SHAPE OPTIMIZATION FOR THE NAVIER–STOKES EQUATIONS BASED ON OPTIMAL CONTROL THEORY

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Abstract. This paper presents a new approach to a shape optimization problem of a body located in the unsteady incompressible viscous flow field based on an optimal control theory. The optimal state is defined by the reduction of drag and lift forces subjected to the body. The state equation used is the transient incompressible Navier–Stokes equations. The shape optimization problem can be formulated to find out geometrical coordinates of the body to minimize the performance function that is defined to evaluate forces subjected to the body. The mixed finite element method by the MINI element is used for the spatial discretization, while the fractional step method with implicit temporal integration is used for the temporal discretization. For the numerical study, the optimal shape of the body which has circular shape as the initial state can be finally obtained as the streamlined shape.
1 INTRODUCTION

The purpose of this paper is to formulate and to solve a shape optimization problem based on the optimal control theory\cite{1}\cite{2}. A formulation concerned with the shape optimization of fluid forces reduction problem\cite{3}\cite{4} of the body located in the incompressible viscous flow is presented. The flow can be assumed to be controlled by the geometrical surface coordinates of the body. The optimal state is defined by the reduction of drag and lift forces subjected to the body. The shape optimization problem is to find out geometrical surface coordinates of the body to minimize the performance function, which shows the magnitude of the forces. The present performance function consists of integration of a square sum of fluid forces and of a square residual sum between state and initial geometrical coordinates. The state equation acts as constraint condition of the performance function. The state equation is expressed as the transient incompressible Navier–Stokes equations. The Lagrange multiplier method is applied to the constraint condition of the performance function and a Lagrange multiplier equation is obtained. This equation is equivalent to the adjoint operator of a linealized version of the state equation, which is referred to as the adjoint equation. The equation is dependent on the state equation and has to be solved backwards in time, starting from a final condition called the transversality condition. It is convenient to obtain the gradient of the performance function with respect to the geometrical surface coordinates, introducing an efficient method for the minimization of the performance function.

The mixed finite element method by the MINI element\cite{5} is used for the spatial discretization of these equations. The fractional step method\cite{6} with implicit temporal integration are used for the temporal discretization of the state and the adjoint equations. As a numerical example, the present shape optimization method is applied to the fluid force reduction problem of a circular cylinder located in the viscous flow. Computed results for optimized case are displayed with the initial state.

2 INCOMPRESSIBLE NAVIER–STOKES EQUATION

Let $\Omega \subset \mathbb{R}^N (N = 2)$ be the spatial domain with the boundary $\partial \Omega = \Gamma$ and $I = [0, T]$ be the temporal domain. In the domain $\Omega \times I$, define velocity $u(x, t) : \Omega \times I \mapsto \mathbb{R}^N$ and pressure $p(x, t) : \Omega \times I \mapsto \mathbb{R}$, non-dimensional incompressible Navier–Stokes equation can be written as follows.

\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \cdot \sigma &= 0 \quad \text{in} \quad \Omega \times I, \\
\nabla \cdot u &= 0 \quad \text{in} \quad \Omega \times I, \\
u(x, 0) &= u_0(x) \quad x \in \Omega, \\
u &= \hat{u} \quad \text{on} \quad \Gamma_1 \cup \Gamma_B \times I, \\
\sigma \cdot n &= \hat{t} \quad \text{on} \quad \Gamma_2 \times I,
\end{align*}
where, $\nu$ is a viscosity coefficient ($\nu = 1/Re$), $Re$ is Reynolds number. $\sigma = -p \mathbf{1} + \nu(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the stress tensor, $\mathbf{u} \cdot \nabla \mathbf{u} = \{\sum_{j=1}^{i=N} u_j u_{i,j}\}_{i=1}^{i=N}$, and $\mathbf{n}$ is the outer normal unit vector at $\Gamma$. The boundary $\Gamma$ is denoted by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_B$.

In this paper, we consider a typical problem in which a solid body of cross-section $B$ with the boundary $\Gamma_B$ is laid in the flow. Throughout this section we assume that this spatial domain $\Omega_s$ is sufficiently smooth. Here the solution $(\mathbf{u}, p)$ depends on the shape of domain $\Omega$. The fluid forces acted on a boundary $\Gamma_B$ are denoted by $(D, L)$, where $D$ and $L$ are drag and lift forces, respectively. The fluid forces $(D, L)$ are obtained by integrating the traction $t$ on $\Gamma_B$.

$$(D, L) = -\int_{\Gamma_B} t \, d\gamma$$

3 SHAPE OPTIMIZATION AND SHAPE DERIVATIVE

In this section, we discuss the shape derivative of solutions to the Navier–Stokes equation. And we will formulate the shape optimization of fluidforce control using the material derivative method[7].

3.1 Performance Funciton

Consider the shape optimization problem[8]:

$$\min J(\mathbf{u}, p, \Omega)$$
subject to equations (1)–(5). In this paper, a fluid force control problem[3][4] is considered, thus, the fluid force is directly used in the performance function. The performance function $J$ is defined by the temporal integration of a square sum of fluid forces as follows.

$$J(\mathbf{u}, p, \Omega) = \frac{1}{2} \int_{t_1}^{t_2} \{q_1 (D - D^*)^2 + q_2 (L - L^*)^2\} \, dt$$

where $D^*, L^*$ are the target drag and lift foeces, respectively. $q_1, q_2$ denote the weighting paremeter. In order to calculate the the gradient of the performance function $J$, we will employ the so–called material derivative mathod. Material derivative concepts are well–known in continuum mechanics and have been applied to shape optimization problems[9].

3.2 Material Derivative Concept

For $|s|$ sufficiently small, let $\Omega_s = F_s(\Omega)$ be the image of $\Omega$ obtained by the mapping $F_s : \mathbb{R}^N \mapsto \mathbb{R}^N$ defined as

$$F_s = \mathbf{x}(s) + s \mathbf{h},$$

where, vector field $\mathbf{h} = \mathbf{h}(\mathbf{x})$ is called speed field[7]. The speed field $\mathbf{h}$ is denoted by the material derivative of mapping $F_s$. 

3
Descriptions of a function can be considered by following twice on difference of region of definition. For \( \varphi_s \in H^1(\Omega) \) (Lagrange description) and \( \varphi^s \in H^1(\Omega_s) \) (material description) let us define

\[
\varphi^s = \varphi_s \circ F_s. \tag{10}
\]

The material derivative of \( \varphi \) for the field \( h \) is given by

\[
\dot{\varphi}(x) = \lim_{s \to 0} \frac{\varphi_s(x + s h) - \varphi(x)}{s}, \quad x \in \Omega. \tag{11}
\]

And the shape derivative of \( \varphi \) is

\[
\varphi'(x) = \lim_{s \to 0} \frac{\varphi_s(x) - \varphi(x)}{s} = \dot{\varphi}(x) - \mathbf{h}(x) \cdot \nabla \varphi(x), \quad x \in \Omega. \tag{12}
\]

Moreover let

\[
J_s = \int_{\Omega_s} \varphi_s \, dx, \quad \varphi_s \in H^1(\Omega_s). \tag{13}
\]

Then

\[
j = \frac{d}{ds} J_s \bigg|_{s=0} = \int_{\Omega_s} (\dot{\varphi} + \varphi \nabla \cdot \mathbf{h}) \, dx = \int_{\Omega_s} \{\varphi' + \nabla \cdot (\mathbf{h} \varphi)\} \, dx \tag{14}
\]

Figure 1. Domain Variation
3.3 Stationary Condition and Variational Formulation

Using above relations, shape optimization problem for incompressible Navier–Stokes equation is formulated. And stationary conditon for its problem is derived by Lagrangian multiplier method. Performance function can be given by the solution of the incompressible Navier–Stokes equation.

Applying the Lagrange multiplier method, this problem can be rewritten the stationary problem of Lagrange functional using the Lagrange multiplies defined by adjoint velocity \( y : \Omega_s \times I \mapsto \mathbb{R}^2 \) and adjoint pressure \( \lambda : \Omega_s \times I \mapsto \mathbb{R} \) as follows.

\[
L = J - \int_I \int_{\Omega_s} y \cdot \left\{ \frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \cdot \sigma \right\} \, dx \, dt + \int_I \int_{\Omega_s} \lambda \nabla \cdot u \, dx \, dt. \tag{15}
\]

Using speed field \( h(\Omega_s) = \partial F_s / \partial s(\Omega) = \partial F_s / \partial s (F^{-1}_s(\Omega_s)) \), Euler derivative \( \dot{L} \) of Lagrange functional \( L \) with respect to the domain variation can be obtained as follows:

\[
\dot{L} = - \int_I \int_{\Omega_s} y' \cdot \left\{ \frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \cdot \sigma \right\} \, dx \, dt \\
+ \int_I \int_{\Omega_s} \lambda' \nabla \cdot u \, dx \, dt \\
- \int_I \int_{\Omega_s} u' \cdot \left\{ - \frac{\partial y}{\partial t} + \nabla u^T y - u \cdot \nabla y + \nabla \lambda - \nu \nabla \cdot (\nabla y + \nabla y^T) \right\} \, dx \, dt \\
+ \int_I \int_{\Omega_s} p' \nabla \cdot y \, dx \, dt \\
- \int_I \int_{\Gamma_2} u' \cdot s \, d\gamma \, dt + \int_I \int_{\Gamma_1} t' \cdot y \, d\gamma \, dt \\
+ \int_I \int_{\Gamma_1} t'_1 \{ y_1 - q_1 (D - D^*) \} \, d\gamma \, dt + \int_I \int_{\Gamma_2} t'_2 \{ y_2 - q_2 (L - L^*) \} \, d\gamma \, dt \\
- < h, G_s >_{\Omega_s} \tag{16}
\]

where, \( (\cdot)' \) shows the shape derivative. \( t' = \sigma' \cdot n \) is the shape derivative of the traction vector, \( s = \{ u y - \lambda 1 + \nu (\nabla y + \nabla y^T) \} \cdot n \) is the adjoint traction.

\[
< h, G_s >_{\Omega_s} = \int_{\Omega_s} \left( h \cdot \nabla G_s + \nabla h \cdot G_s 1 \right) \, dx, \tag{17}
\]

\[
G_s = \int_I \left( y \cdot \left\{ \frac{\partial u}{\partial t} + u \cdot \nabla u - \nabla \cdot \sigma \right\} - \lambda \nabla \cdot u \right) \, dt. \tag{18}
\]

where, \( G_s \) gives sensitivety for domain variation, is called shape derivative density function.

If this shape derivative density function can be obtained, applying minimization technique, speed field \( h \) can be obtained.
If Lagrange functional $L$ is satisfied the stationary condition, then the variational formulation can be written using the following functional spaces as follows:

$$U = \{ u \in (H^1(\Omega_s))^2 \mid u = \hat{u} \text{ on } \Gamma_1 \cup \Gamma_B \}, \quad (19)$$

$$Y' = \{ y' \in (H^1(\Omega_s))^2 \mid y' = 0 \text{ on } \Gamma_1 \cup \Gamma_B \}, \quad (20)$$

$$P = \Lambda' = L^2_0(\Omega_s) \quad (21)$$

$$Y = \{ y \in (H^1(\Omega_s))^2 \mid y = 0 \text{ on } \Gamma_1, \quad y = (q_1(D - D^*), q_2(L - L^*)) \text{ on } \Gamma_B \}, \quad (22)$$

$$U' = \{ u' \in (H^1(\Omega_s))^2 \mid u' = 0 \text{ on } \Gamma_1 \cup \Gamma_B \}, \quad (23)$$

$$\Lambda = P' = L^2_0(\Omega_s), \quad (24)$$

The variational formulation of the state and adjoint equations can be written as follows:

Find $(u, p) \in (U \times P)$ such that

$$\int_{\Omega_s} y' \cdot \left\{ \frac{\partial u}{\partial t} + u \cdot \nabla u \right\} \, dx - \int_{\Omega_s} \nabla y' : p \, 1 \, dx + 2\nu \int_{\Omega_s} D(y') : D(u) \, dx$$

$$= \int_{\Gamma_2} y' \cdot \hat{t} \, d\gamma \quad \forall y' \in Y', \quad (25)$$

$$\int_{\Omega_s} \lambda' \nabla \cdot u \, dx = 0 \quad \forall \lambda' \in \Lambda', \quad (26)$$

$$u(x, 0) = u_0(x) \quad x \in \Omega_s, \quad (27)$$

$$u = \hat{u} \quad \text{on } \Gamma_1 \cup \Gamma_B \times I. \quad (28)$$

$$\sigma \cdot n = \hat{v} \quad \text{on } \Gamma_2 \times I. \quad (29)$$

Find $(y, \lambda) \in (Y \times \Lambda)$ such that

$$\int_{\Omega_s} u' \cdot \left\{ -\frac{\partial y}{\partial t} + (\nabla u)^T y \right\} \, dx + \int_{\Omega_s} \nabla u' : \{ u \, y - \lambda \, 1 \} \, dx$$

$$+ 2\nu \int_{\Omega_s} D(u') : D(y), \, dx = \int_{\Gamma_2} u' \cdot \hat{s} \, d\gamma, \quad \forall u' \in U', \quad (30)$$

$$\int_{\Omega_s} p' \nabla \cdot y \, dx = 0 \quad \forall p' \in P', \quad (31)$$

$$y(x, T) = 0 \quad x \in \Omega_s, \quad (32)$$

$$y = 0 \quad \text{on } \Gamma_1 \times I. \quad (33)$$

$$y = (q_1(D - D^*), q_2(L - L^*)) \quad \text{on } \Gamma_B \times I. \quad (34)$$

$$\{ u \, y - \lambda \, 1 + \nu (\nabla y + \nabla y^T) \} \cdot n = \hat{s} \quad \text{on } \Gamma_2 \times I. \quad (35)$$
where, \( D(v) = (\nabla v + \nabla v^T)/2 \). \( u, p, y, \lambda \) are found, then Lagrange functional \( L \) is equal to performance functional \( J \), following equation can be obtained.

\[
\dot{L} = \dot{J} = - \langle h, G_s \rangle_{\Omega_s} = 0.
\]  

(36)

So \( G_s \) at the derivative of the performance functional in equation (36) is a scalar function of speed field \( h \) which gives infinitesimal variation of domain, shape derivative equation in this problem is obtained. Equations (25)–(29) and (30)–(35) show variational forms of state equation and adjoint equation.

3.4 Solution of Subproblem (36)

The variational problem which gives solution for speed field \( h \in (H^1(\Omega_s))^2 \) can be defined by following the Dirichlet problem.

\[
a_{\Omega_s}(h^*, h) + \langle h^*, G_s \rangle_{\Omega_s} = 0 \quad \forall h^* \in (H^1(\Omega_s))^2,
\]  

(37)

where, \( a_{\Omega_s}(\cdot, \cdot) \) is bi–linear form defined by following equation.

\[
a_{\Omega_s}(u, v) = \int_{\Omega_s} (\alpha u \cdot v + \beta \nabla u : \nabla v) \, dx \\
\forall u, v \in (H^1(\Omega_s))^2. \quad \alpha \geq 0, \beta > 0.
\]  

(38)

Above equation shows that speed field \( h \) is solution of the Dirichlet problem operated by negative shape derivative density \( G \) as a source.

3.5 Numerical Technique

The mixed finite element method by the MINI element[5] is used for the spatial discretization of the state and adjoint equations. The implicit method is used for the temporal discretization of the state and the adjoint equations. The fractional step method[6] with the implicit temporal integration are employed in this paper.

4 NUMERICAL EXAMPLE

4.1 Drag Reduction Problem

The purpose of the present is reduce the drag force, while the lift force and the circular cylinder volume are not kept unchanged in this case. The weighting parameters are chosen as \( q_1 = 1, q_2 = 0 \), thus, this control problem is only drag force reduction.

Figure 3 shows the number of iterations versus the relative performance function \( J \). It is observed that the reduction of the performance function is achieved and 400 iterations need to be converged. It is seen that the amplitude of the drag force is reduced. For this numerical example, the optimal shape of the body which has circular shape as the initial state can be finally obtained as the streamlined shape like a wing shown in
Figure 4. For the drag force to be reduced, the optimal shape has been obtained as flat configuration. However the initial shape has a smooth edge. This means that the present shape optimization method has to be able to treat the apparition of singular points. No particular treatment such as a Spline interpolation has been done for this case. The initial drag force should be positive as the flow direction is uniform. This means that the optimal shape is asymmetric. Iso-pressure contours in optimal is shown in Figure 2.

Figure 2. Iso–contours of pressure

Figure 3. History of performance function
4.2 Drag and Lift Reduction Problem

This control problem is lift and drag force reduction, thus the weighting parameters are $q_1 = 1$, $q_2 = 1$. It is seen that the amplitude of the lift force is reduced. For this numerical example, the optimal shape of the body which has circular shape as the initial state can be finally obtained as shown in Figure 5. This shape is the streamlined one like a wing. The initial lift force should be periodic as the initial shape is symmetric. The optimal shape is almost symmetric. Figure 6, 7 show the time history of the drag and lift force. Iso-pressure contours in optimal ($t = 40$) is shown in Figure 8.
5. CONCLUSION

A formulation for shape optimization of the incompressible Navier–Stokes equations has been presented in this paper. The mixed finite element method employing MINI element has been used for the spatial discretization, while the fractional step method and Crank–Nicolson scheme was used for the temporal discretization. A computational method of optimal shape design for the body located in the transient incompressible viscous fluid flow has been presented to feature a new shape optimization approach based on the optimal control theory. As a numerical example, fluid force reduction problem of a circular cylinder located in the the present shape optimization method is applied to fluid force reduction problem of a circular cylinder located in the viscous flow. The modified transversality condition that is the solution of the steady Lagrange multiplier equation has been proposed. The shape optimization is presented in this peper successful, and it is confirmed that the present shape optimization method is very effective and robust to implement for the incompressible Navier–Stokes equations. Future work will include to expand this technique in three–dimensional configurations.
Figure 8. Iso–contours of pressure
REFERENCES


