PRECONDITIONERS FOR THE NUMERICAL SOLUTION
OF BOUNDARY INTEGRAL EQUATIONS FROM
ACoustics

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Abstract. We extend the preconditioning technique developed by O. Steinbach and W.L. Wendland in [8] to the Helmholtz equation which governs time harmonic acoustic waves. Using layer potentials to represent the diffracted wave the scattering problem is reduced to an integral equation on the surface of the scatterer. This equation can be solved numerically with a Galerkin method. However the matrix is ill-conditioned for fine meshes and close to resonant frequencies. We describe the construction of a preconditioner such that the preconditioned matrix models a compact perturbation of the identity operator. We state the relevant theoretical estimates and show numerical results.

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1 THE BOUNDARY ELEMENT METHOD

Let $\Omega_-$ be a smooth and bounded domain in $\mathbb{R}^3$. Its boundary we denote by $\Gamma$, and $\Omega_+$ is the complement of $\Omega_- \cup \Gamma$. Given a wavenumber $k > 0$, and a smooth incident wave $p^i$ we consider the problem of finding the diffracted wave $p^d$ in $\Omega_+$, satisfying the Neumann boundary condition on $\Gamma$, that is the solution to:

\[
\begin{align*}
\frac{\partial_n p^d}{\partial_n p^i} &= -
\Delta p^d + k^2 p^d = 0 & \text{on } \Gamma \\
|p^d| &= O(\frac{1}{r}) & \text{in } \Omega_+ \\
|\nabla p^d| &= O(\frac{1}{r}) \\
|\partial_r p^d - ikp^d| &= O(\frac{1}{r^2})
\end{align*}
\]  

(1)

When $k^2$ is not an eigenvalue of $-\Delta$ on $\Omega_-$ with Neumann boundary conditions, there is a unique $u$ in $H^{\frac{1}{2}}(\Gamma)$ such that:

\[
\forall y \in \Omega_+ \quad p^d(y) = \int \partial_n(x) G_k(x, y) u(x) dx \quad \text{with} \quad G_k(x, y) = \frac{e^{ik|x-y|}}{|x-y|}
\]  

(2)

We denote by $N_k$ the hyper-singular operator defined by:

\[
N_k : \langle N_k v \rangle(y) = -\partial_n(y) \int \partial_n(x) G_k(x, y) v(x) dx
\]  

(3)

Then $u$ is the unique solution to the equation $N_k u = \partial_n p^i$, which can be put into variational form:

\[
u \in H^{\frac{1}{2}}(\Gamma) \quad \forall v \in H^{\frac{1}{2}}(\Gamma) \quad \langle N_k u, v \rangle = \int \partial_n p^i v
\]  

(4)

For numerical computations we use the expression:

\[
\langle N_k u, v \rangle = \iint_{\Gamma \times \Gamma} G_k(x, y) (\text{rot } u(x) \cdot \text{rot } v(y) - k^2 u(x) n(x) \cdot v(y) n(y)) \, dx \, dy
\]  

(5)

and we solve the variational equation with the Galerkin method. Choosing a triangulation of the surface the standard Galerkin space consists of continuous functions that are affine on each triangle. When we consider a family of triangulations we index it by $h$, the largest diameter of a triangle. Choosing a basis $e_h = (e_h(i))$ of the Galerkin space $S_h$ defined above we define a matrix $A_h(k)$ and a tuple $L_h(k)$ by:

\[
A_h(k)_{ij} = \langle N_k e_h(j), e_h(i) \rangle \quad L_h(k)_i = \int \partial_n p^i e_h(i)
\]  

(6)

Then we solve $A_h(k) U_h = L_h(k)$, in which case $u_h = \sum_i U_h(i) e_h(i)$ is the unique solution of:

\[
u_h \in S_h \quad \forall v \in S_h \quad \langle N_k u_h, v \rangle = \int \partial_n p^i v
\]  

(7)
We consider also Dirichlet boundary conditions:

$$p^d = -p^i \text{ on } \Gamma$$  \hspace{1cm} (8)

In this case, if $k^2$ is not an eigenvalue of the interior Dirichlet problem, there is a unique $u$ in $H^{-\frac{1}{2}}(\Gamma)$ such that:

$$\forall y \in \Omega_+ \quad p^d(y) = \int_{\Gamma} G_k(x, y) u(x) dx$$  \hspace{1cm} (9)

We denote by $S_k$ the single-layer operator defined by:

$$S_k : (S_k v)(y) = \int_{\Gamma} G_k(x, y) v(x) dx$$  \hspace{1cm} (10)

Then $u$ is the unique solution to the equation $S_k u = -p^i$, which can also be put into variational form:

$$u \in H^{-\frac{1}{2}}(\Gamma) \quad \forall v \in H^{-\frac{1}{2}}(\Gamma) \quad \langle S_k u, v \rangle = -\int_{\Gamma} p^i v$$  \hspace{1cm} (11)

This formulation is well suited for the use of a Galerkin method.

2 PRECONDITIONING

We describe the preconditioning technique for the Neumann problem defined above.

For large scale problems the matrix equation $A_h(k) U_h = L_h(k)$ is solved with an iterative method. $A_h(k)$ being complex symmetric we refer to [5] and subsequent publications. Since $N_k$ is a pseudo-differential operator of order 1 the condition number of $A_h(k)$ is at least of order $h^{-1}$ for small $h$. The existence of resonant frequencies also contributes to the deterioration of the conditioning of $A_h(k)$. This in turn slows down the convergence of iterative methods. It is this problem that motivates our search for an adequate preconditioning technique.

Drawing on [8] we construct an implicit preconditioning matrix $Z_h$ such that multiplication by $Z_h$ is of the same complexity as multiplication by $A_h(k)$ (in particular we preserve the possibility of using multipole expansions that have become popular to reduce memory requirements and to speed up the matrix-vector products), and $Z_h A_h(k)$ models a compact perturbation of the identity operator. This is done as follows. For any $k > 0$, let $D_k$ be the double layer operator defined by:

$$D_k : (D_k v)(y) = \int_{\Gamma} \partial_n(x) G_k(x, y) v(x) dx$$  \hspace{1cm} (12)

By virtue of the Calderon formulas we have:

$$S_k : D_k = \frac{1}{4} = D_k^2 + (S_k - S_k) N_k$$  \hspace{1cm} (13)
Hence, up to a compact operator $4\mathcal{S}_{k'}$ is an inverse of $\mathcal{N}_k$. Numerically we exploit this fact by introducing the galerkin matrix $B_h(k')$ of $\mathcal{S}(k')$, and the mass matrix $C_h$:

$$B_h(k')_{ij} = \langle \mathcal{S}_{k'} e_h(j), e_h(i) \rangle \quad (C_h)_{ij} = \langle e_h(j), e_h(i) \rangle = \int_{\Gamma} e_h(j) e_h(i)$$

Then we set :

$$Z_h(k') = C_h^{-1} B_h(k') C_h^{-1}$$

In a sense $Z_h(k')A_h(k)$ is the matrix of $\mathcal{S}_{k'} \mathcal{N}_k$ and is therefore very close to the diagonal matrix with entries $\frac{1}{k'}$. Thus $Z_h(k')$ constitutes a good candidate for preconditioning.

The matrix $Z_h(k')$ depends on some adequate choice of $k'$. In [1] we proved that if $-k^2$ is not an eigenvalue of the Laplace operator on $\Omega_-\text{ with Neumann boundary conditions, and } -k'^2$ is not an eigenvalue of the analogous interior Dirichlet problem, then the spectral condition number of $Z_h(k')A_h(k)$ (defined to be the product of its spectral radius by the spectral radius of its inverse) is uniformly bounded for any regular family of triangulations, as $h \to 0$ (with $k$ and $k'$ fixed). This requires some modifications on the theory developed in [8] since the operators involved, as well as their discrete analogues, are not coercive but rather they satisfy Babuska’s Inf-Sup conditions. In fact we prove the following estimate on the spectral condition number :

$$\kappa(Z_h(k')A_h(k)) \leq C_h \frac{\sup_{u \in S_h} \sup_{v \in S_h} \frac{|\langle \mathcal{S}(k')u,v \rangle|}{\|u\|_2 \|v\|_2 - \frac{1}{2} \inf_{u \in S_h} \sup_{v \in S_h} \frac{|\langle \mathcal{N}(k')u,v \rangle|}{\|u\|_2 \|v\|_2}}}{\inf_{u \in S_h} \sup_{v \in S_h} \frac{|\langle \mathcal{S}(k')u,v \rangle|}{\|u\|_2 \|v\|_2}} \inf_{u \in S_h} \sup_{v \in S_h} \frac{|\langle \mathcal{N}(k')u,v \rangle|}{\|u\|_2 \|v\|_2}} (14)$$

Here $u \neq 0$ and $v \neq 0$ are implicit. Moreover :

$$(C_h)^{-1} = \inf_{u \in S_h} \sup_{v \in S_h} \frac{|\int uv|}{\|u\|_2 \|v\|_2} \inf_{u \in S_h} \sup_{v \in S_h} \frac{|\int uv|}{\|u\|_2 \|v\|_2} (17)$$

$C_h$ is bounded by the square of the norm in $H^\frac{1}{2}(\Gamma)$ of the $L_2$ projector onto $S_h$ ([1], see also [6] and [4]), but $\kappa(Z_h(k')A_h(k))$ is not controlled close to resonant frequencies.

Applying the preconditioner $Z_h(k')$ on a tuple is about as time consuming as applying $A_h(k)$: It requires the application of $B_h(k')$ and the solving of two linear systems with $C(h)$; these systems are real symmetric positive definite, well conditioned and sparse, therefore an unpreconditioned conjugate gradient is enough to solve them, and requires a negligible amount of work compared with straightforward multiplication by $B_h(k')$ or $A_h(k)$.

One of the advantages of representing the scattered wave of a Neumann problem with a double layer potential, rather than through a single layer potential, or a linear combination of a single-layer and a double-layer potential (mixed formulations), is that the method can be used on open surfaces. These surfaces are of interest because many physical objects have a thickness which is very small compared with other dimensions of the
object, and the wave length. Then numerical methods neglecting this fact tend to break
down. Considering instead that the object has zero thickness, and consists of an open
orientable surface $\Gamma$ one obtains a stable method. The unknown field $u$ on $\Gamma$ used to
represent the scattered wave is in this case the solution of :

$$u \in H^{\frac{1}{2}}_{00}(\Gamma) \quad \forall v \in H^{\frac{1}{2}}_{00}(\Gamma) \quad \langle N_k u, v \rangle = \int_\Gamma \partial_n p^i v$$

Here we used the notation :

$$H^{\frac{1}{2}}_{00}(\Gamma) = [H^1_0(\Gamma), H^0(\Gamma)]^{\frac{1}{2}}$$

The standard Galerkin space is defined as above, with the additional requirement that the
field must vanish on the boundary $\partial \Gamma$ of $\Gamma$. In this setting we define the preconditioning
matrix by the same formulas as in the case of a closed surface, remembering to take
only the basis functions that vanish on the boundary. In [1] we proved that this yields
a spectral condition number that is bounded by a power of $|\log(h)|$. The reason for the
logarithmic factor is that the single-layer operator does not operate naturally on the dual
of $H^{\frac{7}{2}}_{00}(\Gamma)$, but on the space $H^{-\frac{1}{2}}(\Gamma)$.

3 NUMERICAL EXPERIMENTS

3.1 Cavity

First we show some results on a spherical cavity. A coarse grid is shown in figure 1. A
spherical wave with wave number $k = 6$ is emitted from the center of the cavity, and we
solve the exterior Neumann problem by the above procedure.

The matrix equation is solved with the Conjugate Gradient (CG) algorithm, and we
evaluate the preconditioner by comparing the performance of the preconditioned CG
algorithm with that of the unpreconditioned one. We plot the number of iterations needed
to achieve a reduction of the $\ell_2$ norm of the residual by a factor of $10^{-4}$, $10^{-5}$ and $10^{-6}$,
for different meshes. That is, for the object represented in figure 1 we have a family
of meshes indexed by some integer $i$ proportional to $1/h$, and we plot the number of
iterations needed as a function of $i$. The integer $i$ varies from 8 to 20 corresponding to
642 and 4002 nodes respectively. In figure 2 there are three graphics, one for each of the
three chosen error thresholds. Each graphic shows two graphs : the upper one corresponds
to the unpreconditioned case, and the lower one corresponds to the preconditioned case.

Taking into account that in the preconditioned case each iteration is about twice as
time consuming as in the unpreconditioned case, we see that for coarse meshes the im-
provement of the convergence rate by the preconditioner is rather modest. However the
number of iterations needed in the preconditioned case increases very slowly with the
mesh refinement, so that for fine meshes the preconditioned algorithm becomes several
times faster than the unpreconditioned one.
Figure 1: Spherical cavity: seen from outside and vertical section

Figure 2: Number of iterations needed to achieve a reduction of the norm of the residual by a factor of $10^{-4}$ (left), $10^{-5}$ (middle) and $10^{-6}$ (right), for mesh refinements 8 to 20.
In some cases the number of iterations needed in the preconditioned case actually decreases as the mesh is refined. For the reduction of the norm of the residual by a factor of $10^{-5}$ we interpret this as stemming from the fact that the Calderon formulas are then more faithfully respected on a discrete level. It can also stem from the fact that -as is well known- the CG algorithm can be rather unstable ; this would be the case for the finest mesh at $10^{-6}$. To illustrate this we show the graph of the function:

$$\rho : n \mapsto \log_{10} \frac{\|A_h(k)U^n - L_h(k)\|}{\|L_h(k)\|}$$

Here $n$ is the iteration and $U^n$ is the approximate solution given by the CG algorithm at the $n$’th iteration. Two graphics are shown in figure 3 corresponding to the mesh refinements $i = 19$ and $i = 20$. Each graphic displays two curves. The upper one is for the unpreconditioned setting, whereas the lower one is for the preconditioned case.

![Graph of $\rho$ for $i = 19$ (left) and $i = 20$ (right).](image)

**Figure 3:** Graph of $\rho$ for $i = 19$ (left) and $i = 20$ (right).

### 3.2 Disc

As stated above the method can be used for so-called open surfaces. We consider diffraction in $\mathbb{R}^3$ by a (two-dimensional) disc with radius 1. The triangulation has 4921 nodes, 240 of which are on the edge of the disc. The mesh is refined close to the edge. The disc is subject to a spherical wave with wave number 24. In figure 4 we show the graph of the function $\rho$ defined as in the preceding subsection, with the proposed preconditioner (lower graph) and without any preconditioner (upper graph).

### 3.3 Sphere

To illustrate the fact that the preconditioner performs well close to resonant frequencies, we consider the scattering by the unit sphere with Dirichlet boundary conditions close to
the resonance $k = \pi$. An eigenvector of the Laplacian on $H^1_0(\Omega_-)$ is then given by:

$$x \mapsto \frac{\sin(k|x|)}{|x|}$$

The mesh considered has 1002 nodes. The diffracted wave is represented with a single layer potential. Thus we are led to solve a single layer equation on the surface of the sphere. This equation can be preconditioned using the hyper-singular operator. In figure 5 we show the number of iterations needed to achieve a reduction of the norm of the residual by $10^{-6}$ for several wave numbers close to the resonance. As can be seen from the graphic the discrete resonance takes place at a value slightly different from $\pi$. The upper (plain) dots correspond to the unpreconditioned case, whereas the lower dots (circles) correspond to the preconditioned case.

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REFERENCES


Figure 5: Resonace of the interior Dirichlet problem
