Discontinuous Galerkin methods for compressible and incompressible flows on space-time adaptive meshes

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In this work the numerical discretization of the partial differential governing equations for dissipative compressible and incompressible flows is dealt within the discontinuous Galerkin (DG) framework along space-time auto-adaptive meshes. We present two main numerical strategies: (1) *fully explicit* ADER-DG methods supplemented by an a posteriori finite-volume subcell limiter on collocated grids and (2) the *semi-implicit* DG methods on *face-based staggered grids.*

In the first part an *arbitrary high-order accurate* ADER Discontinuous Galerkin (DG) method on space-time adaptive meshes (AMR) is outlined for the solution the compressible Navier-Stokes equations and the equations of viscous and resistive magnetohydrodynamics in two and three space-dimensions.

One of the major weakness of high order DG methods lies in the difficulty of limiting discontinuous solutions, which generate spurious oscillations, namely the so-called 'Gibbs phenomenon'. In the present work the nonlinear stabilization of the scheme is locally introduced only for troubled cells on the basis of the *a posteriori* MOOD approach. Moreover, numerical diffusion is considerably reduced also for the *limited* cells by resorting to a proper sub-grid. The method first computes a so-called *candidate solution* by using a high order accurate *unlimited* DG scheme. Then, a set of numerical and physical detection criteria is applied to the candidate solution. Furthermore, only in those cells where at least one of these criteria is violated the computed candidate solution is *locally* rejected. Next, the numerical solution of the *previous* time step is scattered onto cell averages of a suitable *sub-grid* in order to preserve the natural sub-cell resolution of the DG scheme. Subsequently, a more reliable numerical solution is *recomputed a posteriori* by employing a more robust but still very accurate ADER-WENO finite volume scheme on the subgrid averages within that troubled cell. Finally, a high order DG polynomial is reconstructed back from the evolved subcell averages. We apply the whole approach for the first time to the equations of compressible gas dynamics and magnetohydrodynamics in the presence of viscosity, thermal conductivity and magnetic resistivity, therefore extending our family of adaptive ADER-DG schemes to cases for which the numerical fluxes also depend on the gradient of the state vector. The high-resolution properties of the presented numerical scheme are significantly enhanced within a cell-by-cell *Adaptive Mesh Refinement* (AMR) implementation together with time accurate local time stepping (LTS). The afore-mentioned properties are verified against a wide number of non-trivial test cases both for the compressible Navier-Stokes and the viscous and resistive magnetohydrodynamics equations. Recent results regarding the implementation of the presented numerical schemes for the general relativistic MHD equations will be also presented.

In the second part of the talk a new high order semi-implicit discontinuous Galerkin method (SI-DG) is presented for the solution of the *incompressible* Navier-Stokes equations on *staggered* space-time *adaptive* Cartesian grids (AMR) in two and three space-dimensions. The pressure solution is approximated by means of of piecewise polynomials on the main grid, which is dynamically adapted within a *cell-by-cell* AMR framework. According to the time dependent main grid, different face-based spatially *staggered* dual grids are defined for the piece-wise polynomials of the respective velocity components.

Arbitrary high order of accuracy is achieved in space, while a very simple semi-implicit time discretization is obtained via an explicit discretization of the nonlinear convective terms, and an implicit discretization of the pressure gradient in the momentum equation and of the divergence of the velocity field in the continuity equation. The real advantages of the staggered grid arise after substituting the discrete momentum equations into the discrete continuity equation, leading to a linear system for only one unknown, the scalar pressure. The resulting linear pressure system is shown to be symmetric and positive-definite. In order to avoid a quadratic stability condition for the parabolic terms given by the viscous stress tensor, an implicit discretization is also used for the diffusive terms in the momentum equation. Both linear systems for pressure and velocity are very efficiently solved by means of a classical matrix-free conjugate gradient method, for which fast convergence is observed. Moreover, it should be noticed that all test cases shown have been performed without the use of any preconditioner.

The new auto-adaptive staggered DG scheme has been thoroughly verified for polynomial degrees up to $N = 9$ for a large set of non-trivial test problems in two and three space dimensions, for which analytical, numerical or experimental reference solutions exist. To the knowledge of the authors, this is the first staggered semi-implicit DG scheme for the incompressible Navier-Stokes equations on space-time adaptive meshes in two and three space dimensions.

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