A FINITE ELEMENT METHOD TO COMPUTE DAMPED VIBRATION MODES IN DISSIPATIVE ACOUSTICS

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Abstract. We analyze a quadratic eigenvalue problem related to the damped vibrations of an acoustic fluid in a cavity with absorbing walls. The problem is shown to be equivalent to the spectral problem for a non-compact operator and a thorough spectral characterization is given. We consider a discretization based on Raviart-Thomas finite elements and show that it is free of spurious modes and convergent with optimal order. Numerical results for a test case with known exact analytical solution are also included.
1 INTRODUCTION

Numerical modeling of acoustic vibrations is usually done by using pressure to describe the fluid motion. However, if wall damping is considered then modal analysis leads to spectral problems which are neither linear nor quadratic so rather difficult from the mathematical point of view. An alternative approach consists on writing the model in terms of displacements. While we have to deal with a vector function instead of a scalar one, the above mentioned spectral problems are quadratic thus allowing for an easier mathematical analysis.

It is well known that displacement formulation for acoustic fluid suffers from the presence of zero-frequency spurious circulation modes with no physical meaning. Furthermore, discretization by standard finite elements produces approximations of these modes having non-zero frequencies interspersed among the physical ones.

However, as pointed out in [1], this spurious modes can be eliminated by using Raviart-Thomas finite elements to approximate displacements. This method has been proved to be free of spurious modes and to attain optimal order of convergence in structural acoustic problems ([1, 16]). Moreover, numerical experiments showing its performance have been reported for 2D and 3D problems in [5] and [3], respectively.

The present paper concerns the numerical approximation of damped linear oscillations of an acoustic (i.e., inviscid, compressible, barotropic) fluid contained in a cavity. Damping arises due to the presence of a viscoelastic material on the walls which is introduced to dissipate acoustic energy.

This kind of problems are related to interesting engineering goals as that of decreasing the level of noise in aircraft or cars. As in [6, 7, 4], we model the behavior of the damping material following [13], by introducing appropriate boundary conditions on the absorbing walls which make use of an acoustic impedance having a simple form as a function of frequency.

It is well known that when damped systems are subject to a frequency varying periodic excitation, large response peaks are attained for particular values of the external frequency. In most applications it turns out interesting to predict the localization of such peaks. In [7] and [4] it was experimentally shown that the imaginary parts of the obtained eigenvalues practically coincide with the frequencies of the external source for which the response of the damped system is larger. This fact makes it interesting to compute these eigenvalues.

The outline of the paper is as follows: in Section 2 we show that these damped vibration modes are solutions of a quadratic eigenvalue problem. Then we introduce a convenient functional framework to analyze it and show that this non-linear eigenvalue problem is equivalent to the spectral problem for a non self-adjoint, non-compact bounded operator.

In Section 3 we introduce a finite element discretization based on Raviart-Thomas elements and show that the proposed numerical scheme converges and is free of spurious modes. We also obtain optimal order error estimates.

Finally, in Section 4, we show how the quadratic eigenvalue problem can be equivalently
written to be solved by a standard eigensolver (eigs from MATLAB, based on Arnoldi iterations). We also present numerical experiments showing the good performance of the method.

2 STATEMENT OF THE PROBLEM

We consider an inviscid compressible barotropic fluid contained in a rigid cavity, with some or all of its walls covered by a thin layer of a viscoelastic material, to absorb part of the acoustic energy of the fluid.

We denote by \( \Omega \subset \mathbb{R}^n \) (\( n = 2 \) or \( 3 \)) the domain occupied by the fluid, which we suppose polyhedral, with boundary \( \partial \Omega = \Gamma_\Lambda \cup \Gamma_R \). \( \Gamma_\Lambda = \bigcup_{j=1}^J \Gamma_j \), with \( \Gamma_j \) being all the different faces of \( \Omega \) covered by damping material, is called the “absorbing boundary”. \( \Gamma_R \) is the union of the remaining faces and we call it the “rigid boundary”. We assume that \( \Gamma_\Lambda \) is not empty. The unit outer normal vector along \( \partial \Omega \) is denoted by \( \nu \).

We use the following notations for the physical magnitudes: \( P \) for the fluid pressure, \( U \) for the displacement field, \( \rho \) for the fluid density, and \( c \) for the acoustic speed. The dynamic equations for our problem are (see [7, 13]):

\[
\begin{align*}
\rho \frac{\partial^2 U}{\partial t^2} + \nabla P &= 0 \quad \text{in } \Omega, \\
P &= -\rho c^2 \operatorname{div} U \quad \text{in } \Omega, \\
P &= \left( \alpha U \cdot \nu + \beta \frac{\partial U}{\partial t} \cdot \nu \right) \quad \text{on } \Gamma_\Lambda, \\
U \cdot \nu &= 0 \quad \text{on } \Gamma_R.
\end{align*}
\]

The usual kinematic interface condition for rigid walls, \( U \cdot \nu = 0 \), has been relaxed in the absorbing boundary to take into account the effect of the viscoelastic material. Equation (3) models this effect: the fluid pressure on the boundary is in equilibrium with the response of the absorbing walls. This response consists of two terms: the first one is proportional to the normal component of the displacements and accounts for the elastic behavior of the material, whereas the second one is proportional to the normal velocity and models the viscous damping.

The damped vibration modes of the fluid are complex solutions of equations (1)-(4) of the form \( U(x, t) = e^{\lambda t} u(x) \) and \( P(x, t) = e^{\lambda t} p(x) \). They can be found by solving the following non-linear eigenvalue problem:

Find \( \lambda \in \mathbb{C} \), \( u : \Omega \to \mathbb{C}^n \) and \( p : \Omega \to \mathbb{C} \), \((u, p) \neq (0, 0)\), such that:

\[
\begin{align*}
\rho \lambda^2 u + \nabla p &= 0 \quad \text{in } \Omega, \\
p &= -\rho c^2 \operatorname{div} u \quad \text{in } \Omega, \\
p &= (\alpha + \lambda \beta) u \cdot \nu \quad \text{on } \Gamma_\Lambda, \\
u \cdot \nu &= 0 \quad \text{on } \Gamma_R.
\end{align*}
\]
For each damped vibration mode, $\omega := \text{Im } \lambda$ is the vibration angular frequency and $\text{Re } \lambda$ (which is proved below to be negative) the decay rate. For a particular angular frequency $\omega$, coefficients $\alpha$ and $\beta$ in the previous equations are related to the impedance $Z$ of the viscoelastic material through the expression:

$$Z := \beta + \frac{\alpha}{\omega} i.$$

The real and imaginary parts of this impedance can be experimentally measured. For real materials, both coefficients $\alpha$ and $\beta$ actually depend on $\omega$. However, in a first approach, they can be taken as strictly positive constants. Such assumption turns out realistic when a restricted range of frequencies is considered. See for instance [13], where experimental values of the impedance are reported for a typical acoustic insulating material (glasswool).

A variational formulation of problem (5)-(8) involving only displacement variables can be easily obtained. Let

$$\mathcal{V} := \{ \phi \in H(\text{div}, \Omega) : \phi \cdot \nu \in L^2(\partial \Omega) \text{ and } \phi \cdot \nu = 0 \text{ on } \Gamma_R \},$$

endowed with its natural norm $\| \phi \|_\mathcal{V} := (\| \phi \|_{H(\text{div}, \Omega)}^2 + \| \phi \cdot \nu \|_{L^2(\partial \Omega)}^2)^{1/2}$. By integrating by parts (5) multiplied by $\bar{\phi} \in \mathcal{V}$ and by using (6) and (7) we obtain:

Find $\lambda \in \mathbb{C}$ and $u \in \mathcal{V}$, $u \neq 0$, such that:

$$\int_\Omega \rho c^2 \text{div } u \text{div } \bar{\phi} + \int_{\Gamma_A} \alpha u \cdot \nu \bar{\phi} \cdot \nu + \lambda \int_{\Gamma_A} \beta u \cdot \nu \bar{\phi} \cdot \nu$$

$$+ \lambda^2 \int_\Omega \rho u \cdot \bar{\phi} = 0 \quad \forall \phi \in \mathcal{V}. \tag{9}$$

This is a quadratic eigenvalue problem. Clearly $\lambda = 0$ is one of its eigenvalues, with corresponding eigenspace

$$\mathcal{K} := \{ u \in \mathcal{V} : \text{div } u = 0 \text{ in } \Omega \text{ and } u \cdot \nu = 0 \text{ on } \partial \Omega \}.$$

The following lemma, proved in [2], shows that for all the other solutions of (9), the decay rate is strictly negative. This agrees with what is well-known from the physics viewpoint: the effect of the viscoelastic material is to damp the vibrations.

**Lemma 2.1** Let $\lambda \in \mathbb{C}$ and $0 \neq u \in \mathcal{V}$ be a solution of Problem (9). If $\lambda \neq 0$, then $\text{Re } \lambda < 0$.

1 An alternative variational formulation in terms of pressure variables can also be obtained:

$$\int_\Omega \nabla p \cdot \nabla \bar{q} + \frac{\lambda^2}{\alpha + \lambda \beta} \int_{\Gamma_A} \rho p \bar{q} + \frac{\lambda^2}{c^2} \int_\Omega p \bar{q} = 0 \quad \forall q \in H^1(\Omega);$$

however, in this case, the eigenvalue problem turns out to be neither linear nor quadratic.
For the theoretical analysis it is convenient to transform (9) into an equivalent linear eigenvalue problem. This is carried out by introducing the new variable \( v := \lambda u \), as usual in quadratic problems:

Find \( \lambda \in \mathbb{C} \) and \( (u, v) \in V \times L^2(\Omega)^n \), \( (u, v) \neq (0, 0) \), such that:

\[
\int_{\Omega} \rho c^2 \text{div} u \text{div} \tilde{\phi} + \int_{\Gamma} \alpha u \cdot v \tilde{\phi} \cdot v = \lambda \left( -\int_{\Gamma} \beta u \cdot v \tilde{\phi} \cdot v - \int_{\Omega} \rho v \cdot \tilde{\phi} \right) \quad \forall \phi \in V, \tag{10}
\]

\[
\int_{\Omega} \rho v \cdot \tilde{\psi} = \lambda \int_{\Omega} \rho u \cdot \tilde{\psi} \quad \forall \psi \in L^2(\Omega)^n. \tag{11}
\]

It is clear that \( \lambda = 0 \) is also an eigenvalue of problem (10)-(11) with corresponding eigenspace \( K \times \{0\} \). Let \( G \) denote the orthogonal complement of \( K \) in \( V \) and \( G \times L^2(\Omega)^n := G \times L^2(\Omega)^n \). Notice that (see for instance [11])

\[
G = \left\{ u \in V : u = \nabla \varphi \text{ for } \varphi \in H^1(\Omega) \right\}.
\]

We introduce the linear operator \( A : V \times L^2(\Omega)^n \rightarrow V \times L^2(\Omega)^n \) defined by \( A(f, g) := (u, v) \) with \( (u, v) \in G \times L^2(\Omega)^n \) being the solution of

\[
\int_{\Omega} \rho c^2 \text{div} u \text{div} \tilde{\phi} + \int_{\Gamma} \alpha u \cdot v \tilde{\phi} \cdot v = -\int_{\Gamma} \beta f \cdot v \tilde{\phi} \cdot v - \int_{\Omega} \rho g \cdot \tilde{\phi} \quad \forall \phi \in G,
\]

\[
\int_{\Omega} \rho v \cdot \tilde{\psi} = \int_{\Omega} \rho f \cdot \tilde{\psi} \quad \forall \psi \in L^2(\Omega)^n.
\]

It is shown in [2] that \( A \) is a well defined bounded operator. Furthermore, its non-zero eigenvalues are exactly the inverses of the non-zero eigenvalues of problem (10)-(11), with the same corresponding eigenfunctions, and consequently of the original quadratic problem (9).

The operator \( A \) is not compact. In some simple cases, like a cubic 3D domain \( \Omega \) with only one of its faces covered by the viscoelastic material, the eigenvalues and eigenfunctions of \( A \) can be found solving by separation of variables the non-linear spectral problem (5)-(8) (see [4]). In this case, so-called overdamped modes, corresponding to real negative eigenvalues \( \mu \), exist and \(-\frac{\beta}{\alpha}\) is an accumulation point of them.

In more general problems, the spectrum of the operator, \( \sigma(A) \), is not explicitly known. However, it has been shown in [2] that \(-\frac{\beta}{\alpha}\) always belongs to its essential spectrum \( \sigma_{\text{ess}}(A) \) (i.e., the set consisting of all the limit points of \( \sigma(A) \) and the eigenvalues of \( A \) having infinite algebraic multiplicity), thus proving the non-compactness of \( A \).

A thorough spectral characterization of the operator \( A \), which implies the existence of vibration modes of problem (1)-(4), has been given in [2] by following the general analysis carried out by Krein and Langer in [14]:

**Theorem 2.1** The spectrum of \( A \) consists of \( \sigma_{\text{ess}}(A) = \left\{ -\frac{\beta}{\alpha}, 0 \right\} \) and a set of isolated eigenvalues of finite algebraic multiplicity.
3 SPECTRAL APPROXIMATION

In this section we introduce and analyze a finite element method to approximate the solutions of the quadratic eigenvalue problem (9). We prove its convergence and that spurious solutions do not arise with this method. Let us remark that such “spurious modes” are usually present, even in the undamped case (see for instance [12]), when standard finite elements are used in a displacement formulation like this.

To avoid them, we use lowest order Raviart-Thomas elements (see for instance [8, 15]), which proved to be effective in the undamped case (see [1]). Let \( \mathcal{T}_h \) be a regular family of partitions of \( \Omega \) in tetrahedra; \( h \) stands for the maximum diameter of the elements. Let

\[
\mathcal{V}_h := \{ \phi_h \in H(\text{div}, \Omega) : \phi_h|_T \in \mathcal{P}_0^n \oplus \mathcal{P}_0 \times \forall T \in \mathcal{T}_h \text{ and } \phi_h \cdot \nu = 0 \text{ on } \Gamma_h \} \subset \mathcal{V}
\]

(\( \mathcal{P}_k \) denotes the set of polynomials of degree at most \( k \)). Our discrete problem reads:

Find \( \lambda_h \in \mathbb{C} \) and \( u_h \in \mathcal{V}_h, u_h \neq 0 \), such that:

\[
\int_{\Omega} \rho c^2 \text{div} u_h \text{div} \phi_h + \int_{\Gamma_h} \alpha u_h \cdot \nu \phi_h \cdot \nu + \lambda_h \int_{\Gamma_h} \beta u_h \cdot \nu \phi_h \cdot \nu
\]

\[
+ \lambda_h^2 \int_{\Omega} \rho u_h \cdot \phi_h = 0 \quad \forall \phi_h \in \mathcal{V}_h.
\]

We proceed as for the continuous problem; by introducing \( v_h := \lambda_h u_h \), we transform problem (12) into an equivalent linear one:

Find \( \lambda_h \in \mathbb{C} \) and \( (u_h, v_h) \in \mathcal{V}_h \times \mathcal{V}_h, (u_h, v_h) \neq (0, 0) \), such that:

\[
\int_{\Omega} \rho c^2 \text{div} u_h \text{div} \phi_h + \int_{\Gamma_h} \alpha u_h \cdot \nu \phi_h \cdot \nu = \lambda_h \left( -\int_{\Gamma_h} \beta u_h \cdot \nu \phi_h \cdot \nu - \int_{\Omega} \rho v_h \cdot \phi_h \right) \quad \forall \phi_h \in \mathcal{V}_h,
\]

\[
\int_{\Omega} \rho v_h \cdot \psi_h = \lambda_h \int_{\Omega} \rho u_h \cdot \psi_h \quad \forall \psi_h \in \mathcal{V}_h.
\]
The operators $A_h$ are well defined and bounded uniformly on $h$. As in the continuous case, their non-zero eigenvalues are the inverses of the non-zero eigenvalues of problem (13)-(14), with the same corresponding eigenspaces, and consequently of the discrete quadratic problem (12).

Approximation and consistency properties, as well as further regularity of the generalized eigenfunctions of the continuous problem, have been proved in [2]. Then, by using the spectral approximation theory for non-compact operators from [9], the following results about absence of spurious modes and spectral convergence have been obtained:

**Theorem 3.1** Let $K \subset \mathbb{C}$ be a compact set not intersecting $\sigma(A)$. There exists $h_0 > 0$ such that, if $h \leq h_0$, then $K$ does not intersect $\sigma(A_h)$.

**Theorem 3.2** Let $\mu$ be an isolated eigenvalue of $A$ with multiplicity $m$. Let $D \subset \mathbb{C}$ be a closed disk centered at $\mu$, such that $0 \notin D$ and $D \cap \sigma(A) = \{\mu\}$. Let $\mu_{1h}, \ldots, \mu_{mh}$ be the eigenvalues of $A_h$ contained in $D$ (repeated according to their algebraic multiplicities).

There exists $h_0 > 0$ such that, if $h \leq h_0$, then $m(h) = m$. Furthermore, $\lim_{h \to 0} \mu_{kh} = \mu$, for $k = 1, \ldots, m$.

The theory in [16] about non-conforming approximation for non-compact operators could be in principle applied to prove error estimates for this method. However, it would provide estimates depending on $\| (A - A_h)_{|G_h \times G_h} \|_{L^2(\Omega)^n}$, which in our case are not optimal. To avoid this drawback, minor modifications of the proofs in [10] are performed in [2] yielding the following estimates:

**Theorem 3.3** Let $E$ be the invariant space of $A$ corresponding to $\mu$, and $E_h$ the invariant space of $A_h$ corresponding to $\mu_{1h}, \ldots, \mu_{mh}$. There exist constants $h_0 > 0$ and $C > 0$ such that, for $h \leq h_0$, it holds:

$$\hat{\delta}(E_h, E) \leq Ch^s,$$

where $\hat{\delta}(E, E_h)$ is the gap between the invariant subspaces (see for instance [9] for a precise definition).

**Theorem 3.4** There exist constants $h_0 > 0$ and $C > 0$ such that, for $h \leq h_0$, it holds:

$$\left| \mu - \frac{1}{m} \sum_{k=1}^m \mu_{kh} \right| \leq Ch^{2s},$$

$$\left| \frac{1}{m} - \frac{1}{m} \sum_{k=1}^m \frac{1}{\mu_{kh}} \right| \leq Ch^{2s},$$

$$\max_{k=1, \ldots, m} |\mu - \mu_{kh}| \leq Ch^{2s/p},$$

where $p$ is the ascent of the eigenvalue $\mu$ of $A$ (i.e., the length of the longest Jordan chain of $A$ associated with $\mu$).

The constants $s$ in these theorems is a fixed number satisfying $0 < s \leq 1$, which only depend on the geometry of the domain $\Omega$ (for instance, if $\Omega$ is convex, $s = 1$).
4 NUMERICAL RESULTS

To end this paper we present some numerical experiments to show the performance of this method and how the quadratic eigenvalue problem (12) can be efficiently solved by using a standard eigensolver, namely, \texttt{eigs} from \textsc{Matlab} version 5.3. This code is based on the classical \textit{Arnoldi} iteration and allows solving a generalized linear eigenvalue problem with an arbitrary matrix on the left hand side, but a hermitian and positive definite one on the right hand side.

Let \( \{ \phi_j \}_{j=1}^N \) be a nodal basis of \( \mathcal{V}_h \). For \( u_h \in \mathcal{V}_h \), we write \( u_h = \sum_{j=1}^N u_j \phi_j \) in terms of this basis and denote \( u = (u_1, \ldots, u_N) \) the vector of its nodal components. Let \( K := (K_{ij}) \), \( B := (B_{ij}) \) and \( M := (M_{ij}) \), with

\[
K_{ij} := \int_\Omega \rho c^2 \text{div} \phi_i \text{div} \phi_j, \quad B_{ij} := \int_{\Gamma} \phi_i \cdot \nu \phi_j \cdot \nu, \quad \text{and} \quad M_{ij} := \int_\Omega \rho \phi_i \cdot \phi_j,
\]

for \( i, j = 1, \ldots, N \).

Problem (12) can be written in the following matrix form:

\[
K u + (\alpha + \lambda_h \beta) B u + \lambda_h^2 M u = 0. \tag{15}
\]

\( K \) and \( M \) are the stiffness and mass matrices of the fluid, respectively, whereas \( B \) is used to take into account the effect of the absorbing wall. The three matrices are hermitian; furthermore, \( K \) and \( B \) are semipositive and \( M \) positive definite.

The matrix form of the equivalent linear problem (13)-(14), used for the theoretical analysis, is

\[
\begin{pmatrix}
K + \alpha B & 0 \\
0 & M
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \lambda_h
\begin{pmatrix}
-\beta B & -M \\
M & 0
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix},
\]

where \( v = \lambda_h u \). The matrices in the right and left hand sides are singular and indefinite, respectively. Thus this generalized eigenvalue problem cannot be directly solved with \textsc{Matlab} eigensolver \texttt{eigs}. Instead we will transform problem (15) into other suitable equivalent linear one.

For \( \lambda_h \neq 0 \), let \( \mu_h := \frac{1}{\lambda_h} \); then problem (15) is equivalent to

\[
M u + \mu_h \beta B u + \mu_h^2 (K + \alpha B) u = 0.
\]

Now, by introducing \( w = \mu_h u \), we rewrite this problem in the following way

\[
\begin{pmatrix}
M & 0 \\
0 & M
\end{pmatrix}
\begin{pmatrix}
u \\
w
\end{pmatrix}
= \mu_h
\begin{pmatrix}
-\beta B & -(K + \alpha B) \\
M & 0
\end{pmatrix}
\begin{pmatrix}
u \\
w
\end{pmatrix},
\]

which is equivalent to

\[
\begin{pmatrix}
-\beta B & -(K + \alpha B) \\
M & 0
\end{pmatrix}
\begin{pmatrix}
u \\
w
\end{pmatrix}
= \lambda_h
\begin{pmatrix}
M & 0 \\
0 & M
\end{pmatrix}
\begin{pmatrix}
u \\
w
\end{pmatrix}. \tag{16}
\]
This last problem is equivalent to (12), except for $\lambda_h = 0$, and the matrix in its right hand side is hermitian and positive definite. Hence, it can be safely solved by means of the eigensolver \texttt{eigs}.

We apply our method to compute the vibration modes of a 2D rectangular rigid cavity filled with air and with one absorbing wall as that shown in Figure 4. For such simple geometry, problem (9) can be solved by separation of variables (see [7] for details). Apart from $\lambda = 0$, its solutions are

\[
\lambda_j \in \mathbb{C}, \quad u_j(x, y) = \nabla \left( \cos \frac{j\pi x}{a} \cosh \eta_j y \right), \quad j = 0, 1, 2, \ldots
\]

with $\lambda_j$ and $\eta_j \in \mathbb{C}$ being solutions of the so-called dispersion equations:

\[
\eta_j^2 = \frac{\lambda_j^2}{c^2} + \frac{\pi^2}{a^2} j^2, \quad (17)
\]
\[
\eta_j \tanh \eta_j b = -\frac{\rho \lambda_j^2}{(\alpha + \lambda_j \beta)}. \quad (18)
\]

Notice that all the eigenfunctions are oscillatory in the horizontal direction, the integer parameter $j$ in these equations being the number of corresponding half-waves. In general, the system above has an infinite number of complex solutions for each $j \geq 0$. Furthermore, at least for $j$ large enough, it also has real solutions with $\lambda_j$ negative, corresponding to the overdamped modes quoted in §3. These real negative $\lambda_j$ only have one accumulation point at $-\frac{\alpha}{\beta}$ (see [7]).

![Figure 1: Fluid in a cavity with one absorbing wall.](image)

We have taken the geometrical data given in Figure 4 and the following physical data: $\rho = 1 \text{ kg/m}^3$, $c = 340 \text{ m/s}$, $\alpha = 5 \times 10^4 \text{ N/m}^3$ and $\beta = 200 \text{ N s/m}^3$. The two latter
correspond to an ideal very viscous insulating material and were chosen for overdamped modes to occur for $j$ not too large.

We have computed both, overdamped and normally damped vibration modes, the latter being the eigenvalues with non-null imaginary part. We have used uniform meshes obtained by successively refining that in Figure 2. We denote by $N$ the refinement level ($N = 1$ for that in Figure 2, $N = 2$ for the one obtained by halving the meshsize $h$, etc.).

![Figure 2: Initial mesh: $N = 1$.](image)

Table 1 shows the eigenvalues of the discrete problem (16) with lowest positive vibration frequencies: $0 < f := \frac{\omega}{2\pi} < 600$ Hz (recall $\omega := \text{Im} \lambda$). Those with smallest decay rate (i.e., $-\text{Re} \lambda$ small) correspond to the largest response peaks of the damped cavity, which are the magnitudes of interest in most applications.

<table>
<thead>
<tr>
<th>$N = 2$ (604 d.o.f)</th>
<th>$N = 4$ (2360 d.o.f)</th>
<th>$N = 8$ (9328 d.o.f)</th>
<th>‘exact’</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-317.98 \pm 267.76i$</td>
<td>$-320.02 \pm 267.68i$</td>
<td>$-320.54 \pm 267.66i$</td>
<td>$-320.71 + 267.65i$</td>
<td>1.99</td>
</tr>
<tr>
<td>$-259.50 \pm 813.05i$</td>
<td>$-259.28 \pm 813.23i$</td>
<td>$-259.21 \pm 813.27i$</td>
<td>$-259.21 + 813.29i$</td>
<td>2.34</td>
</tr>
<tr>
<td>$-90.32 \pm 1281.55i$</td>
<td>$-90.04 \pm 1281.40i$</td>
<td>$-89.98 \pm 1281.36i$</td>
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</tr>
<tr>
<td>$-299.64 \pm 2176.41i$</td>
<td>$-297.81 \pm 2179.97i$</td>
<td>$-297.36 \pm 2180.85i$</td>
<td>$-297.21 + 2181.15i$</td>
<td>2.00</td>
</tr>
<tr>
<td>$-27.81 \pm 2247.92i$</td>
<td>$-27.47 \pm 2249.79i$</td>
<td>$-27.39 \pm 2250.25i$</td>
<td>$-27.37 + 2250.41i$</td>
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<tr>
<td>$-237.61 \pm 2408.21i$</td>
<td>$-236.93 \pm 2408.97i$</td>
<td>$-236.76 \pm 2409.15i$</td>
<td>$-236.70 + 2409.21i$</td>
<td>2.02</td>
</tr>
<tr>
<td>$-144.34 \pm 3029.85i$</td>
<td>$-143.47 \pm 3025.26i$</td>
<td>$-143.24 \pm 3024.08i$</td>
<td>$-143.16 + 3023.68i$</td>
<td>1.98</td>
</tr>
<tr>
<td>$-13.36 \pm 3269.81i$</td>
<td>$-12.84 \pm 3279.02i$</td>
<td>$-12.73 \pm 3281.31i$</td>
<td>$-12.69 + 3282.07i$</td>
<td>2.01</td>
</tr>
<tr>
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<td>$-304.30 \pm 3583.08i$</td>
<td>$-303.02 \pm 3587.10i$</td>
<td>$-302.60 + 3588.43i$</td>
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</tr>
<tr>
<td>$-281.62 \pm 3723.38i$</td>
<td>$-276.89 \pm 3734.25i$</td>
<td>$-275.78 \pm 3736.92i$</td>
<td>$-275.41 + 3737.81i$</td>
<td>2.01</td>
</tr>
</tbody>
</table>

Table 1: Damped acoustic modes for a rectangular rigid cavity with an absorbing wall.

The table includes the eigenvalues computed with three different meshes (the corresponding number of degrees of freedom (d.o.f) are also given) and the ‘exact’ ones computed by solving the non linear system (17)-(18). An excellent agreement between ‘exact’ and computed values can be observed, even for the coarsest mesh, showing the effectiveness of our method. Finally the table shows the computed orders of convergence which

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compares very well with the theoretically predicted one (since the cavity is convex, \( s = 1 \)), and hence Theorem 3.4 yields a quadratic order of convergence for the eigenvalues.

Table 2 shows some real eigenvalues of the discrete problem (16) which correspond to overdamped modes. The corresponding ‘exact’ eigenvalues and computed orders of convergence are also shown in the table.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( N = 2 )</th>
<th>( N = 4 )</th>
<th>( N = 8 )</th>
<th>‘exact’</th>
<th>order</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-347.04</td>
<td>-343.11</td>
<td>-342.15</td>
<td>-341.83</td>
<td>2.02</td>
</tr>
<tr>
<td>3</td>
<td>-300.35</td>
<td>-297.39</td>
<td>-296.66</td>
<td>-296.41</td>
<td>2.00</td>
</tr>
<tr>
<td>4</td>
<td>-285.58</td>
<td>-282.49</td>
<td>-281.71</td>
<td>-281.45</td>
<td>1.99</td>
</tr>
<tr>
<td>5</td>
<td>-278.36</td>
<td>-274.99</td>
<td>-274.13</td>
<td>-273.84</td>
<td>1.98</td>
</tr>
<tr>
<td>6</td>
<td>-274.16</td>
<td>-270.49</td>
<td>-269.53</td>
<td>-269.21</td>
<td>1.97</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>-1026.37</td>
<td>-1069.12</td>
<td>-1080.41</td>
<td>-1084.23</td>
<td>1.96</td>
</tr>
<tr>
<td>3</td>
<td>-1721.86</td>
<td>-1854.08</td>
<td>-1892.02</td>
<td>-1905.19</td>
<td>1.90</td>
</tr>
<tr>
<td>4</td>
<td>-2295.75</td>
<td>-2576.38</td>
<td>-2664.69</td>
<td>-2696.36</td>
<td>1.83</td>
</tr>
<tr>
<td>5</td>
<td>-2766.17</td>
<td>-3249.14</td>
<td>-3417.56</td>
<td>-3480.48</td>
<td>1.75</td>
</tr>
<tr>
<td>6</td>
<td>-3156.62</td>
<td>-3876.48</td>
<td>-4153.53</td>
<td>-4261.77</td>
<td>1.68</td>
</tr>
</tbody>
</table>

Table 2: Overdamped modes for a rectangular rigid cavity with an absorbing wall.

For the physical parameters considered in this example, the system of equations (17)-(18) (and consequently the continuous problem) has real solutions if and only if \( j \geq 2 \). In that case, for each \( j \), the system has two different pairs of solutions, one with \(-\frac{2\alpha}{\beta} < \lambda_j < -\frac{\alpha}{\beta}\) and the other with \(\lambda_j < -\frac{2\alpha}{\beta}\). Thus two sequences of eigenvalues are obtained, the former converges to \(-\frac{\alpha}{\beta}\) and the latter diverges to \(-\infty\) (see [7]).

Once more a very good agreement between ‘exact’ and computed values can be observed and the orders of convergence are close to the theoretical one. However, the quality of the approximation deteriorates for larger numbers of half-waves \( j \), particularly for the second sequence.
REFERENCES


