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STABILITY OF THERMOELASTIC LAYERED COMPOSITE IN AXIAL MOVEMENT

NIKOLAY BANICHUK[1], EVGENY MAKEEV[2], PEKKA NEITTAANMAKI[3] AND TERO TUOVINEN[4]

[1] Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russian Federation
Prospekt Vernadskogo, 101-1, Moscow 119526 Russia
banichuk@ipmnet.ru

[2] Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, Russian Federation
Prospekt Vernadskogo, 101-1, Moscow 119526 Russia
makeev@ipmnet.ru

[3] Faculty of Information Technology, University of Jyvaskyla, Finland
Mattilanniemi 2, 40014 Jyvaskyla
pekka.neittaanmaki@jyu.fi

[4] Faculty of Information Technology, University of Jyvaskyla, Finland
Mattilanniemi 2, 40014 Jyvaskyla
tero.tuovinen@jyu.fi

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Abstract. In this paper we study mechanical model for a layered thermoelastic composite and its stability in high-speed axially movement. We consider statical forms of the loss of stability and apply some approaches, based on averaging of the thermomechanical characteristics of layered composite materials and find some effective modulus of the considered composite structures.

1 INTRODUCTION

In our previous studies (see e.g., Banichuk et al. [1], [2], [3], [4], [5], [6]), we have considered many aspects of mathematical modelling of axially moving materials. Examples include the processing of paper or steel, fabric, rubber or some other continous material, and looping systems such as band saws and timing belts. In this article, we have will extend our studies focusing now on layered composites and their thermoelastic stabilities in movement.

Frequently used models for systems of axially moving material have been travelling flexible strings, membranes, beams, and plates. The dynamic and stability aspects discussed in this paper were first reviewed in the article by Mote [7]. Natural frequencies are commonly analyzed together with the stability. It was realized early on that the vibration problem for an axially moving continuum is not the a conventional one. Because of the longitudinal continuity of the material, the equation of motion for transverse vibration will contain additional terms, representing a Coriolis force and a centripetal force acting on the material. As a consequence, the resonant frequencies will be dependent on the longitudinal velocity of the axially moving continuum, as was noted by Archibald and Emslie [8], as well as Swope and Ames [9], Simpson [10], and Mujumber and Douglas [11].

Travelling beams have been further analyzed by e.g. Parker [12] in his study on gyroscopic continua, and by Kong and Parker [13], where an approximate analytical expression was derived for the eigenfrequencies of a moving beam with small flexural stiffness. Response predictions have been made for particular cases where the excitation assumes special forms, such as harmonic support motion, see Miranker [14], or a constant transverse point force as presented by Chonan [15]. Arbitrary excitations and initial conditions were analyzed with the help of a modal analysis and the Green's function method in the article by Wickert and Mote [16]. As a result, the critical speeds for travelling strings and beams were explicitly determined. Travelling strings and beams on an elastic foundation have been investigated by, e.g., Bhat et al. [17], Perkins [18], Wickert [19] and Parker [20].

The loss of stability was studied with an application of dynamic and static approaches in the article by Wickert [21]. It was shown by means of numerical analysis that in all cases instability occurs when the frequency is zero and the critical velocity coincides with the corresponding velocity obtained from static analysis. The dynamical properties of moving plates have been studied by Shen et al. [22] and by Shin et al. [23], and the properties of a moving paper web have been studied in the two-part article by Kulachenko et al. [24, 25]. Critical regimes and other problems of stability analysis have been studied e.g. by Wang [26] and by Sygulski [27].

The results indicating that axially moving beams experience divergence instability at a sufficiently high beam velocity have been obtained also for beams interacting with external media; see, e.g., study by Chang and Moretti [28]. In a study by Banichuk et al. [29], the authors extended these ideas to a two-dimensional model of the web, considered as a moving plate under homogeneous tension but without external media. The most straightforward and efficient way to study stability is to use a linear stability analysis.

In an article by Hatami et al. [30], the free vibration of a moving orthotropic rectangular plate was studied at sub- and supercritical speeds, and its flutter and divergence instabilities at supercritical speeds. The study is limited to simply supported boundary conditions at all edges. For the solution of equations of orthotropic moving material, many necessary fundamentals can be found in the work by Marynowski et. al (see e.g. [31] and [32]).

In the present study, we will limit our focus to moving layered composites. We consider the statical forms of the loss of stability and apply some approaches, based on averaging of the thermomechanical characteristics of layered composite materials and find some effective modulus of the considered composite structures. We will perform the studies mainly using analytical approaches. The article is structured in the following manner.

First, we will formulate equations for the instability of a homogeneous thermoelastic continuous panel in the Section 2. In the Section 3, we consider a layered thermoelastic composites. Finally, in the Section 4 we will draw conclusions.

The summarizing result of this article is that in the case of the considered multilayered composite, we have realized the technique of averaging of thermomechanical properties of different layers and found the effective composite characteristics. This has been done using basic material properties and taking into account that there is no sliding and discrepancies between layers. By using the found averaged effective modulus including combined effective characteristics, the required values were obtained and used in the final formulas for critical velocities and temperatures of a moving composite.

2 STABILITY OF THE HOMOGENEOUS THERMOELASTIC CONTINUOUS PANEL IN AXIAL MOVEMENT

The homogeneous panels are mechanically simple supported at the inflow ($x = -\ell$) and out-flow ($x = \ell$) boundaries of the panels. The panels are travelling at a constant velocity V_0 in the x- direction of the rectangular global coordinate system and are loaded by axial tension T_0 and thermal loads. The length 2ℓ and the total thickness H are supposed to be given, while $-\ell < x < \ell$ and $-H/2 < z < H/2$.

Free transverse vibrations of a homogeneous panel axially moving with constant velocity and loaded by axial tension and heated by some temperature are described by the following equation for transverse displacement w and simply supported boundary conditions

$$m\left(\frac{\partial^2 w}{\partial t^2} + 2V_0\frac{\partial^2 w}{\partial x \partial t} + V_0^2\frac{\partial^2 w}{\partial x^2}\right) = \left(T_0 - \frac{EH}{1-\nu}\varepsilon_\theta\right)\frac{\partial^2 w}{\partial x^2} - D\frac{\partial^4 w}{\partial x^4} \tag{1}$$

$$(w)_{x=-\ell} = 0, \qquad \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=-\ell} = 0, \qquad (w)_{x=\ell} = 0, \qquad \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=\ell} = 0 \tag{2}$$

where m, E, ν, D are, respectively, the mass per unit area, Young's modular, Poisson ratio, bending rigidity ($D = EH/12(1-\nu^2)$) and the deformation ε_θ is defined as

$$\varepsilon_\theta = \alpha_\theta \theta, \qquad \theta = \theta_a - \theta_0. \tag{3}$$

Here α_0 is a linear expansion coefficient, θ is the temperature discrepency, θ_0 is the temperature of zero deformation, θ_a is the actual temperature.

In a stationary case, when

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial t^2} = 0 \tag{4}$$

the transverse displacement $w = w(x)$ satisfies the equation

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \lambda\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0 \tag{5}$$

where parameter λ (eigenvalue) is given by the expression

$$\lambda = \frac{1}{D}\left(mV_0^2 + \frac{EH}{(1-\nu)}\alpha_\theta\theta - T_0\right) = f\left(V_0^2, \theta\right). \tag{6}$$

If we introduce a new unknown variable $\Psi(x)$ as

$$\Psi(x) = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \qquad -\ell \le x \le \ell \tag{7}$$

we obtain the following spectral problem

$$\frac{\mathrm{d}^2 \Psi}{\mathrm{d}x^2} + \lambda \Psi = 0 \tag{8}$$

$$\Psi(-\ell) = 0, \qquad \Psi(\ell) = 0. \tag{9}$$

Here the value λ plays the role of an eigenvalue. A nontrivial solution to the formulated eigenvalue problem can be represented as

$$\Psi(x) = C_1 \sin\left(\sqrt{\lambda}\left(\frac{x+\ell}{2}\right)\right) + C_2 \cos\left(\sqrt{\lambda}\left(\frac{x+\ell}{2}\right)\right) \tag{10}$$

with two arbitrary coefficients C_1 and C_2 and an unknown value λ. Taking into account (9) and (10) we will have $C_2 = 0$ and

$$\lambda = \left(\frac{j\pi}{\ell}\right)^2, \qquad j = 1, 2, \ldots \tag{11}$$

$$\Psi(x) = C_1 \sin\left(\frac{j\pi}{2\ell}(x+\ell)\right) \tag{12}$$

with arbitrary constant C_1.

Thus, for given problem parameters D, E, ν, H, ℓ, T_0, m, V_0, α_θ we obtain the critical temperature θ^{div} of instability (divergence of buckling)

$$\theta^{\mathrm{div}} = \frac{(1-\nu)}{EH\alpha_\theta}\left\{D\left(\frac{\pi}{\ell}\right)^2 + T_0 - mV_0^2\right\} \tag{13}$$

and

$$\lambda_{\min} = \left(\frac{\pi}{\ell}\right)^2 \tag{14}$$

corresponding the minimal $j = 1$ in the equation (11).

Analogously we find the critical instability velocity (squared) $(V_0^2)^{\mathrm{div}}$ as:

$$\left(V_0^2\right)^{\mathrm{div}} = \frac{1}{m}\left\{D\left(\frac{\pi}{\ell}\right)^2 + T_0 - \frac{EH\theta\alpha_\theta}{(1-\nu)}\right\} \tag{15}$$

where D, E, ν, H, ℓ, θ, α_θ, m are considered as a given positive parameters. The safety domain for stability in the values (θ, V_0^2) is defined by the inequality

$$f\left(V_0^2, \theta\right) < \lambda_{\min} = \left(\frac{\pi}{\ell}\right)^2 \tag{16}$$

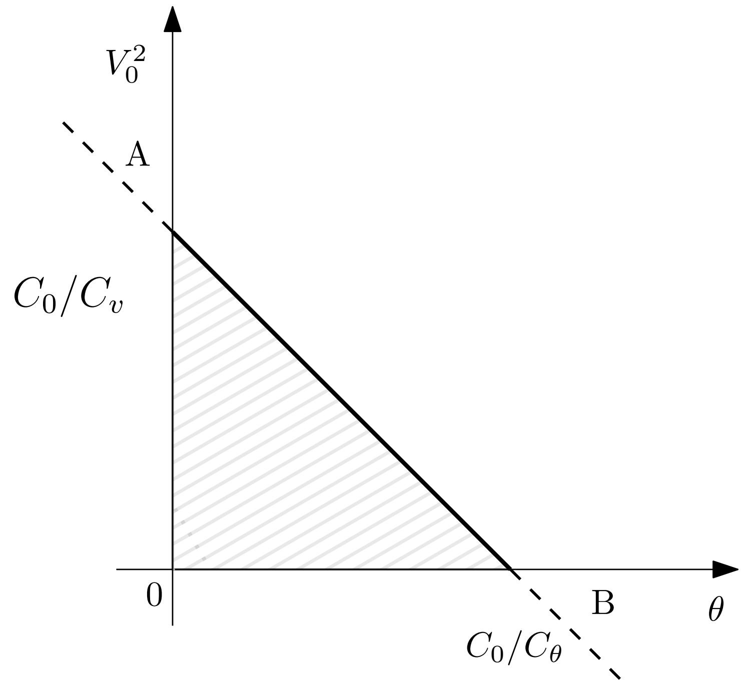

Figure 1: Safety domain OAB.

that is reduced to the condition

$$F\left(V_0^2, \theta\right) \equiv \frac{1}{D}\left(\frac{\ell}{\pi}\right)^2 f\left(V_0^2, \theta\right) = C_v V_0^2 + C_\theta \theta - C_0 < 0 \tag{17}$$

where

$$C_v = \frac{m}{D}\left(\frac{\ell}{\pi}\right)^2$$
$$C_\theta = \frac{EH\alpha_\theta}{D(1-\nu)}\left(\frac{\ell}{\pi}\right)^2 \tag{18}$$
$$C_0 = \frac{T_0}{D}\left(\frac{\ell}{\pi}\right)^2 + 1$$

The safety domain of the values V_0^2, θ has a triangular shape OAB shown in the Figure 1.

3 LAYERED THERMOELASTIC COMPOSITE

Consider the layered panel that is symmetrically composed with respect to a middle plane (see Figure 2) and consist of $2n+1$ (odd number) thermoelastic layers characterized by mass per unit area m_i, Young's modulus E_i, Poisson ratio ν_i, coefficient $(\alpha_\theta)_i$ and distances h_i from the symmetry of internal panel structure, i.e.

$$E(z) = E(-z), \qquad \nu(z) = \nu(-z), \qquad \alpha_\theta(z) = \alpha_\theta(-z) \tag{19}$$

and derive the expressions for effective moduli D^{ef}, ν^{ef}, $\varepsilon_\theta^{\mathrm{ef}}$ and m^{ef}. To this end we apply the formulas for stresses and strains and the expression for bending moment

$$\int_{-H/2}^{H/2} \sigma_x z \mathrm{d}z = \left(2 \int_0^{H/2} \frac{z^2 E(z) \mathrm{d}z}{1 - (\nu(z))^2} \right) \frac{\partial^2 w}{\partial x^2} = D^{\mathrm{ef}} \left(\frac{\partial^2 w}{\partial x^2} \right). \tag{20}$$

Thus we find the expression for effective bending rigidity in the form

$$D^{\mathrm{ef}} = 2 \int_0^{H/2} \frac{z^2 E(z)}{1 - (\nu(z))^2} \mathrm{d}z. \tag{21}$$

Using mechanical and geometric characteristics of the panel layers E_i, ν_i, h_i we evaluate the integral in the equation (21). We will have the following formula

$$D^{\mathrm{ef}} = \frac{2}{3} \frac{E_{n+1}}{1 - \nu_{n+1}^2} h_{n+1}^3 + \frac{2}{3} \sum_{i=1}^{n} \frac{E_i}{1 - \nu_i^2} \left(h_i^3 - h_{i+1}^3 \right). \tag{22}$$

In an analogous manner, we derive the formulas for an effective Poisson's ratio ν^{ef} and for effective thermal deformation $\varepsilon_\theta^{\mathrm{ef}}$ of a nonhomogeneous isotropic layered panel. We have

$$\nu^{\mathrm{ef}} = \frac{2}{D^{\mathrm{ef}}} \int_0^{H/2} \frac{z^2 \nu(z) E(z)}{1 - (\nu(z))^2} \mathrm{d}z =$$

$$= \frac{2}{3D^{\mathrm{ef}}} \left[\frac{\nu_{n+1} E_{n+1} h_{n+1}^3}{1 - \nu_{n+1}^2} + \sum_{i=1}^{n} \frac{E_i \nu_i}{1 - \nu_i^2} \left(h_i^3 - h_{i+1}^3 \right) \right] \tag{23}$$

$$\varepsilon_\theta^{\mathrm{ef}} = \frac{2}{H} \int_0^{H/2} \alpha_\theta(z) \theta \mathrm{d}z = \frac{2}{H} \left\{ (\alpha_\theta)_{n+1} \theta_{n+1} h_{n+1} + \sum_{i=1}^{n} (\alpha_\theta)_i \, \theta_i \left(h_i - h_{i+1} \right) \right\} \tag{24}$$

Besides that we have the following expression for m^{ef}:

$$m^{\mathrm{ef}} = m_{n+1} + 2 \sum_{i=1}^{n} m_i. \tag{25}$$

We derive also the corresponding formula for a joint expression

$$a(z) = \frac{HE(z)}{1 - \nu(z)} \varepsilon_\theta(z) = \frac{HE(z)}{1 - \nu(z)} \alpha_\theta(z) \theta(z) \tag{26}$$

in the following form

$$a^{\mathrm{ef}} = \frac{2}{H} \int_0^{H/2} a(z) \mathrm{d}z = 2 \left\{ \frac{(\alpha_\theta)_{n+1} E_{n+1} \theta_{n+1}}{1 - \nu_{n+1}} h_{n+1} + \sum_{i=1}^{n} \frac{(\alpha_\theta)_i E_i \theta_i}{1 - \nu_i} \left(h_i - h_{i+1} \right) \right\}. \tag{27}$$

The last formula contains many particular cases. Thus, if the Poisson's ratio and the temperature are the same for all materials, i.e.

$$\nu_1 = \nu_2 = \nu_3 = \cdots = \nu_{n+1} = \nu, \tag{28}$$

$$\theta_1 = \theta_2 = \theta_3 = \cdots = \theta_{n+1} = \theta \tag{29}$$

then we will have

$$a^{\text{ef}} = \frac{2\theta}{1-\nu} \left\{ (\alpha_\theta)_{n+1} E_{n+1} h_{n+1} + \sum_{i=1}^{n} (\alpha_\theta)_i E_i (h_i - h_{i+1}) \right\} \tag{30}$$

If besides (28), (29) the Young's modulus are equal for all layers, i.e.

$$E_1 = E_2 = E_3 = \cdots = E_{n+1} = E \tag{31}$$

then we obtain

$$a^{\text{ef}} = \frac{2E\theta}{1-\nu} \left\{ (\alpha_\theta)_{n+1} h_{n+1} + \sum_{i=1}^{n} (\alpha_\theta)_i (h_i - h_{i+1}) \right\}. \tag{32}$$

To use the obtained results (18)-(32) for a stability analysis we will take (13), (15) and suppose that

$$D = D^{\text{ef}}, \qquad \nu = \nu^{\text{ef}}, \qquad a = a^{\text{ef}}, \qquad m = m^{\text{ef}} \tag{33}$$

in the equations (13), (15). We will have

$$\left(V_0^2\right)^{\text{div}} = \frac{1}{m^{\text{ef}}} \left\{ D^{\text{ef}} \left(\frac{\pi}{\ell}\right)^2 + T_0 - a^{\text{ef}} \right\} \tag{34}$$

$$\theta^{\text{div}} = \frac{1}{b^{\text{ef}}} \left\{ D^{\text{ef}} \left(\frac{\pi}{\ell}\right)^2 + T_0 - a^{\text{ef}} \right\} \tag{35}$$

where

$$b^{\text{ef}} = \left(\frac{EH\alpha_\theta}{1-\nu}\right)^{\text{ef}} = 2\left\{ \frac{(\alpha_\theta)_{n+1} E_{n+1}}{1-\nu_{n+1}} + \sum_{i=1}^{n} \frac{(\alpha_\theta)_i E_i}{1-\nu_i} (h_i - h_{i+1}) \right\} \tag{36}$$

4 SOME NOTES AND CONCLUSIONS

In this paper the stability problems have been studied for homogeneous thermoelastic panels and for nonhomogeneous layered composites that perform axial movement. The critical velocities of instability and critical temperatures of buckling have been presented in an analytical form. The safety domains of principal parameters were obtained and presented in the paper. This result can be used in the engineering practice for tuning the parameters of mechanical systems to optimize the efficiency of production processes.

In the case of the considered multilayered composite we realized the technique of averaging of thermomechanical properties of different layers and found the effective composite

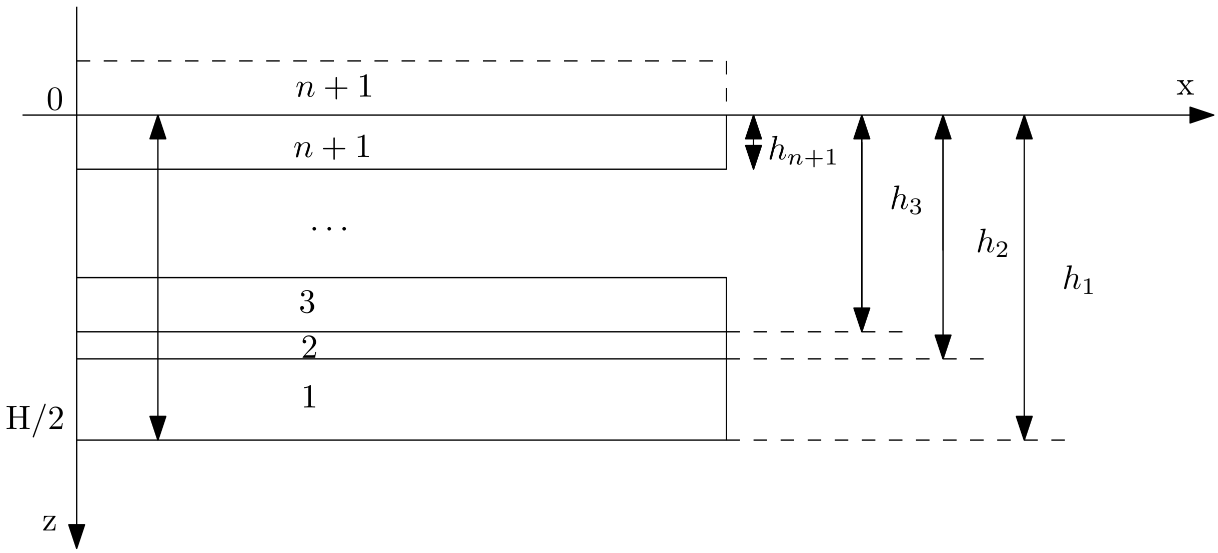

Figure 2: Cross-sectional of the panel modelled as a layered continuous composite.

characteristics. This has been done using basic material properties and taking into account that there is no sliding and discrepancies between layers. Using the found averaged effective modulus including combined effective characteristics the required values were obtained and used in the final formulas for critical velocities and temperatures of a moving composite.

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