METHOD OF MOMENTS FOR THE EVALUATION OF SCATTERING FROM DIELECTRIC FRACTAL SURFACES

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Abstract. This paper presents the rationale for the use of method of moments for evaluation of electromagnetic field scattered by dielectric random fractal surfaces. Derivation of the algorithm is given along with a reliable fractal profile generation procedure. Choice of key parameters pertinent to method of moments is also discussed. Numerical results are finally used to investigate the behaviour of the electromagnetic scattering from fractal surfaces.
1 INTRODUCTION

The study of scattering from natural surfaces plays an important role in both remote sensing and telecommunication applications. During the last fifty years different approximated analytical methods for the evaluation of scattering from rough surfaces, e.g., Kirchhoff approach (KA), the small perturbation method (SPM), the integral equation method (IEM), and so on [1-2], have been developed. Different analytical methods rely on different approximations to get a closed form expression of the scattered field, and therefore they have different regions of validity. The advantage of analytical methods is that they shed light onto the scattering dependence on surface electromagnetic and geometric properties. However, their drawback is that there are natural surfaces that do not belong to the region of validity of any one of them. To evaluate the field scattered by such surfaces, numerical methods must be used [3-4]. Since, in principle, numerical methods get rid of any approximation, they are also useful to assess the limits of validity of analytical methods. Most numerical methods for the evaluation of scattering from natural rough surfaces are based on the method of moments (MoM) [5], which is particularly convenient for the external problem at hand.

Analytical and numerical methods reported in literature usually share the same scattering surface description. As a matter of fact, the surface is often described by means of a random process with prescribed (usually gaussian) probability density function (pdf) and prescribed (usually gaussian, exponential, or a proper combination of them) autocorrelation function [1-4]. However, it has been recently shown that a new description, based on fractal geometry, is able to better model natural surfaces [6]. As a matter of fact, the fractional Brownian motion (fBm) fractal model has been shown to properly describe statistical scale-invariance and spectral properties of natural surfaces. However, use of fBm within analytical and numerical scattering models is certainly not straightforward. Modelling of natural surfaces by means of the fBm model in conjunction with some analytical methods for the evaluation of the electromagnetic scattering has been recently presented (see, e.g., Refs.[7-8]). In this paper, we explore use of MoM numerical algorithm for the evaluation of scattering from an fBm surface.

In order to generate an fBm surface with prescribed fractal parameters (fractal dimension and topothesy [6, 8]), we use a spectral method; the point matching method is employed to evaluate the equivalent current densities over the generated fBm sample surface, and subsequently the scattered far field is computed. The surface backscattering coefficient is finally computed by repeatedly applying the latter algorithm to several generated fBm sample surfaces (with the same fractal parameters) and evaluating the variance of the resulting scattered field.
In this paper we also give criteria to choose the algorithm parameters and we assess the validity of the proposed algorithm. In particular, numerical results are compared and validated against those obtained under the hypotheses of the small perturbation scattering theory available together with the applicability limits in the current literature [8]. Finally, we use the proposed algorithm to study the scattering dependence on surface fractal parameters. Our simulations refer to a (topologically) one-dimensional (1-D) profile embedded in a two-dimensional (2-D) space. Physically, this corresponds to assume that both the electromagnetic field and the surface height are constant along a fixed direction. Extension to the case of a 2-D surface embedded in a three-dimensional (3-D) space is not conceptually difficult, but any simulation run requires a much longer computational time. Furthermore, scattering results obtained for 1-D profiles give also a good indication of scattering dependence on 2-D surface parameters. The proposed model is general: in fact, it applies to dielectric surfaces (whereas the restriction to metallic ones is straightforward) and to random surfaces (whereas the restriction to deterministic surfaces is trivial).

2 FRAC TAL SURFACE DESCRIPTION AND GENERATION

An fBm profile is defined by the property that height increment over a fixed distance \( \tau \) is a zero-mean Gaussian random variable whose variance is proportional to \( \tau^{2H} \), where \( H \) is the Hurst coefficient, related to the fractal dimension \( D \) by the equality \( D = 2 - H \) (for topologically 1-D profiles). Therefore, an fBm profile is characterised by its fractal dimension and by the variance proportionality constant \( s^2 \). It can be also shown that its spectrum is equal to \( S_0/\kappa^\alpha \), where \( \kappa \) is the spatial frequency \( \alpha=1+2H \) and \( S_0 \) is related to \( s^2 \) and \( H \) by [8]

\[
S_0 = s^2 \frac{\pi H}{\cos(\pi H)} \frac{1}{\Gamma(1-2H)}.
\]

Finally, we recall that, in the case of electromagnetic scattering models, bandlimited fBm profiles must be considered, i.e., profiles satisfying above definition only in a wide but limited range of \( \tau \) values (or, equivalently, showing a power law spectrum only for a wide but limited range of \( \kappa \) values).

An efficient way to generate an fBm sample profile is the spectral method: a white Gaussian profile is first generated, and then it is filtered by a filter whose transfer function is the squared root of the desired spectrum. In Fig.1a a profile generated by using the spectral method by setting \( D=1.3 \) and \( S_0=10^{-3} \) m\(^2\)-2\( H \) is plotted. It is a crucial and non-trivial task to verify that the generated profile complies with any theoretical requirement reported above. To
this end three tests can be performed: the first test compares the spectral behaviour of the surface sample with the theoretical one; the second test checks that fractal parameters of the sample profile agree with prescribed ones; and the third test fits the sample height differences distribution with the theoretical one.

According to this test procedure, in Fig.1b a log-log plot of the profile spectrum, evaluated by using an averaged periodogram estimator with pre-whitening (pre-whitening is used to minimise the effect of spectral leakage from smaller frequencies, which is a very critical problem for power law spectra estimation), is reported; for reference purposes, the ideal spectrum is also plotted: the latter obviously appears as a straight line. In Fig.1c the variogram of the generated profile (i.e., the increments' variance versus the distance $\tau$) is plotted on a log-log plane. It is with a very good approximation a straight line whose slope is 1.4, i.e., equal to $2H$, according to the fBm definition. Finally, a histogram of the height differences between points at a fixed distance equal to 1 m is plotted in Fig.1d. It very well fits a zero-mean, $s^2$ variance Gaussian distribution, as prescribed by the fBm definition.

**Fig.1:** fBm with $D=1.3$ and $S_0=10^{-3}$ m$^{2-2H}$: (a) sample profile, (b) log-log plot of the spectrum, (c) log-log plot of the variogram, and (d) plot of the histogram of the height differences between points at a fixed distance equal to 1 m.
3 METHOD OF MOMENTS AND FRACTAL SURFACES

Once the scattering surface has been generated, surface tangential fields must be evaluated, so that, from their knowledge, the scattered field can be calculated. To this aim, we use the MoM, whose rationale is here briefly recalled. We refer to the geometry depicted in Fig.2.

\[ \nabla \cdot \mathbf{H} = \mathbf{J} \]

\[ \mathbf{E} \]

Fig.2: Geometry of the problem.

If the incident field is horizontally polarised, surface electric and magnetic fields can be obtained by solving the following integral equation pair [3,9]:

\[ E^i(r) = \frac{1}{2} E(r) + \int \left\{ j \omega \mu_0 \phi_0(r, r') J_s(r') + E(r') \left[ \hat{n}' \cdot \nabla \phi_0(r, r') \right] \right\} dl' \]

\[ 0 = \frac{1}{2} E(r) - \int \left\{ j \omega \mu_0 \phi_1(r, r') J_s(r') + E(r') \left[ \hat{n}' \cdot \nabla \phi_1(r, r') \right] \right\} dl' \]

wherein \( \mathbf{E}^i = \mathbf{E}^i \hat{y} \) is the incident electric field, \( \mathbf{J}_s = \hat{n} \times \mathbf{H} = J_s \hat{y} \) the equivalent electric surface current density, \( \hat{n} \) is the outgoing normal to the surface,

\[ \phi_{0,1}(r, r') = -\frac{j}{4} H^{(2)}_0 \left( k_{0,1} |r - r'| \right) \]

is the Green's function in the 2-D space, \( k_0, k_1 \) are the free-space propagation constants of the upper and lower media, respectively, and \( H^{(2)}_0(\cdot) \) is the zero order Hankel function of the second kind. Eqs.(2) are obtained as a simple application of the equivalence theorem, and the integrals are understood as Cauchy principal values. Formulation (2) assumes the lower space to be homogeneous, illimitated and with the same magnetic permittivity \( \mu_0 \) of the upper one.
By using rectangular pulse basis functions and the point matching method, the integral equation pair (2) can be converted into the following matrix equation in the unknowns $E$ and $J_s$ [3]:

$$
\begin{bmatrix}
A_{11}^{ij} & A_{12}^{ij} \\
A_{21}^{ij} & A_{22}^{ij}
\end{bmatrix}
\begin{bmatrix}
E \\
J_s
\end{bmatrix}
=
\begin{bmatrix}
E_i \\
0
\end{bmatrix},
$$

(4)

The expressions of the elements of the matrixes $A_{ij}$ are reported in [3] and in [9]. If the incident field is vertically polarised, equations analogous to (2-4) are obtained, in which electric fields and current densities must be replaced by magnetic ones; in addition, $\mu_0$ must be replaced by $\varepsilon_0$ and by $\varepsilon_1$ in the first and the second of eqs.(2), respectively.

Once tangential fields are known, the backscattered field, $hh$ polarisation, is given by

$$
E^s = \int_{-\infty}^{+\infty} i(x') g(x') \exp[jkz(x') \cos \vartheta] \exp(jkx' \sin \vartheta) dx'
$$

(5)

where

$$
i(x) = \frac{k_0 \exp(-j(k_0r - \pi/4))}{\sqrt{8\pi k_0 \zeta}} \left\{ J_s \left[ x, z(x) \right] - E \left[ x, z(x) \right] \hat{n} \cdot \hat{k}_s \right\} \sqrt{1 + (dz/dx)^2},
$$

(6)

$\hat{k}_s$ is the unit vector indicating the scattering direction, $\zeta$ is the free-space intrinsic impedance, $z = z(x)$ describes the scattering surface profile, $g(\cdot)$ is the (slow varying) illumination function, which is supposed to be negligible for $|x| \geq L_s / 2$, and $L_s$ is the profile length. Analogous expressions are obtained for $vv$ polarisation.

In order to apply above method to random rough surfaces, a large number of independent sample profiles, with prescribed statistics, must be generated. The MoM evaluation of the corresponding scattered fields and subsequent averaging allows the computation of the surface normalised (to the profile length) radar cross section (NRCS).

Examination of the choice of the method key-parameters is now in order: profile sampling rate $\Delta x$ (equal to the width of the adopted rectangular pulse basis functions), profile length $L_s$, and number of independent realisations $N$. This problem was studied in [3] with reference to classical (non fractal) scattering surfaces: we now show how above approach [3] must be modified when a fractal scattering surface is considered. With regard to profile sampling rate and number of realisations, the same considerations of the classical case are valid. Hence, a sampling step smaller than $\lambda/8$ and a number of realisations equal to at least 64 are needed. With regard to the profile length, the fractal case requires more attention. As a matter of fact, for classical surfaces a profile length much larger than the profile correlation length is usually
chosen. This choice is not applicable to the case of fBm profile, because its correlation length is proportional to its length. However, it can be demonstrated [9] that it is sufficient to choose a profile length much larger than the correlation length of \( i(\cdot) \). This condition is verified by an fBm if

\[
k^2 s^2 \cos^2 \vartheta L_i^{2H} > 6. \tag{7}
\]

Condition (7) is derived in [9]. In order to verify this condition, we plot in Fig.3 the MoM computed NRCS of a profile with \( H=0.75 \) and \( S_0=10^{-3} \), evaluated by using \( L_c=5\lambda, \ L_s=10\lambda, \ L_s=20\lambda \ (\lambda=0.3 \text{ m}), \) corresponding to \( k^2 s^2 \cos^2 \vartheta \tau^{2H} \) equal to about 0.8, 2.4 and 6.8, respectively, for \( \vartheta=0 \). Both \( hh \) (triangles) and \( vv \) (squares) polarisations are considered. It can be verified [8-9] that considered surface parameters satisfy SPM validity conditions, i.e.,

\[
k^2 \sigma^2 \leq \frac{k^2 S_0}{2\pi H \kappa_{\text{min}}^{2H}} \ll 1 \tag{8}
\]

so that SPM can be used for reference purposes (solid line). It turns out that the criterion we propose leads to satisfactory numerical results. In eq.(8) \( \sigma \) is the profile height standard deviation and \( \kappa_{\text{min}} \) is the lower cut-off frequency, related to illuminated patch size.

### 4 RESULTS AND DISCUSSION

In this Section we use the MoM to evaluate the NRCS of fBm surfaces with different fractal parameters as a function of incidence angle, in order to analyse the scattering dependence on the surface parameters \( S_0 \) and \( H \). The parametric study is performed varying fractal parameters inside ranges typical of natural Earth surfaces.

Let us start from the dependence on \( S_0 \). In Fig. 4 we plot the NRCS (evaluated by using the MoM) of fBm surfaces with \( H=0.75 \) and (a) \( S_0=10^{-4} \text{ m}^2\text{-2H} \), (b) \( S_0=10^{-3} \text{ m}^2\text{-2H} \), (c) \( S_0=10^{-1} \text{ m}^2\text{-2H} \) as a function of the incidence angle, for \( hh \) (triangles) and \( vv \) (squares) polarisations. For reference purposes, results obtained by using the SPM are reported as solid lines. The following considerations are in order.
Fig. 3: NRCS of a profile with $H=0.75$ (i.e., $D=1.25$) and $S_0=10^{-3} \, \text{m}^{2-2H}$, evaluated by using (a) $L_s=5\lambda$, (b) $L_s=10\lambda$, and (c) $L_s=20\lambda$ ($\lambda=0.3$ m, $\Delta=\lambda/16$, $N_s=80$).
Fig. 4: NRCS of fBm surfaces with $H=0.75$ and (a) $S_0=10^{-4}$ m$^{2-2H}$, (b) $S_0=10^{-3}$ m$^{2-2H}$, (c) $S_0=10^{-1}$ m$^{2-2H}$, as a function of the incidence angle, for $hh$ (triangles) and $vv$ (squares) polarisations. Results obtained by using the SPM are reported as solid lines.
First of all, we note that at high incidence angles the scattering increases with $S_0$, while at near vertical incidence the scattering decreases with $S_0$. This is in agreement with intuition, since, for a fixed $H$, $S_0$ is directly related to $s$ and hence to the surface slope.

Furthermore, we note that for values of $S_0$ smaller than $10^{-2}$ m$^{2-2H}$ (Figs. 4a-b) and incidence angles greater than $10^\circ$, MoM computations are in very good agreement with SPM theoretical results, at variance of computations with $S_0=10^{-1}$ m$^{2-2H}$ (Fig. 4c). These results are in agreement with condition (8).

Let us now move to the dependence on $H$. In Fig.5 we plot the NRCS (evaluated by using the MoM) of fBm surfaces with $S_0=10^{-3}$ m$^{2-2H}$ and (a) $H=0.95$, (b) $H=0.75$, (c) $H=0.5$ as a function of the incidence angle, for $hh$ (triangles) and $vv$ (squares) polarisations. Again, for reference purposes, results obtained by using the SPM are reported as solid lines. We note that, as $H$ decreases, i.e., $D$ increases, the scattered power increases at large incidence angles ($\theta>10^\circ$) and (very slightly) decreases at near vertical incidence. Again, this is in agreement with intuition, since a higher fractal dimension corresponds to a rougher surface. In all three cases, eq.(8) is satisfied, and in fact simulation results are in agreement with SPM theoretical results for $\theta>10^\circ$. In particular, $vv$ polarisation return is higher than the $hh$ polarisation one, in agreement with the SPM theoretical mode.

5 CONCLUSIONS

In this paper, use of the method of moment for the evaluation of scattering from fractal surfaces is explored. Criteria to choose profile sampling frequency, profile length, and number of realisation to be used in the evaluation of NRCS are provided. Finally, some significant results have been presented, as examples of applications of the proposed method.

REFERENCES


**Fig. 5:** NRCS of fBm surfaces with $S_0=10^{-3} \text{ m}^2-2H$ and (a) $H=0.95$, (b) $H=0.75$, (c) $H=0.5$ as a function of the incidence angle, for $hh$ (triangles) and $vv$ (squares) polarisations. Results obtained by using the SPM are reported as solid lines.