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PROBABILISTIC STRENGTH ANALYSIS OF FIBER COMPOSITE STRUCTURES USING LAMINATION PARAMETERS

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Abstract. An approach is presented that allows using lamination parameters for probabilistic analyses of the onset of material failure of composite structures. Lamination parameters allow for describing the scatter of ply orientations with 12 parameters, independently of the number of plies. The probabilistic analysis is split into two steps. First, the stochastic distribution of lamination parameters due to scattering ply orientations is determined analytically. Secondly, the stochastic distribution of the structural response (buckling load, material failure) is determined using lamination parameters as random input parameters. The drawback of utilizing lamination parameters is that the information of a discrete stacking gets lost as the lamination parameters vary. A discrete stacking however is required to evaluate strength criteria like, for instance, the Tsai-Wu criterion. In the current contribution, for each strength evaluation a discrete stacking, which corresponds to a varied set of lamination parameters, is determined by optimization. Then, for each stacking the Tsai-Wu criterion is evaluated. The proposed approach requires a couple of steps, which are not required when directly considering ply orientation as random parameter. Still, the approach offers efficiency gain, which is demonstrated by examples with analytical and numerical objective functions.

1 INTRODUCTION

The structural response of composite structures show relatively large scatter of structural response. Therefore, probabilistic design approaches are promising to reduce conservatism in the sizing of composites structures [1]. One source of scatter is the scatter of fibre orientations of each ply. When fibre orientations are considered as random parameters within a probabilistic analysis, the number of parameters obviously increases with the number of plies. When applying gradient based probabilistic approaches to structural response function for which the derivatives are not given analytically, the computational effort increases with the number of parameters and hence, with number of plies. Lamination parameters allow for describing the stiffness of any laminate with only 12 parameters. Therefore, lamination parameters have been widely used for optimization of composite structures (see, e.g., [2, 3, 4]). In a similar manner, lamination parameters also can be used for gradient-based probabilistic analyses of composite structures [5].

When lamination parameters vary within an optimization or probabilistic analysis, the information of fiber orientations is lost. This prevents applying typical failure criteria for composites structures, which are evaluated on ply level. Therefore, lamination parameters have mostly be used for stiffness problems like buckling or aeroelastic stability. However, the more layers a laminate consists of, i.e. the thicker the laminate is, the more it is the material strength, which is decisive for the structural performance. Ijsselmuiden et al. [6] incorporated the Tsai-Wu failure criterion into the lamination parameter design space. They show that their approach accurately represents the factor of safety of practical laminates under in-plane loading; however, for bending dominated problems it may be too conservative. To the authors knowledge, there is no other failure criterion for composites defined in the space of lamination parameters.

In the current contribution, buckling and material strength of composite structure are considered within the probabilistic analysis with lamination parameters. A discrete layup is determined by an optimization algorithm any time the set of lamination parameters varies. Then, a failure criterion is evaluated on ply level. The applicability of the new approach is demonstrated with an analytic example and the efficiency improvement is demonstrated by application to a use case which involves nonlinear finite element simulations for buckling and strength evaluation.

2 PROBABILISTIC ANALYSIS WITH LAMINATION PARAMETERS

The objective of the probabilistic analyses in this paper is to determine the stochastic distribution of the structural response (buckling and/or material failure) due to random ply orientations, as shown in figure 1. The use of lamination parameters requires to split the probabilistic analysis into two steps, which is described in the following section.

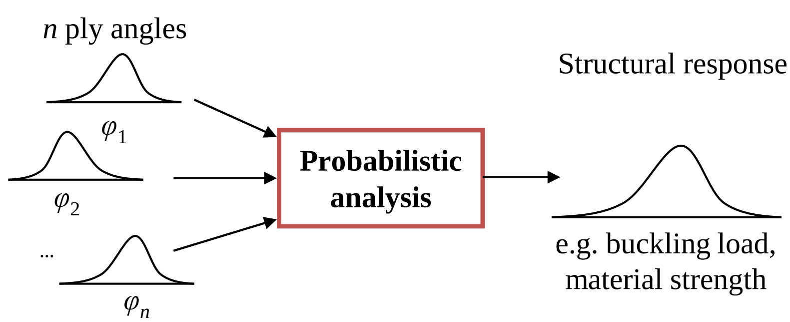

Figure 1: Probabilistic analysis with ply angles as random parameters

2.1 Two step approach using lamination parameters

Tsai and Hahn [7] introduced the concept of lamination parameters (where in [7] the lamination parameters are referred to as geometric factors.) The basic concept is to decompose the stiffness matrix ("ABD matrix") into a set of parameters $\boldsymbol{\xi}$, which are only dependent on the ply orientations, and another set of parameters $\mathbf{U}$, which are only dependent on the material properties. For the definition of these parameters see the appendix. The number of lamination parameters that describe the ply orientation is always smaller or equal 12, no matter how many plies a laminate consists of. Therefore, the use of lamination parameters allows for reducing the number of parameters involved in an analysis, if the number of plies exceeds 12.

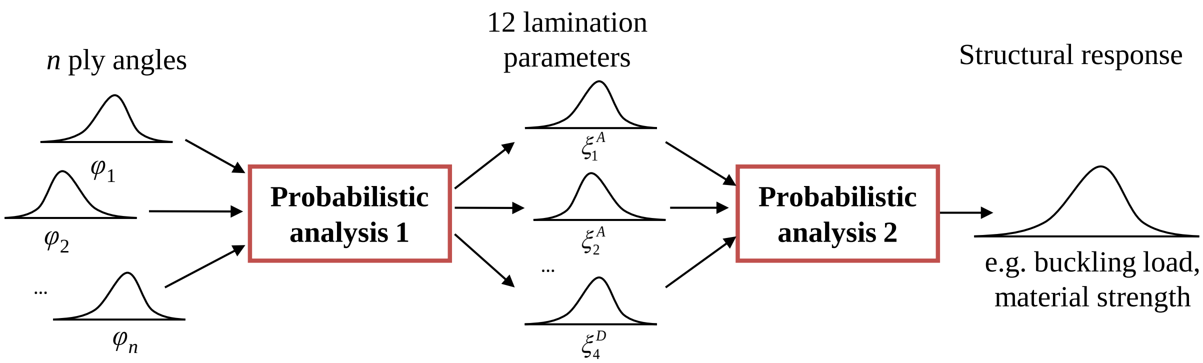

Figure 2: Two-step probabilistic analysis with ply angles as random parameters, using lamination parameters

Using lamination parameters for a probabilistic analysis requires to split the analysis into two steps, as shown in figure 2. In the first step, the joint stochastic distribution of lamination parameters ξ_j^k is determined based on the stochastic distributions of ply orientations φ_i. This first analysis can be performed analytically as shown in [5], and it is therefore very fast. As shown in [5], the lamination parameters approximately follow gaussian distribution as the number of plies increases. Hence, their distribution is fully described by the mean values, variances and co-variances.

In the second step, the lamination parameters ξ_j^k are the random input parameters and the stochastic distribution of the structural response is determined. Since this response may be result form nonlinear finite element analyses, the second step typically required much more computing time. Depending on the probabilistic approach used, it is beneficial to have a low number of random parameters in order to keep the computational cost low. This is what is achieved by using lamination parameters, if the number of plies exceeds 12.

2.2 Probabilistic approach

Consider the objective function $g(\mathbf{x})$, which is a function of realizations $\mathbf{x}$ of the random vector $\mathbf{X}$ with the probability density function $f_{\mathbf{X}}(\mathbf{x})$. The objective of the probabilistic approaches is to determine the stochastic distribution $F_g(g)$ of the objective function g

due to the scatter of $\mathbf{X}$. In the current paper, the objective function is given by the failure load of a composite structure and the random vector contains the ply orientation (when using the one-step approach) or the lamination parameters (when using the two-step approach).

One of the simplest and fastest probabilistic approaches is the first-order second-moment (FOSM) method, which utilizes a first-order Taylor series expansion of the objective function g at the mean vector $\boldsymbol{\mu}$ of $\mathbf{X}$. Inserting the Taylor series into the definition of the mean value μ_g of g yields the following approximation.

$$\mu_g \approx g(\boldsymbol{\mu}) \tag{1}$$

In a similar manner, the variance σ_g^2 of g is approximated.

$$\sigma_g^2 \approx \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial g(\boldsymbol{\mu})}{\partial x_i} \frac{\partial g(\boldsymbol{\mu})}{\partial x_j} \mathrm{cov}(X_i, X_j) \tag{2}$$

Using a second-order Taylor series yields a higher order approximation of mean and variance, like the incomplete second order approach (ISOA) suggested in [1]. In any case, approaches based on Taylor series require the derivatives of the objective function. If these are not given explicitly (for instance because the objective function results from a nonlinear finite element analysis), the derivatives must be estimated by finite differences. When using central differences, $2n + 1$ evaluations of the objective function are required, where n is the number of random parameters.

For validation of the approach, the Monte Carlo method is used. For a Monte Carlo simulation, a large number of realizations $\mathbf{x}^{(\mathbf{i})}$ of the random vector $\mathbf{X}$ is generated according to its distribution $f_{\mathbf{X}}(\mathbf{x})$. For each realization, the objective function is evaluated $g(\mathbf{x}^{(i)}) = g_i$ and thereby a discrete distribution of g is obtained. The Monte Carlo method is typically the computationally most expensive method, but it also provides a very robust and easy to use algorithm. Several techniques exist which improve the efficiency of Monte Carlo simulations, such as importance sampling or response surface methods (see, e.g., [8]).

3 STRENGTH EVALUATION FOR LAMINATION PARAMETERS

Lamination parameters mostly have been used for the optimization of stiffness problems such as the mass minimization with buckling constraints [4], maximization of buckling load for constant volume [9], and the aerodynamic design optimization of wings [10], [11]. Considering material strength is difficult since because as lamination parameter vary (within optimization or probabilistic analysis) the information of an associated stacking sequence is lost. Therefore, it is not possible to determine the stresses in each unidirectional ply, which are required for evaluating failure criteria.

Ijsselmuiden et al. [6] found a way to incorporate the Tsai-Wu failure criterion into the lamination parameter design space, which requires several conservative assumptions. Especially for laminates subject to bending it is overly conservative. In structure where material failure occurs after local buckling, bending plays an important role.

When performing optimization with lamination parameters, at the end of the optimization a certain stacking is desired as result. It has been proposed to determine such a stacking sequence by an subsequent optimization, which seeks the set of ply orientations which matches the lamination parameters obtain from the structural optimization best (see, e.g, [4]). In the current paper, a similar approach is followed to obtain a discrete stacking each time the material strength of a laminate is evaluated.

In the second step of the two-step approach, FOSM or ISOA are used to determine the stochastic distribution of material failure. For that, finite difference steps are performed for each lamination parameter. Each time the failure criterion is evaluated for varied lamination parameters, the following optimization problem is solved.

$$\min_{\boldsymbol{\varphi}} \|\boldsymbol{\xi}_{\text{target}} - \boldsymbol{\xi}(\boldsymbol{\varphi})\| \tag{3}$$

Here, $\boldsymbol{\xi}_{\text{target}}$ is the set of lamination parameters for which material failure needs to be evaluated, and $\boldsymbol{\xi}(\boldsymbol{\varphi})$ is the set of lamination parameters that is varied to match the $\boldsymbol{\xi}_{\text{target}}$ considered, by varying the ply orientations $\boldsymbol{\varphi}$. For the current paper, the interior point method is used as optimization algorithm.

The function (5) that transforms a layup of n_p plies to a set of lamination parameters is a nonlinear function $\mathbb{R}^{n_p} \to \mathbb{R}^{12}$ (where typically $n_p > 12$, because otherwise there is no efficiency gain by using lamination parameters). The inverse function is a surjective function, which means that multiple stackings have the same set of lamination parameters. In the current paper, the staking sequence that corresponds to the mean stacking is used as start vector for the optimization, which increases the chance to find the solution which is closest to the mean. But determining one stacking sequence for one set of lamination parameters and evaluating material strength only for this one stacking is a simplification, which needs to be emphasized.

4 NUMERICAL EXAMPLES AND RESULTS

The approach described in the previous section is applied to two examples, one which allows a fast, analytic evaluation of the objective function, and one with a more realistic, time consuming objective function. In all cases the ply angles are assumed to be uniformly distributed in an interval of $\pm 5°$.

4.1 Analytical objective function

As a first example, the stacking sequence $[45, 90, -45, 0]_{5s}$ is considered with a ply thickness of $0.125mm$ and the material properties given in table 1.

The laminate is subject to the section force $N_x = 1430N/mm$ in load case A and the section forces $N_x = 660N/mm, N_y = 1080N/mm, N_{xy} = 540N/mm, M_x = 300N$ in load case B. The stresses in each ply are determined by classical laminate theory and the Tsai-Wu criterion [12] is used to evaluate failure, indicated by a reserve factor (RF) below 1.

The two-step approach is used with FOSM and ISOA in the second analysis. For validation, two Monte Carlo simulation are carried out with 400 samples. In the "1-step

Table 1: Generic material properties used for the analytic example

stiffness properties		strength properties	
E_{11}	$140000 N/mm^2$	X_T	$2000 N/mm^2$
E_{22}	$12000 N/mm^2$	X_C	$1500 N/mm^2$
ν_{12}	0.26	Y_T	$70 N/mm^2$
G_{12}	$5800 N/mm^2$	Y_C	$230 N/mm^2$
		S_L	$90 N/mm^2$

Table 2: Results of the probabilistic analyses of the analytic

	load case A		load case B	
	mean RF	stdv of RF	mean MFL	stdv MFL
1-step Monte Carlo	1.085	0.020	1.137	0.025
2-step Monte Carlo	1.086	0.020	1.134	0.024
FOSM	1.104	0.013	1.137	0.018
ISOA	1.100	0.013	1.135	0.018

Monte Carlo" simulation the ply orientations are sampled, and in the "2-step Monte Carlo" the lamination parameters are sampled based on the distribution obtained in the first step of the two-step approach. The results of these analyses are summarized in table 2 and in figure 3. For the cumulative distribution plots of the FOSM and ISOA results, the RF is assumed to be normally distributed.

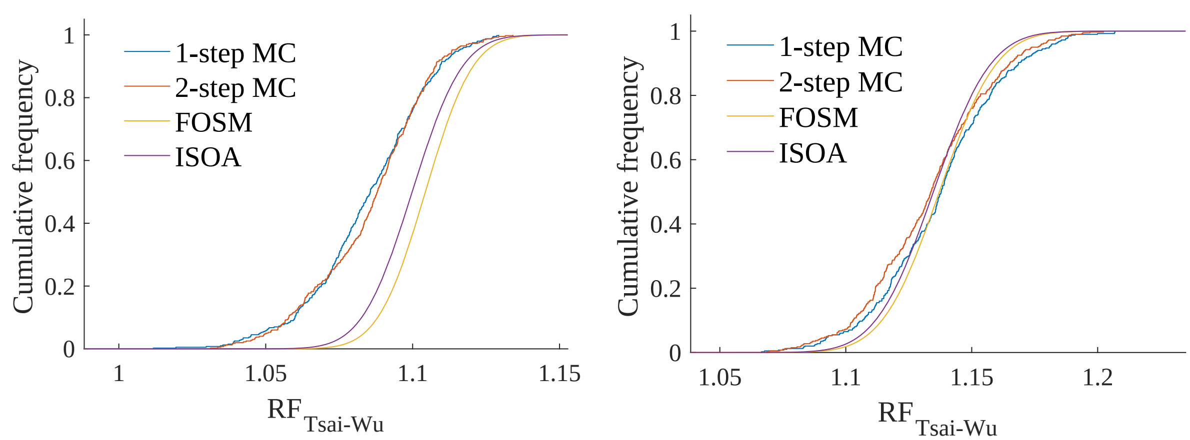

Figure 3: Cumulative distribution function of the reserve factor for the analytic example with load case A (left) and load case B (right)

For load case B, all results are in good agreement, which indicate that the proposed method is valid. For load case A, the results obtained from FOSM and ISOA differ from the ones given by the Monte Carlos simulations. This deviation does not originate from the two-step procedure or from the embedded optimization for evaluating material failure, because the 2-step Monte Carlo uses the same approach as FOSM and ISOA. Hence, the deviation must originate from the underlying assumptions of FOSM and ISOA, i.e. linear/quadratic objective function and Gaussian distribution of the objective function.

4.2 Numerical objective function

The example considered is the stiffened composite panel investigated by Nagendra et al. [13]. The dimensions, the laminate stackings and material properties are summarized in table 3 and table 4. The panel is subject to axial loading (in direction of the stringers). The loaded edges are clamped, and the longitudinal edges are simply supported.

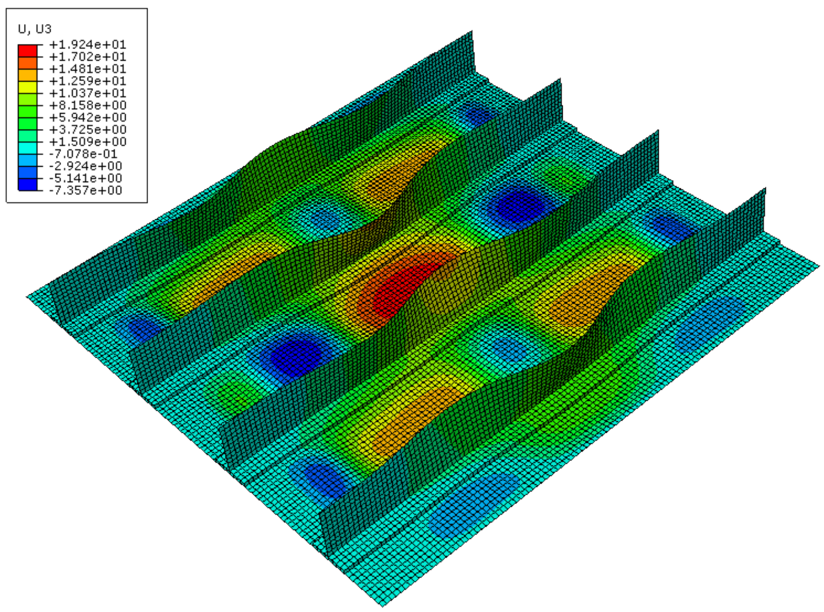

Figure 4: Postbuckling pattern of the considered stiffened composite panel

The finite element model shown in figure 4 consists of linear shell elements with reduced integration. The lowest buckling load, which for this panel corresponds to local skin buckling, is determined by a linear eigenvalue analysis. The onset of material failure in the post-buckling regime is determined in a displacement driven, nonlinear analysis with artificial stabilization, using the Hashin criterion [15].

The panel skin buckles locally (no stringer buckling) before the first material failure occurs. As the load increases, the panel buckles globally, accompanied by a drop in the load-displacement curve given in figure 5.

The two-step probabilistic analysis is carried out for both, the skin buckling load and the onset of material failure using FOSM and ISOA. All stringers are assumed to have the same random stacking sequence. Hence, the scattering ply orientations are described by 12 lamination parameters for the skin and 12 lamination parameters for the stringers, leading to 24 random parameters. For the central difference steps for FOSM and ISOA,

Table 3: Characteristics of the panel considered, from [13]

dimensions	
panel length	$762mm$
panel width	$812.8mm$
stringer spacing	$203.2mm$
stringer foot width	$60.96mm$
stringer flange height	$82.55mm$
stacking	
skin stacking	$[\pm 45, 90_4, (\pm 45)_5]_s$
stringer stacking	$[(\pm 45)_4, 0_2, (\pm 45)_2, (0_4, \pm 45)_3, 0_2]$
ply thickness	$0.132mm$

Table 4: Material properties used for the panel analyses

stiffness properties	from [13]	strength properties	from [14]
E_{11}	$127553.8N/mm^2$	X_T	$2326.2N/mm^2$
E_{22}	$11307.47N/mm^2$	X_C	$1200.1N/mm^2$
ν_{12}	0.3	Y_T	$62.3N/mm^2$
G_{12}	$5998.48N/mm^2$	Y_C	$199.8N/mm^2$
		S_L	$92.3N/mm^2$

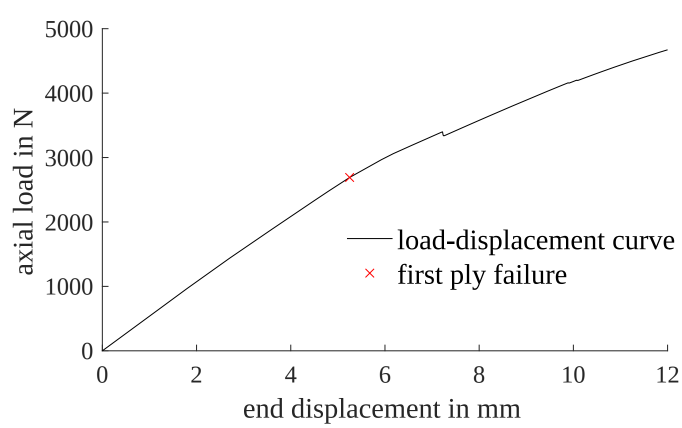

Figure 5: Axial load over end shortening of the perfect panel

Table 5: Results of the probabilistic analyses of the Nagendra panel in N

	mean BL	stdv of BL	mean MFL	stdv MFL
Monte Carlo	1461	8.7	2625	17.7
FOSM	1458	8.4	2622	9.2
ISOA	1459	8.5	2626	10.2

$24 \times 2 + 1 = 49$ nonlinear finite element (FE) simulations are performed. Each FE analysis took approximately 2 hours including the automated post-processing, which is very time consuming due to the large number of plies for which the failure criterion had to be evaluated. For validation a Monte Carlo simulation with 200 realizations is carried out following the one-step approach. The results are summarized in table 5 and figure 6. The distribution of buckling loads show good agreement where a certain deviation is found for the distributions of the material failure load. For both structural responses, the scatter is extremely small.

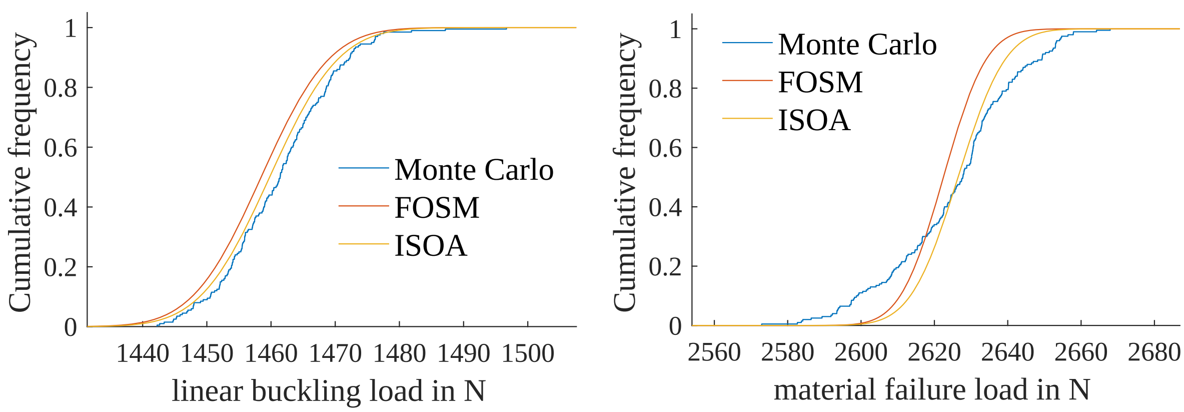

Figure 6: Cumulative distribution function of the linear buckling load (left) and material failure (right) of the stiffened panel for different approaches

5 CONCLUSIONS

A two-step procedure is presented that allows using lamination parameters for probabilistic analysis of buckling load and material failure load of composite structures with scattering ply orientations. In difference to directly considering ply orientations as random parameters, the two-step procedure require a couple of additional steps that need to be implemented. It however allows a huge efficiency gain if the following conditions are fulfilled:

1. the number of plies is considerably larger than 12,
2. the evaluation of the objective function (e.g. buckling load, material failure) is very expensive,

3. the computational cost of the probabilistic approach depends on the number of random parameters.

In the example with numerical objective function, the computational time is only driven by the number of finite element simulations. The two-step approach required 49 FE simulations, where the Monte Carlo simulation required 200 simulations, which is still a relatively small sample size for Monte Carlo.

In some cases, the two-step approach using FOSM or ISOA showed a certain deviation from the direct/one-step Monte Carlo simulation. From the analytic example it is seen that this deviation originates from the assumptions of FOSM and ISOA, but not from the two-step approach.

In all examples considered, the scatter of the response is very small even though the scatter of ply orientations was chosen realistically. In probabilistic analyses, which incorporate multiple random parameters, the influence of scattering ply orientations might be negligible. This is subject of future investigation, because to the authors knowledge all publishes probabilistic analyses of composite structures considered component with much fewer layers.

APPENDIX

Lamination parameters are defined as

$$\begin{aligned}
\xi^A_{[1,2,3,4]} &= \frac{1}{h}\int_{-h/2}^{h/2} [\cos(2\varphi),\ \cos(4\varphi),\ \sin(2\varphi),\ \sin(4\varphi)]dz \\
\xi^B_{[1,2,3,4]} &= \frac{4}{h^2}\int_{-h/2}^{h/2} [\cos(2\varphi),\ \cos(4\varphi),\ \sin(2\varphi),\ \sin(4\varphi)]z\ dz \\
\xi^D_{[1,2,3,4]} &= \frac{12}{h^3}\int_{-h/2}^{h/2} [\cos(2\varphi),\ \cos(4\varphi),\ \sin(2\varphi),\ \sin(4\varphi)]z^2\ dz
\end{aligned} \tag{4}$$

For a discrete layup, lamination parameters are determined from

$$\begin{aligned}
\xi^A_{[1,2,3,4]} &= \frac{1}{h}\sum_{i=1}^{n} [\cos(2\varphi_i),\ \cos(4\varphi_i),\ \sin(2\varphi_i),\ \sin(4\varphi_i)]t_{p,i} \\
\xi^B_{[1,2,3,4]} &= \frac{4}{h^2}\sum_{i=1}^{n} [\cos(2\varphi_i),\ \cos(4\varphi_i),\ \sin(2\varphi_i),\ \sin(4\varphi_i)]z_i\, t_{p,i} \\
\xi^D_{[1,2,3,4]} &= \frac{12}{h^3}\sum_{i=1}^{n} [\cos(2\varphi_i),\ \cos(4\varphi_i),\ \sin(2\varphi_i),\ \sin(4\varphi_i)](z_i^2 t_{p,i} + \frac{t_{p,i}^3}{12})
\end{aligned} \tag{5}$$

The entries the stiffness matrix (ABD matrix) are determined by

$$\begin{pmatrix} A_{11} \\ A_{22} \\ A_{12} \\ A_{66} \\ A_{16} \\ A_{26} \end{pmatrix} = h \begin{pmatrix} 1 & \xi^A_1 & \xi^A_2 & 0 & 0 \\ 1 & -\xi^A_1 & \xi^A_2 & 0 & 0 \\ 0 & 0 & -\xi^A_2 & 1 & 0 \\ 0 & 0 & -\xi^A_2 & 0 & 1 \\ 0 & \frac{1}{2}\xi^A_3 & \xi^A_4 & 0 & 0 \\ 0 & \frac{1}{2}\xi^A_3 & -\xi^A_4 & 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} B_{11} \\ B_{22} \\ B_{12} \\ B_{33} \\ B_{13} \\ B_{23} \end{pmatrix} = \frac{h^2}{4} \begin{pmatrix} 0 & \xi_1^B & \xi_2^B & 0 & 0 \\ 0 & -\xi_1^B & \xi_2^B & 0 & 0 \\ 0 & 0 & -\xi_2^B & 0 & 0 \\ 0 & 0 & -\xi_2^B & 0 & 0 \\ 0 & \frac{1}{2}\xi_3^B & \xi_4^B & 0 & 0 \\ 0 & \frac{1}{2}\xi_3^B & -\xi_4^B & 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} \tag{7}$$

and

$$\begin{pmatrix} D_{11} \\ D_{22} \\ D_{12} \\ D_{33} \\ D_{13} \\ D_{23} \end{pmatrix} = \frac{h^3}{12} \begin{pmatrix} 1 & \xi_1^D & \xi_2^D & 0 & 0 \\ 1 & -\xi_1^D & \xi_2^D & 0 & 0 \\ 0 & 0 & -\xi_2^D & 1 & 0 \\ 0 & 0 & -\xi_2^D & 0 & 1 \\ 0 & \frac{1}{2}\xi_3^D & \xi_4^D & 0 & 0 \\ 0 & \frac{1}{2}\xi_3^D & -\xi_4^D & 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} \tag{8}$$

Here, h is the total laminate thickness and the vector $\mathbf{U}$ is determined from the stiffness matrix of a unidirectional ply.

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{pmatrix} = \frac{1}{8} \begin{bmatrix} 3 & 3 & 2 & 4 \\ 4 & -4 & 0 & 0 \\ 1 & 1 & -2 & -4 \\ 1 & 1 & 6 & -4 \\ 1 & 1 & -2 & 4 \end{bmatrix} \begin{pmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \end{pmatrix} \tag{9}$$

with

$$\begin{array}{lll} Q_{11} = \frac{E_{11}}{1-\nu_{12}\nu_{21}} & Q_{22} = \frac{E_{22}}{1-\nu_{12}\nu_{21}} & \\ Q_{12} = \nu_{12} Q_{22} & Q_{66} = G_{12} & \nu_{21} = \nu_{12}\frac{E_{22}}{E_{11}} \end{array} \tag{10}$$

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