respective model. However, the dissipation energy of simple models and the dissipation energy of the fractional-derivative model are not equal and, as expected, the differences are significant, especially for the simple Maxwell model.

Chevron braces are used to connect the dampers with the structure. The braces are made of HEB 200 stainless steel profiles of which the parameters are: $EA = 1.60105 \times 10^9$ N and $EJ = 1.1685 \times 10^7$ Nm^2.

4.2 Comparison of dynamic characteristics of considered frame

Results of the solution to the eigenvalue problems are presented in Tables 5 – 10. The real and the complex conjugate numbers are obtained as eigenvalues. In Table 5 the values of the first three complex conjugate eigenvalues are given for all of the considered models (the two parameter model denotes the simple Kelvin or Maxwell model).

Kelvin model of damper with		Maxwell model of damper with	
7 parameters	2 parameters	7 parameters	2 parameters
-0.18461 ± i 3.30499	-0.1224 ± i 3.3754	-0.18260 ± i 3.30439	-0.25113 ± i 3.31731
-0.99448 ± i 9.39577	-0.9846 ± i 9.28452	-1.01395 ± i 9.50962	-0.58872 ± i 9.83667
-2.09561 ± i 16.8189	-2.4740 ± i 16.0635	-1.76745 ± i 17.2242	-0.56505 ± i 17.0933

Table 5: The first three complex conjugate eigenvalues of frame with dampers

Kelvin model of damper with 7 parameters $[\sec^{-1}]$		Maxwell model of damper with 7 parameters $[\sec^{-1}]$	
-0.377771	-2.90147	-0.166298	-1.83572
-0.382183	-2.93738	-0.167949	-1.85359
-0.383780	-2.95781	-0.168588	-1.86274
-0.386515	-3.01417	-0.169621	-1.89241
-0.389423	-27.3841	-0.170717	-15.3707
-0.391331	-28.5853	-0.171435	-16.1435
-0.392254	-29.2224	-0.171781	-16.6836
-0.395517	-30.4773	-0.173008	-17.2797
-2.72596	-31.9943	-1.73882	-17.9245
-2.80627	-33.8003	-1.78531	-18.8592
-2.84271	-35.3870	-1.78985	-20.0361
-2.85854	-36.0392	-1.80968	-20.5792

Table 6: Real eigenvalues of frame with dampers

Given in Tables 6 and 7 are all real eigenvalues obtained for all of the models considered. The real eigenvalues for both of the generalized models could be divided into three groups, each with eight elements. The values of elements in one group are of the same order as the eigenvalues of the eigenproblem obtained for one separate damper. For the generalized Kelvin model of damper with seven parameters the above-mentioned problem has the following form:

$$\left(\begin{bmatrix} k_0 + k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix} + s \begin{bmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 + c_2 & -c_2 \\ 0 & -c_2 & c_2 + c_3 \end{bmatrix} \right) \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} , \qquad (33)$$

from which we obtain: $s_1 = -0.3975\,\text{sec}^{-1}$, $s_2 = -3.0518\,\text{sec}^{-1}$ and $s_3 = -41.418\,\text{sec}^{-1}$. In the case of the generalized Maxwell model the equations of the above-mentioned eigenproblem are uncoupled and the real eigenvalues are approximately equal to the inverse of the relaxation time of the Maxwell element (with a minus sign), i.e., in the case under consideration, $s_1 = -k_1 / c_1 = -22.588\,\text{sec}^{-1}$, $s_2 = -1.911\,\text{sec}^{-1}$ and $s_3 = -0.1738\,\text{sec}^{-1}$. A similar remark is true for the real eigenvalues of the frame with dampers modeled using the simple Kelvin model and presented in Table 7.

The reason why real eigenvalues exist for the frame with dampers modeled by the simple Kelvin model, given in Table 8, is completely different. The loss factor of the simple Kelvin model is a linear function of excitation frequency. This means that higher modes of vibration are more strongly damped, compared with lower ones, and could be overdamped, which happens in the case under consideration where eight modes of vibration are overcritically damped. In other cases, all modes are undercritically damped. It is a qualitative difference, compared with other damper models.

Maxwell model of damper with 2 parameters [sec^{-1}]	
-3.75157	-4.57767
-4.09211	-4.79722
-4.27776	-4.98507
-4.53600	-5.32095

Table 7: Real eigenvalues for frame with dampers modeled by the simple Maxwell model

Kelvin model of damper with 2 parameters [sec^{-1}]			
-231.120	-254.940	-14840.1	-14847.5
-240.435	-258.708	-14845.1	-14848.7
-245.480	-261.561	-14846.0	-14849.6
-250.975	-262.479	-14846.9	-14855.5

Table 8: Real eigenvalues for frame with dampers modeled by the simple Kelvin model

The first three natural frequencies of the frame with dampers modeled using different models are presented in Table 9. As is easily verified, the damper model does not significantly change the first three natural frequencies of the frame. The maximal difference is 6.1%. However, as is obvious from Table 10, the influence of the damper model on the non-dimensional damping ratios is substantial because the maximal difference is of the order of 80%.

Natural frequency	Kelvin model of damper with		Maxwell model of damper with	
	7 parameters [rad/sec]	2 parameters [rad/sec]	7 parameters [rad/sec]	2 parameters [rad/sec]
ω_1	3.31014	3.37757	3.30944	3.32681
ω_2	9.44825	9.33658	9.56352	9.85427
ω_3	16.9490	16.2529	17.3146	17.1026

Table 9: The natural frequencies of frame with dampers

Damping ratio	Kelvin model of damper with		Maxwell model of damper with	
	7 parameters	2 parameters	7 parameters	2 parameters
γ_1	0.0557702	0.0362529	0.0551757	0.0754865
γ_2	0.105256	0.105455	0.106023	0.0597427
γ_3	0.123642	0.152223	0.102078	0.0330390

Table 10: Non-dimensional damping ratios of frame with dampers

A comparison of the first and second modes of vibration is shown in Figures 3. The dashed lines represent the shape of vibration of frame without dampers while the real and imaginary parts of eigenvector for the frame with dampers are shown by the solid line and the solid line with crosses, respectively. Moreover, the imaginary part of eigenvectors is multiplied by 10 in order to show this quantity in detail. The calculation is done for dampers modeled using the Kelvin model with seven parameters. It is easy to observe that the real part of both eigenvectors is very similar to the respective mode of vibration of the frame without dampers.

The frequency response functions are also calculated and one example is shown in Figures 4 where the frequency response function corresponding to the horizontal displacement of the eighth storey is shown. The frequency response curve for the frame with dampers modeled using the generalized Kelvin model (the solid line) and the simple Kelvin model (the dashed line) is shown.

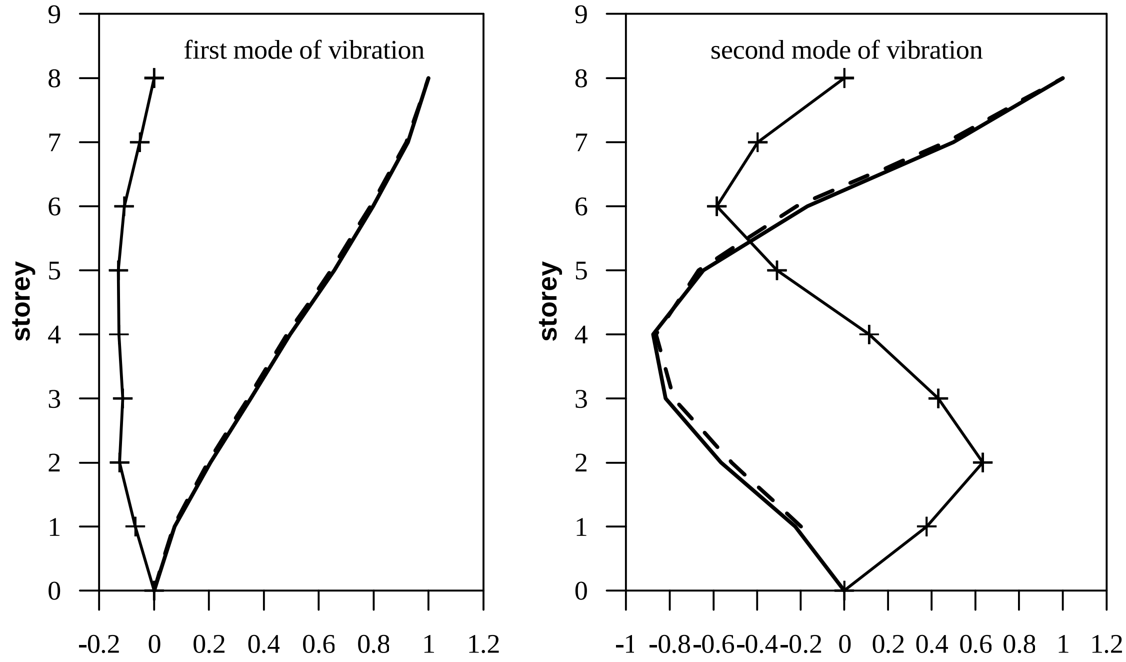

Figure 3: A comparison of modes of vibration for a frame without dampers (the dashed lines) with the real part of eigenvector for frame with dampers modeled by the generalized Kelvin model (the solid line). The imaginary part of the eigenvector is multiplied by 10 and shown by the solid line with crosses

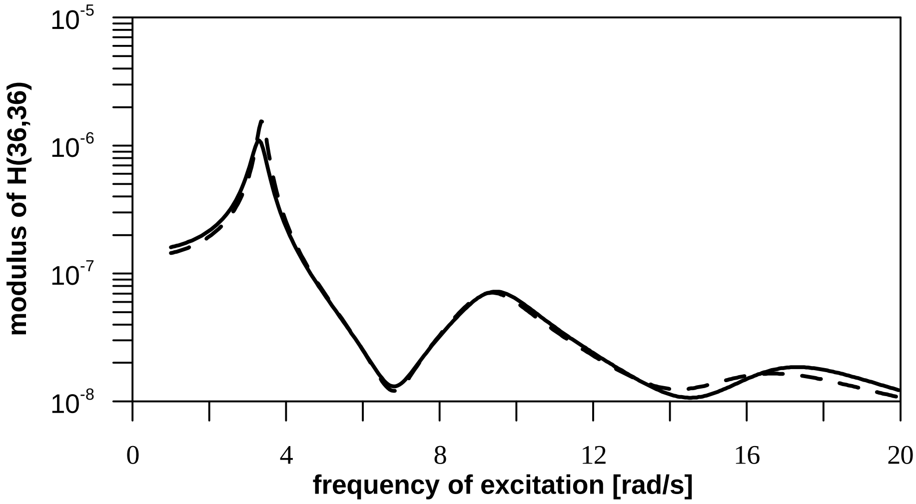

Figure 4: Frequency response function of frame with dampers modeled by the generalized Kelvin model (the solid line) and by the simple Kelvin model (the dashed line)

5 CONCLUSIONS

Several models of dampers are used in this paper to describe the dynamic behaviour of frame structures with VE dampers. A family of generalized rheological models, including the very often used simple Kelvin and Maxwell models, are compared in detail. The comparison is made in the frequency domain for a carefully selected frame structure with VE dampers. The finite element method is used to derive equations of motion.

Several conclusions can be formulated on the basis of the results of numerical analysis presented above. The most important ones are listed below.

- Different models are able to correctly describe the dynamic behaviour of VE dampers. The seven-parameter Kelvin model and the seven-parameter Maxwell model provide almost identical results. This conclusion is in agreement with the results presented by Singh and Chang [4] where the generalized Kelvin and Maxwell models are used as models of VE dampers.
- The simple Kelvin and the simple Maxwell models are not able to correctly describe, in the frequency domain, the dynamic behavior of frames with VE dampers. In particular, relative differences concerning the non-dimensional damping ratios are large.
- The linear eigenvalue problems must be solved in order to determine the dynamic characteristics of the frame with VE dampers. The solution procedure for this problem is much simpler than the solution procedure for the nonlinear eigenvalue problem obtained when the fractional-derivative Kelvin model or the complex modulus model are used as the VE damper model.
- There are some qualitative differences between the results obtained. For the frame with a fixed number of VE dampers the total number of eigenvalues and eigenvectors depends on the selected model of dampers and the number of parameters of the models. Both the real and complex eigenvalues are obtained. The number of real eigenvalues depends on dampers model.

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6 APPENDIX A

In this appendix the explicit form of the matrices used to describe the generalized Kelvin model of the VE damper is given.

$$\tilde{\mathbf{K}}_{zz} = \begin{bmatrix} k_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k_m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathbf{C}}_{zz} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_m & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{A.1}$$

$$\tilde{\mathbf{K}}_{zw} = \begin{bmatrix} -k_0 & 0.....0.......0 \\ 0 & 0.....0.......0 \\ 0 & 0.....0.....-k_m \\ 0 & 0.....0.......0 \end{bmatrix}, \qquad \tilde{\mathbf{C}}_{zw} = \begin{bmatrix} 0 & 0.....0.......0 \\ 0 & 0.....0.......0 \\ 0 & 0.....0.....-c_m \\ 0 & 0.....0.......0 \end{bmatrix}, \tag{A.2}$$

$$\tilde{\mathbf{K}}_{ww} = \begin{bmatrix} k_0+k_1 & -k_1.............0 & 0 & 0.............0 & 0 \\ -k_1 & k_1+k_2..........0 & 0 & 0.............0 & 0 \\ \multicolumn{5}{c}{...} \\ 0 & 0............-k_{i-1} & k_{i-1}+k_i & -k_i..........0 & 0 \\ \multicolumn{5}{c}{...} \\ 0 & 0................0 & 0 & 0.......-k_{m-1} & k_{m-1}+k_m \end{bmatrix}, \tag{A.3}$$

$$\tilde{\mathbf{C}}_{ww} = \begin{bmatrix} c_1 & -c_1 \ldots\ldots\ldots\ldots 0 & 0 & 0 \ldots\ldots\ldots 0 & 0 \\ -c_1 & c_1 + c_2 \ldots\ldots\ldots 0 & 0 & 0 \ldots\ldots\ldots 0 & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 \ldots\ldots\ldots -c_{i-1} & c_{i-1} + c_i & -c_i \ldots\ldots\ldots 0 & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ 0 & 0 \ldots\ldots\ldots\ldots 0 & 0 & 0 \ldots\ldots -c_{m-1} & c_{m-1} + c_m \end{bmatrix}, \tag{A.4}$$

$$\mathbf{K}_d = \mathbf{T}_d^T \tilde{\mathbf{K}}_d \mathbf{T}_d = \begin{bmatrix} \mathbf{K}_{zz} & \mathbf{K}_{zw} \\ \mathbf{K}_{wz} & \mathbf{K}_{ww} \end{bmatrix}, \qquad \mathbf{C}_d = \mathbf{T}_d^T \tilde{\mathbf{C}}_d \mathbf{T}_d = \begin{bmatrix} \mathbf{C}_{zz} & \mathbf{C}_{zw} \\ \mathbf{C}_{wz} & \mathbf{C}_{ww} \end{bmatrix}, \tag{A.5}$$

$$\mathbf{K}_{zz} = \begin{bmatrix} c^2 k_0 & csk_0 & 0 & 0 \\ csk_0 & s^2 k_0 & 0 & 0 \\ 0 & 0 & c^2 k_m & csk_m \\ 0 & 0 & csk_m & s^2 k_m \end{bmatrix}, \qquad \mathbf{C}_{zz} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 c_m & \mathrm{csc}_m \\ 0 & 0 & \mathrm{csc}_m & \mathrm{csc}_m \end{bmatrix}, \tag{A.6}$$

$$\mathbf{K}_{zw} = \mathbf{K}_{wz}^T = \begin{bmatrix} -ck_0 & 0 \ldots\ldots 0 \ldots\ldots\ldots & 0 \\ -sk_0 & 0 \ldots\ldots 0 \ldots\ldots\ldots & 0 \\ 0 & 0 \ldots\ldots 0 \ldots\ldots & -ck_m \\ 0 & 0 \ldots\ldots 0 \ldots\ldots & -sk_m \end{bmatrix}, \quad \mathbf{C}_{zw} = \mathbf{C}_{wz}^T = \begin{bmatrix} 0 & 0 \ldots\ldots 0 \ldots\ldots\ldots & 0 \\ 0 & 0 \ldots\ldots 0 \ldots\ldots\ldots & 0 \\ 0 & 0 \ldots\ldots 0 \ldots\ldots & -cc_m \\ 0 & 0 \ldots\ldots 0 \ldots\ldots & -sc_m \end{bmatrix}, \tag{A.7}$$

$$\mathbf{K}_{ww} = \tilde{\mathbf{K}}_{ww}, \qquad \mathbf{C}_{ww} = \tilde{\mathbf{C}}_{ww}. \tag{A.8}$$

7 APPENDIX B

In this Appendix the explicit form of matrices used to describe the generalized Maxwell model of the VE damper is given.

$$\tilde{\mathbf{K}}_{zz} = \begin{bmatrix} k_0 + \sum_{i=1}^{m} k_i & 0 & -k_0 & 0 \\ 0 & 0 & 0 & 0 \\ -k_0 & 0 & k_0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathbf{C}}_{zz} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sum_{i=1}^{m} c_i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{B.1}$$

$$\tilde{\mathbf{K}}_{zw} = \begin{bmatrix} -k_1 & -k_2 \ldots\ldots -k_i \ldots\ldots\ldots & -k_m \\ 0 & 0 \ldots\ldots\ldots\ 0 \ldots\ldots\ldots & 0 \\ 0 & 0 \ldots\ldots\ldots\ 0 \ldots\ldots\ldots & 0 \\ 0 & 0 \ldots\ldots\ 0 \ldots\ldots\ldots & 0 \end{bmatrix}, \qquad \tilde{\mathbf{C}}_{zw} = \begin{bmatrix} 0 & 0 \ldots\ldots\ldots\ 0 \ldots\ldots\ldots & 0 \\ 0 & 0 \ldots\ldots\ldots\ 0 \ldots\ldots\ldots & 0 \\ -c_1 & -c_2 \ldots\ldots -c_i \ldots\ldots\ldots & -c_m \\ 0 & 0 \ldots\ldots\ 0 \ldots\ldots\ldots & 0 \end{bmatrix}, \tag{B.2}$$

$$\tilde{\mathbf{K}}_{ww} = diag(k_1,\ k_2, \ldots\ldots\ldots, k_m,), \qquad \tilde{\mathbf{C}}_{ww} = diag(c_1,\ c_2, \ldots\ldots\ldots, c_m,), \tag{B.3}$$

$$\mathbf{K}_d = \mathbf{T}_d^T \tilde{\mathbf{K}}_d \mathbf{T}_d = \begin{bmatrix} \mathbf{K}_{zz} & \mathbf{K}_{zw} \\ \mathbf{K}_{wz} & \mathbf{K}_{ww} \end{bmatrix}, \qquad \mathbf{C}_d = \mathbf{T}_d^T \tilde{\mathbf{C}}_d \mathbf{T}_d = \begin{bmatrix} \mathbf{C}_{zz} & \mathbf{C}_{zw} \\ \mathbf{C}_{wz} & \mathbf{C}_{ww} \end{bmatrix}, \tag{B.4}$$

$$\mathbf{K}_{ww} = \tilde{\mathbf{K}}_{ww}, \qquad \mathbf{C}_{ww} = \tilde{\mathbf{C}}_{ww}, \tag{B.5}$$

$$\mathbf{K}_{zz} = \begin{bmatrix} c^2\left(k_0 + \sum_{i=1}^{m} k_i\right) & cs\left(k_0 + \sum_{i=1}^{m} k_i\right) & -c^2 k_0 & -csk_0 \\ cs\left(k_0 + \sum_{i=1}^{m} k_i\right) & s^2\left(k_0 + \sum_{i=1}^{m} k_i\right) & -csk_0 & -s^2 k_0 \\ -c^2 k_0 & -csk_0 & c^2 k_0 & csk_0 \\ -csk_0 & -s^2 k_0 & csk_0 & s^2 k_0 \end{bmatrix}, \qquad (B.6)$$

$$\mathbf{C}_{zz} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 \sum_{i=1}^{m} c_i & cs \sum_{i=1}^{m} c_i \\ 0 & 0 & cs \sum_{i=1}^{m} c_i & s^2 \sum_{i=1}^{m} c_i \end{bmatrix}, \qquad \mathbf{K}_{zw} = \mathbf{K}_{zw}^T = \begin{bmatrix} -ck_1 - ck_2 \ldots\ldots - ck_i \ldots\ldots\ldots - ck_m \\ -sk_1 - sk_2 \ldots\ldots - sk_i \ldots\ldots\ldots - sk_m \\ 0 \quad 0\ldots\ldots\ldots \quad 0 \ \ldots\ldots\ldots \quad 0 \\ 0 \quad 0\ldots\ldots\ldots \quad 0 \ \ldots\ldots\ldots \quad 0 \end{bmatrix}, \qquad (B.7)$$

$$\mathbf{C}_{zw} = \mathbf{C}_{zw}^T = \begin{bmatrix} 0 \quad 0\ldots\ldots\ldots \quad 0 \ \ldots\ldots\ldots \quad 0 \\ 0 \quad 0\ldots\ldots\ldots \quad 0 \ \ldots\ldots\ldots \quad 0 \\ -cc_1 - cc_2 \ldots\ldots - cc_i \ldots\ldots\ldots - cc_m \\ -sc_1 - sc_2 \ldots\ldots - sc_i \ldots\ldots\ldots - sc_m \end{bmatrix}. \qquad (B.8)$$

8 APPENDIX C

The explicit form of the vectors and matrices used to describe the simple Kelvin model of VE damper is:

$$\tilde{\mathbf{q}}_d(t) = \tilde{\mathbf{q}}_z(t) = col(\tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t), \tilde{q}_4(t)) \ , \qquad (C.1)$$

$$\tilde{\mathbf{K}}_d = \tilde{\mathbf{K}}_{zz} = k_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \tilde{\mathbf{C}}_d = \tilde{\mathbf{C}}_{zz} = c_1 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad (C.2)$$

$$\mathbf{q}_d(t) = \mathbf{q}_z(t) = col(q_1(t), q_2(t), q_3(t), q_4(t)) \ , \qquad (C.3)$$

$$\mathbf{K}_d = \mathbf{K}_{zz} = k_1 \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}, \quad \mathbf{C}_d = \mathbf{C}_{zz} = c_1 \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}, \qquad (C.4)$$

The explicit form of the matrices used to describe the simple Maxwell model of VE damper is:

$$\tilde{\mathbf{q}}_d(t) = col(\tilde{\mathbf{q}}_z(t), \ \tilde{\mathbf{q}}_w(t)) \ , \qquad (C.5)$$

$$\tilde{\mathbf{q}}_z(t) = col(\tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t), \tilde{q}_4(t)) \ , \qquad \tilde{\mathbf{q}}_w(t) = col(\tilde{q}_{w,1}(t)) \ , \qquad (C.6)$$

$$\tilde{\mathbf{K}}_{zz} = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{K}}_{zw} = \tilde{\mathbf{K}}_{wz}^T = \begin{bmatrix} -k_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{K}}_{ww} = [\,k_1\,], \tag{C.7}$$

$$\tilde{\mathbf{C}}_{zz} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{C}}_{zz} = \tilde{\mathbf{C}}_{wz}^T = \begin{bmatrix} 0 \\ 0 \\ -c_1 \\ 0 \end{bmatrix}, \quad \tilde{\mathbf{C}}_{ww} = [\,c_1\,], \tag{C.8}$$

$$\mathbf{q}_d(t) = col(\mathbf{q}_z(t),\ \mathbf{q}_w(t)), \tag{C.9}$$

$$\mathbf{q}_z(t) = col(q_1(t), q_2(t), q_3(t), q_4(t)), \quad \mathbf{q}_w(t) = col(q_{w,1}(t)), \quad \mathbf{K}_d = \begin{bmatrix} \mathbf{K}_{zz} & \mathbf{K}_{zw} \\ \mathbf{K}_{wz} & \mathbf{K}_{ww} \end{bmatrix}, \tag{C.10}$$

$$\mathbf{K}_d = \begin{bmatrix} \mathbf{K}_{zz} & \mathbf{K}_{zw} \\ \mathbf{K}_{wz} & \mathbf{K}_{ww} \end{bmatrix}, \quad \mathbf{C}_d = \begin{bmatrix} \mathbf{C}_{zz} & \mathbf{C}_{zw} \\ \mathbf{C}_{wz} & \mathbf{C}_{ww} \end{bmatrix}, \tag{C.11}$$

$$\mathbf{K}_{zz} = k_1 \begin{bmatrix} c^2 & cs & 0 & 0 \\ cs & s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{K}_{zw} = \mathbf{K}_{zw}^T = k_1 \begin{bmatrix} -c \\ -s \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{K}_{ww} = [\,k_1\,], \tag{C.12}$$

$$\mathbf{C}_{zz} = c_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 & cs \\ 0 & 0 & cs & s^2 \end{bmatrix}, \quad \mathbf{C}_{zw} = \mathbf{C}_{zw}^T = c_1 \begin{bmatrix} 0 \\ 0 \\ -c \\ -s \end{bmatrix}, \quad \mathbf{C}_{ww} = [\,c_1\,]. \tag{C.13}$$

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