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INHOMOGENEOUS LOCAL BOUNDARY CONDITIONS IN NONLOCAL PROBLEMS

BURAK AKSOYLU[1] AND GEORGE A. GAZONAS[2]

[1] U.S. Army Research Laboratory, Attn:RDRL-WMM-B,
Aberdeen Proving Ground, MD 21005, USA &
Wayne State University, Department of Mathematics, Detroit, MI 48202, USA
burak@wayne.edu and www.math.wayne.edu/∼burak

[2] U.S. Army Research Laboratory, Attn:RDRL-WMM-B,
Aberdeen Proving Ground, MD 21005, USA.
george.a.gazonas.civ@mail.mil

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Abstract. In prior work, we have presented novel governing operators with homogeneous boundary conditions (BC). Here, we extend the construction to inhomogeneous BC. The construction of the operators is inspired by peridynamics. They agree with the original peridynamics operator in the bulk of the domain and simultaneously enforce local Dirichlet and Neumann BC. We present exact solutions and utilize the resulting error to verify numerical experiments.

1 INTRODUCTION

We consider the following nonlocal (NL) wave equations with inhomogeneous local Dirichlet and local Neumann boundary condition (BC), respectively:

$$u^{\mathtt{D}}_{tt}(x,t) + \mathcal{M}_{\mathtt{D}} u^{\mathtt{D}}(x,t) = b^{\mathtt{D}}(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.1a}$$
$$u^{\mathtt{D}}(\pm 1,t) = \alpha^{\mathtt{D}}_{\pm}(t), \tag{1.1b}$$
$$u^{\mathtt{D}}(x,0) = \phi_{\mathtt{D}}(x), \tag{1.1c}$$
$$u^{\mathtt{D}}_{t}(x,0) = \psi_{\mathtt{D}}(x), \tag{1.1d}$$
$$u^{\mathtt{N}}_{tt}(x,t) + \mathcal{M}_{\mathtt{N}} u^{\mathtt{N}}(x,t) = b^{\mathtt{N}}(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.2a}$$
$$u^{\mathtt{N}}_{x}(\pm 1,t) = \alpha^{\mathtt{N}}_{\pm}(t), \tag{1.2b}$$
$$u^{\mathtt{N}}(x,0) = \phi_{\mathtt{N}}(x), \tag{1.2c}$$
$$u^{\mathtt{N}}_{t}(x,0) = \psi_{\mathtt{N}}(x), \tag{1.2d}$$

on the domain $\Omega := (-1,1)$ for some $T > 0$. The problems (1.1) and (1.2) fall into the class of initial boundary value problems. We have studied the above NL wave equations with homogeneous local BC in prior work [1, 2, 4]. The main purpose of this study is to extend the treatment to inhomogeneous BC. A more comprehensive study is in preparation.

In earlier work, we proved that the NL diffusion operator is a function of the classical operator [7]. This observation opened a gateway to incorporate local BC to NL problems on bounded domains. The main tool we used to define the novel governing operators was functional calculus, in which we replaced the classical governing operator by a suitable function of it. We provided the operator-theoretic treatment of (1.1) and (1.2) by resorting to an abstract version [2, 7]. More precisely, we utilized an operator-differential equation in $L^2(\Omega)$ in the form

$$\begin{aligned} \frac{\mathrm{d}^2 u^{\mathtt{BC}}}{\mathrm{d}t^2}(t) + \mathcal{M}_{\mathtt{BC}} u^{\mathtt{BC}}(t) &= b^{\mathtt{BC}}(t), \\ u^{\mathtt{BC}}(0) &= \phi_{\mathtt{BC}}, \\ \frac{\mathrm{d}u^{\mathtt{BC}}}{\mathrm{d}t}(0) &= \psi_{\mathtt{BC}}, \end{aligned} \tag{1.3}$$

where $\mathtt{BC} \in \{\mathtt{D}, \mathtt{N}\}$ and $\mathtt{D}$ and $\mathtt{N}$ denote the Dirichlet and Neumann BC. Here the $\mathbb{R}$-valued functions $u^{\mathtt{BC}} = u^{\mathtt{BC}}(x,t),\ b^{\mathtt{BC}} = b^{\mathtt{BC}}(x,t) : \Omega \times [0,T] \to \mathbb{R}$ are associated with their $L^2(\Omega)$-valued counterparts $u^{\mathtt{BC}} = u^{\mathtt{BC}}(t),\ b^{\mathtt{BC}} = b^{\mathtt{BC}}(t) : [0,T] \to L^2(\Omega)$ through $[u^{\mathtt{BC}}(t)](x) := u^{\mathtt{BC}}(x,t)$. If the forcing function

$$b^{\mathtt{D}} \in \mathcal{C}^0([0,T], L^2(\Omega)) \quad \text{and} \quad b^{\mathtt{N}} \in \mathcal{C}^1([0,T], L^2(\Omega))$$

we proved that the solution to (1.3) [2, Thm. 8] [7, Thm. 1]

$$u^{\mathtt{D}} \in \mathcal{C}^2([0,T], L^2(\Omega)) \quad \text{and} \quad u^{\mathtt{N}} \in \mathcal{C}^3([0,T], L^2(\Omega)). \tag{1.4}$$

The spatial and temporal behaviors are different. For instance by $b^{\mathtt{D}} \in \mathcal{C}^0([0,T], L^2(\Omega))$, we mainly mean a continuous function in time variable and a square integrable function in space variable. Namely, $b^{\mathtt{D}}(x,\cdot) \in \mathcal{C}^0([0,T])$ for $x \in \Omega$ and $b^{\mathtt{D}}(\cdot,t) \in L^2(\Omega)$ for $t \in [0,T]$.

As pointed out in (1.4), the solutions $u^{\mathtt{D}}$ and $u^{\mathtt{N}}$ to (1.3), possess $\mathcal{C}^2\big([0,T], L^2(\overline{\Omega})\big)$ and $\mathcal{C}^3\big([0,T], L^2(\overline{\Omega})\big)$ regularity, respectively. In the x-variable, we use the density of $\mathcal{C}^2(\overline{\Omega})$ in $L^2(\Omega)$ and the density of $\mathcal{C}^3(\overline{\Omega})$ in $L^2(\Omega)$ for the Dirichlet and Neumann problems, respectively. When we construct the computational framework and identify the relations between the forcing function, boundary and initial conditions, the space chosen is $\mathcal{C}^2\big([0,T], \mathcal{C}^2(\overline{\Omega})\big)$ and $\mathcal{C}^3\big([0,T], \mathcal{C}^3(\overline{\Omega})\big)$ for the Dirichlet and Neumann problems, respectively. On the other hand, we have the isomorphism

$$\mathcal{C}^s\big([0,T], \mathcal{C}^s(\overline{\Omega})\big) \cong \mathcal{C}^s(\overline{\Omega} \times [0,T]), \quad s = 2, 3.$$

Consequently, the main space in which low level construction takes place is $\mathcal{C}^s(\overline{\Omega} \times [0,T])$; see Sec. 3.

2 THE CONVOLUTION AND THE GOVERNING OPERATORS

In this section, we explain the key steps in construction of the governing operator $\mathcal{M}_{\mathtt{BC}}$. We observe that the peridynamics governing operator contains a convolution operator.

First, we construct the convolution operators $\mathcal{C}_{\mathtt{a}}$ and $\mathcal{C}_{\mathtt{p}}$ with antiperiodic and periodic BC, respectively, using the eigenfunctions

$$e_k^{\mathtt{a}}(x) := \frac{1}{\sqrt{2}} e^{i\pi(k+\frac{1}{2})x}, \quad k \in \mathbb{N}, \quad \text{and} \quad e_k^{\mathtt{p}}(x) := \frac{1}{\sqrt{2}} e^{i\pi k x}, \quad k \in \mathbb{N},$$

of the classical operator $A_{\mathtt{a}}$ and $A_{\mathtt{p}}$ in which the BC information is already encoded. For a given kernel function $C \in L^2(\Omega)$, the convolution operator, for $u \in L^2(\Omega)$, is defined as

$$\mathcal{C}_{\mathtt{BC}} u(x) := \sqrt{2} \sum_{k\in\mathbb{N}} \langle e_k^{\mathtt{BC}} | C \rangle \, \langle e_k^{\mathtt{BC}} | u \rangle \, e_k^{\mathtt{BC}}(x), \quad \mathtt{BC} \in \{\mathtt{a}, \mathtt{p}\},$$

where $\langle \cdot | \cdot \rangle$ denotes the $L^2(\Omega)$ inner product. The operators $\mathcal{C}_{\mathtt{BC}}$ turn out to be bounded functions of the classical operator $A_{\mathtt{BC}}$, thereby maintaining the connection to $A_{\mathtt{BC}}$.

In this study, we consider only the operators $\mathcal{M}_{\mathtt{D}}$ and $\mathcal{M}_{\mathtt{N}}$. Hence, in the rest of the discussion, we set $\mathtt{BC} \in \{\mathtt{D}, \mathtt{N}\}$. The operator $\mathcal{M}_{\mathtt{BC}}$ is constructed using functional calculus on the classical self-adjoint operator $A_{\mathtt{BC}}$. We are in search of a suitable regulating function $f_{\mathtt{BC}} : \sigma(A_{\mathtt{BC}}) \to \mathbb{R}$ that would connect the NL operator $\mathcal{M}_{\mathtt{BC}}$ to $A_{\mathtt{BC}}$, i.e., $\mathcal{M}_{\mathtt{BC}} = f_{\mathtt{BC}}(A_{\mathtt{BC}})$. We want this regulating function to be bounded so that the end product $\mathcal{M}_{\mathtt{BC}}$ is a bounded operator. Eventually, we end up with the NL governing operator $\mathcal{M}_{\mathtt{BC}}$ that is densely defined in $L^2(\Omega)$ with a domain that encodes the prescribed BC, bounded, and self-adjoint. We can therefore conclude that the operator $\mathcal{M}_{\mathtt{BC}}$ has a unique bounded extension to $L^2(\Omega)$. Consequently, we find that a construction involving densely defined operators provides a suitable framework for treating local BC in the NL wave equation.

We want to elaborate on the choice of $f_{\mathtt{BC}}$. Since we want keep a close proximity to peridynamics, we want $f_{\mathtt{BC}}$ to be inspired by the theory of peridynamics. In prior work, we discovered that the peridynamics governing operator for the case $\Omega = \mathbb{R}$ is a function of the classical operator [7]. We reuse that regulating function for the case of $\Omega = (-1, 1)$. We define $\mathbb{N}_{\mathtt{D}} := \mathbb{N} \setminus \{0\}$ and $\mathbb{N}_{\mathtt{N}} := \mathbb{N}$. Our choice of regulating functions are

$$f_{\mathtt{BC}} : \sigma(A_{\mathtt{BC}}) \to \mathbb{R}, \quad f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}}) = \langle 1 | C \rangle - \sqrt{2} \begin{cases} \langle e_{k/2}^{\mathtt{p}} | C \rangle & \text{if } k \in \mathbb{N}_{\mathtt{BC}} \text{ is even}, \\ \langle e_{(k-1)/2}^{\mathtt{a}} | C \rangle & \text{if } k \in \mathbb{N}_{\mathtt{BC}} \text{ is odd}. \end{cases}$$

Utilizing the convolution operators $\mathcal{C}_{\mathtt{a}}$ and $\mathcal{C}_{\mathtt{p}}$ obtained by functional calculus on $A_{\mathtt{a}}$ and $A_{\mathtt{p}}$, respectively, defining $c := \langle 1 | C \rangle$, we proved in [1, 3] that

$$\begin{aligned} f_{\mathtt{D}}(A_{\mathtt{D}}) u^{\mathtt{D}} &= \big(c - \mathcal{C}_{\mathtt{a}} P_e - \mathcal{C}_{\mathtt{p}} P_o\big) u^{\mathtt{D}} = \mathcal{M}_{\mathtt{D}} u^{\mathtt{D}}, \\ f_{\mathtt{N}}(A_{\mathtt{N}}) u^{\mathtt{N}} &= \big(c - \mathcal{C}_{\mathtt{p}} P_e - \mathcal{C}_{\mathtt{a}} P_o\big) u^{\mathtt{N}} = \mathcal{M}_{\mathtt{N}} u^{\mathtt{N}}, \end{aligned}$$

where we denote the orthogonal projections that give the even and odd parts, respectively, by $P_e, P_o : \ L^2(\Omega) \ \to \ L^2(\Omega)$, whose definitions are

$$P_e u(x) := \frac{u(x) + u(-x)}{2}, \quad P_o u(x) := \frac{u(x) - u(-x)}{2}.$$

The crucial step in the construction of $\mathcal{M}_{\mathtt{BC}}$ is the application of the spectral theorem for bounded operators. Namely, for $u^{\mathtt{BC}} = \sum_k \langle e_k^{\mathtt{BC}} | u^{\mathtt{BC}} \rangle\, e_k^{\mathtt{BC}}$, we have

$$\mathcal{M}_{\mathtt{BC}} u^{\mathtt{BC}} = f_{\mathtt{BC}}(A_{\mathtt{BC}}) u^{\mathtt{BC}} = \sum_{k \in \mathbb{N}_{\mathtt{BC}}} f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})\, \langle e_k^{\mathtt{BC}} | u^{\mathtt{BC}} \rangle\, e_k^{\mathtt{BC}}. \tag{2.1}$$

For an extended discussion on the treatment of general NL problems using functional calculus, see [5].

Integral representation of the series (2.1) is more convenient for implementation. We presented such representations in [1] and the governing operators take the form

$$\begin{aligned}
\big(\mathcal{M}_{\mathtt{BC}} - c\big) u^{\mathtt{BC}}(x,t) &= -\int_\Omega K_{\mathtt{BC}}(x,x') u^{\mathtt{BC}}(x',t)\,\mathrm{d}x', \\
K_{\mathtt{D}}(x,x') &:= \frac{1}{2}\big\{\big[\widehat{C}_{\mathtt{a}}(x'-x) + \widehat{C}_{\mathtt{a}}(x'+x)\big] + \big[\widehat{C}_{\mathtt{p}}(x'-x) - \widehat{C}_{\mathtt{p}}(x'+x)\big]\big\}, \\
K_{\mathtt{N}}(x,x') &:= \frac{1}{2}\big\{\big[\widehat{C}_{\mathtt{p}}(x'-x) + \widehat{C}_{\mathtt{p}}(x'+x)\big] + \big[\widehat{C}_{\mathtt{a}}(x'-x) - \widehat{C}_{\mathtt{a}}(x'+x)\big]\big\},
\end{aligned}$$

where we denote the periodic and antiperiodic extensions of $C(x)$ from $(-1,1)$ to $(-2,2)$, respectively, as follows

$$\widehat{C}_{\mathtt{p}}(x) := \begin{cases} C(x+2), & x \in (-2,-1), \\ C(x), & x \in (-1,1), \\ C(x-2), & x \in (1,2), \end{cases} \qquad \widehat{C}_{\mathtt{a}}(x) := \begin{cases} -C(x+2), & x \in (-2,-1), \\ C(x), & x \in (-1,1), \\ -C(x-2), & x \in (1,2). \end{cases}$$

3 FORCING FUNCTION, BC, AND INITIAL VALUE RELATIONSHIPS

In order to find the suitable forcing function that enforces the prescribed BC, we need to identify the governing ordinary differential equation (ODE) on the boundary. We assume that $u^{\mathtt{D}} \in \mathcal{C}^2(\overline{\Omega} \times [0,T])$, $u^{\mathtt{N}} \in \mathcal{C}^3(\overline{\Omega} \times [0,T])$, and $b^{\mathtt{D}} \in \mathcal{C}^0(\overline{\Omega} \times [0,T])$, $b^{\mathtt{N}} \in \mathcal{C}^1(\overline{\Omega} \times [0,T])$. On the boundary, we denote the displacement, the stress and the forcing functions by

$$\begin{aligned}
u_{\pm}^{\mathtt{D}}(t) &:= \lim_{x \to \pm 1} u^{\mathtt{D}}(x,t) \quad &\text{and} \quad b_{\pm}^{\mathtt{D}}(t) &:= \lim_{x \to \pm 1} b^{\mathtt{D}}(x,t) \\
u_{x,\pm}^{\mathtt{N}}(t) &:= \lim_{x \to \pm 1} \frac{\partial u^{\mathtt{N}}}{\partial x}(x,t) \quad &\text{and} \quad b_{x,\pm}^{\mathtt{N}}(t) &:= \lim_{x \to \pm 1} \frac{\partial b^{\mathtt{N}}}{\partial x}(x,t).
\end{aligned}$$

In order to investigate the behavior of the solution on the boundary, first we study the action of the governing operator $\mathcal{M}_{\mathtt{BC}}$ on the boundary. By the Lebesgue Dominated Convergence Theorem and the design of the kernel functions $K_{\mathtt{BC}}(x,x')$, we have

$$\begin{aligned}
\lim_{x \to \pm 1} \big(\mathcal{M}_{\mathtt{D}} - c\big) u^{\mathtt{D}}(x,t) &= -\lim_{x \to \pm 1} \int_\Omega K_{\mathtt{D}}(x,x') u^{\mathtt{D}}(x',t)\,\mathrm{d}x' \\
&= -\int_\Omega \lim_{x \to \pm 1} K_{\mathtt{D}}(x,x') u^{\mathtt{D}}(x',t)\,\mathrm{d}x' \qquad = 0, \\
\lim_{x \to \pm 1} \frac{\partial}{\partial x} \big(\mathcal{M}_{\mathtt{N}} - c\big) u^{\mathtt{N}}(x,t) &= -\lim_{x \to \pm 1} \frac{\partial}{\partial x} \int_\Omega K_{\mathtt{N}}(x,x') u^{\mathtt{N}}(x',t)\,\mathrm{d}x' \\
&= -\int_\Omega \lim_{x \to \pm 1} \frac{\partial K_{\mathtt{N}}}{\partial x}(x,x') u^{\mathtt{N}}(x',t)\,\mathrm{d}x' \quad = 0.
\end{aligned}$$

We see that the governing equations (1.1a) and (1.2a) under the action of $\lim_{x\to\pm 1}$ and $\lim_{x\to\pm 1}\frac{\partial}{\partial x}$, respectively, reduce to the following ODE:

$$\frac{\mathrm{d}^2 u^{\mathtt{D}}_{\pm}}{\mathrm{d}t^2}(t) + c u^{\mathtt{D}}_{\pm}(t) = b^{\mathtt{D}}_{\pm}(t), \qquad t \in (0,T), \tag{3.1}$$

$$\frac{\mathrm{d}^2 u^{\mathtt{N}}_{x,\pm}}{\mathrm{d}t^2}(t) + c u^{\mathtt{N}}_{x,\pm}(t) = b^{\mathtt{N}}_{x,\pm}(t), \quad t \in (0,T). \tag{3.2}$$

In order to obtain a unique solution to (3.1) and (3.2), we need to prescribe the two initial values $u^{\mathtt{D}}_{\pm}(0)$ and $\frac{\mathrm{d}u^{\mathtt{D}}_{\pm}}{\mathrm{d}t}(0)$ and $u^{\mathtt{N}}_{x,\pm}(0)$ and $\frac{\mathrm{d}u^{\mathtt{N}}_{x,\pm}}{\mathrm{d}t}(0)$, respectively.

By taking $\lim_{x\to\pm 1}$ in (1.1c) and (1.1d) and $\lim_{x\to\pm 1}\frac{\partial}{\partial x}$ in (1.2c) and (1.2d), we immediately identify the initial displacement and velocity for the Dirichlet problem and initial stress and stress rate for the Neumann problem as

$$u^{\mathtt{D}}_{\pm}(0) = \phi_{\mathtt{D}}(\pm 1) \quad \text{and} \quad \frac{\mathrm{d}u^{\mathtt{D}}_{\pm}}{\mathrm{d}t}(0) \;\; = \psi_{\mathtt{D}}(\pm 1), \tag{3.3}$$

$$u^{\mathtt{N}}_{x,\pm}(0) = \phi'_{\mathtt{N}}(\pm 1) \quad \text{and} \quad \frac{\mathrm{d}u^{\mathtt{N}}_{x,\pm}}{\mathrm{d}t}(0) = \psi'_{\mathtt{N}}(\pm 1). \tag{3.4}$$

Putting together (3.1) and (3.3), we arrive at the initial value problem (IVP) on the boundary for the Dirichlet problem:

$$\begin{aligned} &\frac{\mathrm{d}^2 u^{\mathtt{D}}_{\pm}}{\mathrm{d}t^2}(t) + c u^{\mathtt{D}}_{\pm}(t) = b^{\mathtt{D}}_{\pm}(t), \quad t \in (0,T), \\ &u^{\mathtt{D}}_{\pm}(0) = \phi_{\mathtt{D}}(\pm 1) \quad \text{and} \quad \frac{\mathrm{d}u^{\mathtt{D}}_{\pm}}{\mathrm{d}t}(0) = \psi(\pm 1). \end{aligned} \tag{3.5}$$

Similarly, putting (3.2) and (3.4) together, we arrive at the IVP on the boundary for the Neumann problem:

$$\begin{aligned} &\frac{\mathrm{d}^2 u^{\mathtt{N}}_{x,\pm}}{\mathrm{d}t^2}(t) + c u^{\mathtt{N}}_{x,\pm}(t) = b^{\mathtt{N}}_{x,\pm}(t), \quad t \in (0,T) \\ &u^{\mathtt{N}}_{x,\pm}(0) = \phi'_{\mathtt{N}}(\pm 1) \quad \text{and} \quad \frac{\mathrm{d}u^{\mathtt{N}}_{x,\pm}}{\mathrm{d}t}(0) = \psi'_{\mathtt{N}}(\pm 1). \end{aligned} \tag{3.6}$$

On the other hand, the BC (1.1b) and (1.2b) demand a solution from (3.5) and (3.6) that are equal to $\alpha^{\mathtt{D}}_{\pm}(t)$ and $\alpha^{\mathtt{N}}_{\pm}(t)$, respectively. Hence, we identify the initial displacement and velocity, for the Dirichlet problem and initial stress and initial stress rate, for the Neumann problem, as well as the corresponding forcing functions. When the following choices are made,

$$\text{Dirichlet:} \quad b^{\mathtt{D}}_{\pm}(t) \;\; = \frac{\mathrm{d}^2 \alpha^{\mathtt{D}}_{\pm}}{\mathrm{d}t^2}(t) + c\alpha^{\mathtt{D}}_{\pm}(t), \quad \phi_{\mathtt{D}}(\pm 1) = \alpha^{\mathtt{D}}_{\pm}(0), \quad \psi_{\mathtt{D}}(\pm 1) = \frac{\mathrm{d}\alpha^{\mathtt{D}}_{\pm}}{\mathrm{d}t}(0), \tag{3.7}$$

$$\text{Neumann:} \quad b^{\mathtt{N}}_{x,\pm}(t) = \frac{\mathrm{d}^2 \alpha^{\mathtt{N}}_{\pm}}{\mathrm{d}t^2}(t) + c\alpha^{\mathtt{N}}_{\pm}(t), \quad \phi'_{\mathtt{N}}(\pm 1) = \alpha^{\mathtt{N}}_{\pm}(0), \quad \psi'_{\mathtt{N}}(\pm 1) = \frac{\mathrm{d}\alpha^{\mathtt{N}}_{\pm}}{\mathrm{d}t}(0), \tag{3.8}$$

the IVP (3.5) for the Dirichlet problem takes the form

$$\frac{\mathrm{d}^2 u^{\mathtt{D}}_{\pm}}{\mathrm{d}t^2}(t) + c u^{\mathtt{D}}_{\pm}(t) = \frac{\mathrm{d}^2 \alpha^{\mathtt{D}}_{\pm}}{\mathrm{d}t^2}(t) + c\alpha^{\mathtt{D}}_{\pm}(t), \quad t \in (0,T)$$
$$u^{\mathtt{D}}_{\pm}(0) = \alpha^{\mathtt{D}}_{\pm}(0) \quad \text{and} \quad \frac{\mathrm{d}u^{\mathtt{D}}_{\pm}}{\mathrm{d}t}(0) = \frac{\mathrm{d}\alpha^{\mathtt{D}}_{\pm}}{\mathrm{d}t}(0).$$

Similarly, the IVP (3.6) for the Neumann problem takes the form

$$\frac{\mathrm{d}^2 u^{\mathtt{N}}_{x,\pm}}{\mathrm{d}t^2}(t) + c u^{\mathtt{N}}_{x,\pm}(t) = \frac{\mathrm{d}^2 \alpha^{\mathtt{N}}_{\pm}}{\mathrm{d}t^2}(t) + c\alpha^{\mathtt{N}}_{\pm}(t), \quad t \in (0,T)$$
$$u^{\mathtt{N}}_{x,\pm}(0) = \alpha^{\mathtt{N}}_{\pm}(0) \quad \text{and} \quad \frac{\mathrm{d}u^{\mathtt{N}}_{x,\pm}}{\mathrm{d}t}(0) = \frac{\mathrm{d}\alpha^{\mathtt{N}}_{\pm}}{\mathrm{d}t}(0).$$

Consequently, we guarantee that the solutions to (3.5) and (3.6) are exactly $\alpha^{\mathtt{D}}_{\pm}(t)$ and $\alpha^{\mathtt{N}}_{\pm}(t)$, respectively. As seen above, the way we enforce inhomogeneous local BC is by the use of a forcing function on the boundary only (not in the interior of Ω). This is a major difference between enforcing local and nonlocal BC.

Remark 3.1 *Since $u^{\mathtt{D}} \in \mathcal{C}^2(\overline{\Omega} \times [0,T])$, the choices $(3.7)_2$ and $(3.7)_3$ correspond to the continuity of $u^{\mathtt{D}}$ and $u^{\mathtt{D}}_t$, respectively, at the corner points $(\pm 1, 0)$. More precisely, they are implications for the following interchange of limits.*

$$\phi_{\mathtt{D}}(\pm 1) = \lim_{x\to\pm 1}\lim_{t\to 0} u^{\mathtt{D}}(x,t) = \lim_{t\to 0}\lim_{x\to\pm 1} u^{\mathtt{D}}(x,t) = \alpha^{\mathtt{D}}_{\pm}(0)$$
$$\psi_{\mathtt{D}}(\pm 1) = \lim_{x\to\pm 1}\lim_{t\to 0} u^{\mathtt{D}}_t(x,t) = \lim_{t\to 0}\lim_{x\to\pm 1} u^{\mathtt{D}}_t(x,t) = \frac{\mathrm{d}\alpha^{\mathtt{D}}_{\pm}}{\mathrm{d}t}(0).$$

Similarly, since $u^{\mathtt{N}} \in \mathcal{C}^3(\overline{\Omega} \times [0,T])$, the choices $(3.8)_2$ and $(3.8)_3$ correspond to the continuity of $u^{\mathtt{N}}_x$ and $u^{\mathtt{D}}_{xt}$, respectively, at the corner points $(\pm 1, 0)$.

$$\phi'_{\mathtt{N}}(\pm 1) = \lim_{x\to\pm 1}\lim_{t\to 0} \frac{\partial u^{\mathtt{N}}}{\partial x}(x,t) = \lim_{t\to 0}\lim_{x\to\pm 1} \frac{\partial u^{\mathtt{N}}}{\partial x}(x,t) = \alpha^{\mathtt{N}}_{\pm}(0)$$
$$\psi'_{\mathtt{N}}(\pm 1) = \lim_{x\to\pm 1}\lim_{t\to 0} \frac{\partial u^{\mathtt{N}}_t}{\partial x}(x,t) = \lim_{t\to 0}\lim_{x\to\pm 1} \frac{\partial u^{\mathtt{N}}_t}{\partial x}(x,t) = \frac{\mathrm{d}\alpha^{\mathtt{N}}_{\pm}}{\mathrm{d}t}(0).$$

4 EXACT SOLUTIONS WITH HOMOGENEOUS BC

Thanks to functional calculus, it is possible to find exact solutions to (1.1) and (1.2). The expressions for the solution to (1.1) and (1.2) are given as [2, 7]

$$u^{\mathtt{BC}}(x,t) = \cos\left(t\sqrt{\mathcal{M}_{\mathtt{BC}}}\right)\phi_{\mathtt{BC}}(x) + \frac{\sin\left(t\sqrt{\mathcal{M}_{\mathtt{BC}}}\right)}{\sqrt{\mathcal{M}_{\mathtt{BC}}}}\psi_{\mathtt{BC}}(x) + \int_0^t \frac{\sin\left((t-\tau)\sqrt{\mathcal{M}_{\mathtt{BC}}}\right)}{\sqrt{\mathcal{M}_{\mathtt{BC}}}} b^{\mathtt{BC}}(x,\tau)\,\mathrm{d}\tau. \tag{4.1}$$

Using the Hilbert basis and the spectral theorem for bounded operators, we can turn expression (4.1) into the following series representation.

$$u^{\mathtt{BC}}(x,t) = \sum_{k\in\mathbb{N}_{\mathtt{BC}}} \cos\left(t\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}\right) \langle e_k^{\mathtt{BC}}|\phi_{\mathtt{BC}}\rangle\, e_k^{\mathtt{BC}}(x) + \sum_{k\in\mathbb{N}_{\mathtt{BC}}} \frac{\sin\left(t\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}\right)}{\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}} \langle e_k^{\mathtt{BC}}|\psi_{\mathtt{BC}}\rangle\, e_k^{\mathtt{BC}}(x) +$$
$$\sum_{k\in\mathbb{N}_{\mathtt{BC}}} \left[\int_0^t \frac{\sin\left((t-\tau)\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}\right)}{\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}} \langle e_k^{\mathtt{BC}}|b^{\mathtt{BC}}(\tau)\rangle\, \mathrm{d}\tau\right] e_k^{\mathtt{BC}}(x).$$

We collapse the series by using the orthonormality of $e_k^{\mathtt{BC}}$. For instance, the choice of

$$b^{\mathtt{BC}}(x,t) \equiv 0, \quad \phi_{\mathtt{BC}}(x) = e_m^{\mathtt{BC}}(x), \quad \psi_{\mathtt{BC}}(x) \equiv 0, \tag{4.2}$$

for some $m \in \mathbb{N}\setminus\{0\}$, leads to

$$u^{\mathtt{BC}}(x,t) = \cos\left(t\sqrt{f_{\mathtt{BC}}(\lambda_m^{\mathtt{BC}})}\right) e_m^{\mathtt{BC}}(x).$$

4.1 Classical Exact Solutions with Homogeneous BC

We also study the local analogs of the problems (1.1) and (1.2). We consider the classical wave equation with homogeneous Dirichlet and Neumann BC with the same choice given in (4.2)

$$\begin{aligned} v_{tt}^{\mathtt{BC}}(x,t) - \frac{4}{\pi^2} v_{xx}^{\mathtt{BC}}(x,t) = 0, \quad (x,t) \in \Omega\times(0,T),\\ v^{\mathtt{D}}(\pm 1,t) = 0 \quad \text{or} \quad v_x^{\mathtt{N}}(\pm 1,t) = 0,\\ v^{\mathtt{BC}}(x,0) = e_m^{\mathtt{BC}}(x),\\ v_t^{\mathtt{BC}}(x,0) = 0, \end{aligned} \tag{4.3}$$

for some $m \in \mathbb{N}\setminus\{0\}$. It is possible to obtain a closed form solution using d'Alembert's formula together with the method of images or reflections. After some algebra, we obtain

$$v^{\mathtt{BC}}(x,t) = \cos\left(t\sqrt{m^2}\right) e_m^{\mathtt{BC}}(x).$$

Since the classical governing equations (4.3) contain the classical operators $A_{\mathtt{BC}}$, the regulating function is nothing but the identity function. Using the expression of the spectrum $\sigma(A_{\mathtt{BC}}) = \{k^2 : k \in \mathbb{N}_{\mathtt{BC}}\}$, we have

$$f_{\mathtt{BC}}^{\text{classi}}(\lambda_k^{\mathtt{BC}}) = \lambda_k^{\mathtt{BC}} = k^2, \quad k \in \mathbb{N}_{\mathtt{BC}}.$$

Even though $f_{\mathtt{BC}}^{\text{classi}} : \sigma(A_{\mathtt{BC}}) \to \mathbb{R}$ is not a bounded function, the solution expression obtained from the formula (4.1) still captures the expression obtained from d'Alembert's formula. This is due to the instance of the spectral theorem for Sturm-Liouville operators.

5 EXACT SOLUTIONS WITH INHOMOGENEOUS BC

We treat inhomogeneous BC by the method of shifting the data. We define a shift function $G^{\mathtt{BC}}(x,t)$ that satisfies the BC:

$$G^{\mathtt{D}}(\pm 1,t) = \alpha^{\mathtt{D}}_{\pm}(t) \quad \text{and} \quad G^{\mathtt{N}}_x(\pm 1,t) = \alpha^{\mathtt{N}}_{\pm}(t) \tag{5.1}$$

$G^{\mathtt{BC}}(x,t)$ can be any function that satisfies (5.1). A practical choice is

$$G^{\mathtt{D}}(x,t) = \frac{1-x}{2}\alpha^{\mathtt{D}}_{-}(t) + \frac{1+x}{2}\alpha^{\mathtt{D}}_{+}(t) \tag{5.2}$$

$$G^{\mathtt{N}}(x,t) = \frac{(1-x)^2}{4}\alpha^{\mathtt{N}}_{-}(t) + \frac{(1+x)^2}{4}\alpha^{\mathtt{N}}_{+}(t). \tag{5.3}$$

We assume that the boundary data have the following regularity

$$\alpha^{\mathtt{D}}_{\pm} \in \mathcal{C}^2([0,T]) \quad \text{and} \quad \alpha^{\mathtt{N}}_{\pm} \in \mathcal{C}^3([0,T]). \tag{5.4}$$

As a result of (5.4), the shift function should have the following regularity.

$$G^{\mathtt{D}} \in \mathcal{C}^2([0,T], L^2(\Omega)) \quad \text{and} \quad G^{\mathtt{N}} \in \mathcal{C}^3([0,T], L^2(\Omega)).$$

Eventually, we end up with an equivalent IVP with homogeneous BC by defining

$$w^{\mathtt{BC}}(x,t) := u^{\mathtt{BC}}(x,t) - G^{\mathtt{BC}}(x,t). \tag{5.5}$$

Combining (1.1b) and (1.2b) with (5.1), we obtain the homogeneous BC, i.e., $w^{\mathtt{D}}(\pm 1,t) = 0$ and $w^{\mathtt{N}}_x(\pm 1,t) = 0$. Substituting the expression for $u^{\mathtt{BC}}(x,t)$ from (5.5) into (1.1) and (1.2), we arrive at the equivalent problem with homogeneous BC:

$$\begin{aligned}
w^{\mathtt{BC}}_{tt}(x,t) + \mathcal{M}_{\mathtt{BC}} w^{\mathtt{BC}}(x,t) &= b^{\mathtt{BC},w}(x,t), \quad (x,t) \in \Omega \times (0,T),\\
w^{\mathtt{D}}(\pm 1,t) = 0 \quad &\text{or} \quad w^{\mathtt{N}}_x(\pm 1,t) = 0,\\
w^{\mathtt{BC}}(x,0) &= \phi^{w}_{\mathtt{BC}}(x),\\
w^{\mathtt{BC}}_t(x,0) &= \psi^{w}_{\mathtt{BC}}(x),
\end{aligned}$$

where we define

$$\begin{aligned}
b^{\mathtt{BC},w}(x,t) &:= b^{\mathtt{BC}}(x,t) - G^{\mathtt{BC}}_{tt}(x,t) - \mathcal{M}_{\mathtt{BC}} G^{\mathtt{BC}}(x,t)\\
\phi^{w}_{\mathtt{BC}}(x) &:= \phi_{\mathtt{BC}}(x) - G^{\mathtt{BC}}(x,0)\\
\psi^{w}_{\mathtt{BC}}(x) &:= \psi_{\mathtt{BC}}(x) - G^{\mathtt{BC}}_t(x,0).
\end{aligned}$$

Then, the explicit expression for the solution $u^{\mathtt{BC}}(x,t)$ from (4.1) takes the form

$$\begin{aligned}
&u^{\mathtt{BC}}(x,t) = G^{\mathtt{BC}}(x,t) + \cos(t\sqrt{\mathcal{M}_{\mathtt{BC}}})\big(\phi_{\mathtt{BC}}(x) - G^{\mathtt{BC}}(x,0)\big) +\\
&\frac{\sin(t\sqrt{\mathcal{M}_{\mathtt{BC}}})}{\sqrt{\mathcal{M}_{\mathtt{BC}}}}\big(\psi_{\mathtt{BC}}(x) - G^{\mathtt{BC}}_t(x,0)\big) +\\
&\int_0^t \frac{\sin\big((t-\tau)\sqrt{\mathcal{M}_{\mathtt{BC}}}\big)}{\sqrt{\mathcal{M}_{\mathtt{BC}}}}\big(b^{\mathtt{BC}}(x,\tau) - G^{\mathtt{BC}}_{tt}(x,\tau) - \mathcal{M}_{\mathtt{BC}} G^{\mathtt{BC}}(x,\tau)\big)\, \mathrm{d}\tau.
\end{aligned}$$

The corresponding series representation takes the form

$$u^{\mathtt{BC}}(x,t) = G^{\mathtt{BC}}(x,t) + \sum_{k\in\mathbb{N}_{\mathtt{BC}}} \cos\left(t\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}\right) \langle e_k^{\mathtt{BC}} | \phi_{\mathtt{BC}} - G^{\mathtt{BC}}(\cdot,0)\rangle\, e_k^{\mathtt{BC}}(x) +$$
$$\sum_{k\in\mathbb{N}_{\mathtt{BC}}} \frac{\sin\left(t\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}\right)}{\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}} \langle e_k^{\mathtt{BC}} | \psi_{\mathtt{BC}} - G_t^{\mathtt{BC}}(\cdot,0)\rangle\, e_k^{\mathtt{BC}}(x) +$$
$$\sum_{k\in\mathbb{N}_{\mathtt{BC}}} \left[\int_0^t \frac{\sin\left((t-\tau)\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}\right)}{\sqrt{f_{\mathtt{BC}}(\lambda_k^{\mathtt{BC}})}} \langle e_k^{\mathtt{BC}} | b^{\mathtt{BC}}(\cdot,\tau) - G_{tt}^{\mathtt{BC}}(\cdot,\tau) - \mathcal{M}_{\mathtt{BC}} G^{\mathtt{BC}}(\cdot,\tau)\rangle\, \mathrm{d}\tau\right] e_k^{\mathtt{BC}}(x). \tag{5.6}$$

To find an exact solution with inhomogeneous BC, we make the following choices for the series representation (5.6):

$$\begin{aligned} b^{\mathtt{BC}}(x,t) &= G_{tt}^{\mathtt{BC}}(x,t) + \mathcal{M}_{\mathtt{BC}} G^{\mathtt{BC}}(x,t), \\ \phi_{\mathtt{BC}}(x) &= G^{\mathtt{BC}}(x,0), \\ \psi_{\mathtt{BC}}(x) &= G_t^{\mathtt{BC}}(x,0). \end{aligned}$$

With this choice, note that all the terms in (5.6) vanish except the first term. Eventually, we arrive at the exact solution

$$u^{\mathtt{BC}}(x,t) = G^{\mathtt{BC}}(x,t). \tag{5.7}$$

6 NUMERICAL EXPERIMENTS

We employ a collocation method with linear basis functions to discretize the governing equations (1.1a) and (1.2a). We choose a family of kernel functions with horizon δ

$$C(x) := \begin{cases} \dfrac{2}{\delta^m}\left(1 - \left|\dfrac{x}{\delta}\right|\right), & x \in (-\delta,\delta) \\ 0, & \text{otherwise}, \end{cases}$$

with $m = 0,\dots,3$. The scaling $1/\delta^m$ is inserted to capture the local operator when $m = 3$; see [6]. We use the same boundary data for Dirichlet and Neumann problems:

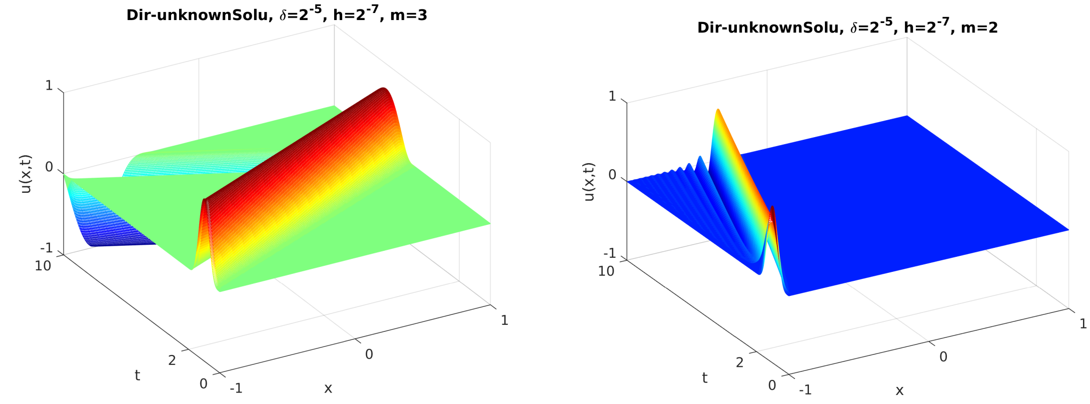
$$\alpha_-^{\mathtt{BC}}(t) := \begin{cases} \dfrac{1}{4}(1-\cos(\pi t))^2, & t \in [0,2] \\ 0, & t \in (2,10] \end{cases} \quad \text{and} \quad \alpha_+^{\mathtt{BC}}(t) := 0, \quad t \in [0,10],$$

where

$$u^{\mathtt{D}}(\pm 1,t) = \alpha_\pm^{\mathtt{D}}(t) \quad \text{and} \quad \frac{\partial u^{\mathtt{N}}}{\partial x}(\pm 1,t) = \alpha_\pm^{\mathtt{N}}(t).$$

Note that $\alpha_\pm^{\mathtt{BC}}(t) \in \mathcal{C}^3([0,10])$. For discretization, we use an adaptive mesh with mesh spacing h and δ inside and outside the bulk, respectively. Hence, mesh nodes are

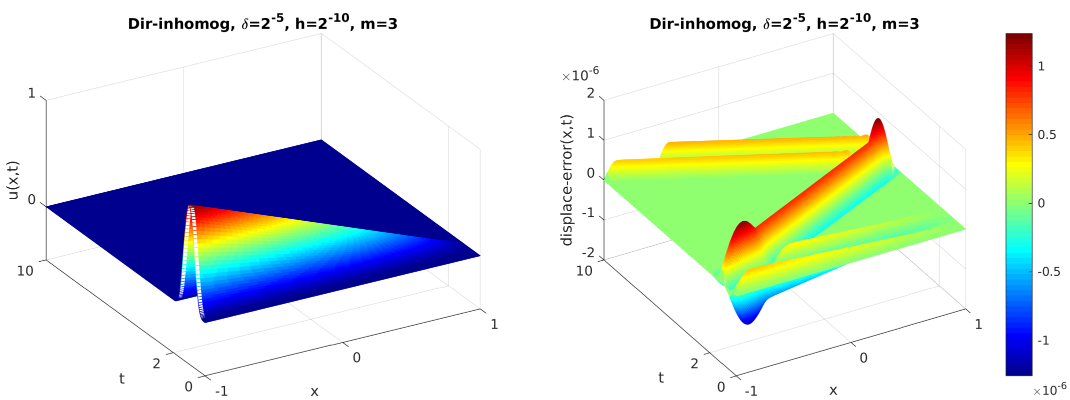
$$\Omega_h := \{-1, -1+\delta, -1+\delta+h, \dots, -h, 0, h, \dots, 1-\delta-h, 1-\delta, 1\}.$$



(a) The scaling $m = 3$ captures the local solution. No dispersion arises.

(b) The scaling $m = 2$ gives rise to dispersion.

Figure 6.1: Displacement of the Dirichlet problems with unknown solution. Note that $\delta = 4h$.



(a) Approximate displacement $u^{\mathtt{D}}(x,t)$.

(b) Displacement error $e^{\mathtt{D}}(x,t)$.

Figure 6.2: Displacement of the Dirichlet problem with known exact solution.

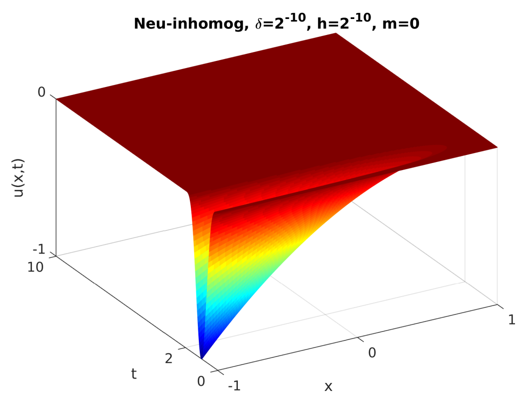
For time integration, we employ the Newmark scheme with $\Delta t = 10^{-3}$. We define the pointwise error between the exact and the approximate displacement

$$e^{\mathtt{BC}}(x_i, t_j) := G^{\mathtt{BC}}(x_i, t_j) - u^{\mathtt{BC}}(x_i, t_j), \tag{6.1}$$

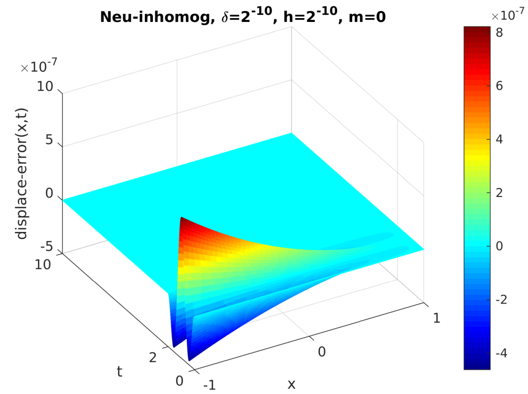
where $u^{\mathtt{BC}}$ denotes the approximate displacement. On the other hand, for the Neumann problem, we also define the stress error by

$$e_{\text{stress}}(x_i, t_j) := G_x^{\mathtt{N}}(x_i, t_j) - s(x_i, t_j), \tag{6.2}$$

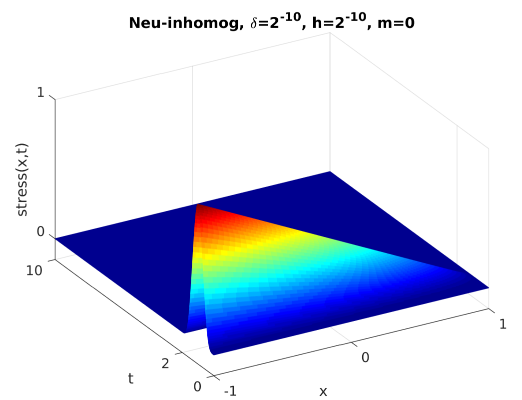
where $s(x_i, t_j)$ denotes the approximate stress computed by a central difference scheme.



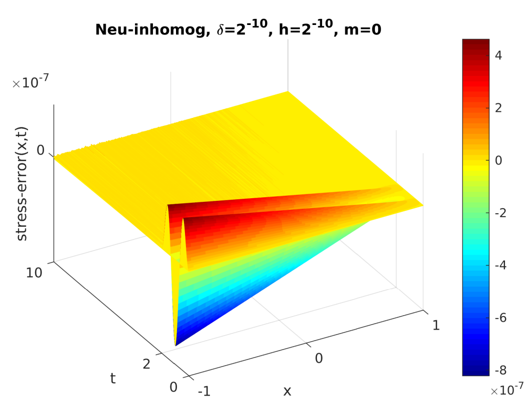
(a) Approximate displacement $u^{\mathtt{N}}(x,t)$.



(b) Displacement error $e^{\mathtt{N}}(x,t)$.



(c) Approximate stress $s(x,t)$.



(d) Stress error $e_{\text{stress}}(x,t)$.

Figure 6.3: Displacement and stress of the Neumann problem with known exact solution.

6.1 Dirichlet Problem with Unknown Solution

We report experiments of the Dirichlet problem (1.1) with unknown solution. We choose zero initial data, i.e., $u^{\mathtt{D}}(x,0) = u_t^{\mathtt{D}}(x,0) = 0$, and zero forcing function in the interior so that the wave propagation is initiated only by the boundary data. Reflecting on $(3.7)_1$, the forcing function becomes

$$b^{\mathtt{D}}(\pm 1,t) = \frac{\mathrm{d}^2 \alpha_{\pm}^{\mathtt{D}}}{\mathrm{d}t^2}(t) + c\alpha_{\pm}^{\mathtt{D}}(t) \quad \text{and} \quad b^{\mathtt{D}}(x,t) = 0, \quad x \neq \pm 1.$$

When the kernel function is scaled with $1/\delta^3$, we observe a wave pattern reminiscent of the classical wave equation. We also observe a boundary reflection agreeing with the classical equation, i.e., a reflection pattern with opposite sign. On the other hand, when the scaling is $1/\delta^2$, we observe a dispersive wave pattern common to NL operators. See Fig. 6.1. For both cases, we choose $\delta = 4h$ with $h = 2^{-7}$ indicating that the computation is NL.

6.2 Dirichlet and Neumann Problem with Known Exact Solution

We verify the accuracy of the numerical solution by employing the exact solutions given in (5.7) and the shift functions in (5.2) and (5.3). The forcing functions are chosen as

$$b^{\mathtt{BC}}(x,t) = G^{\mathtt{BC}}_{tt}(x,t) + \mathcal{M}_{\mathtt{BC}} G^{\mathtt{BC}}(x,t), \quad x \in \overline{\Omega},\ t \in [0,10].$$

We choose $\Delta t = h = \mathcal{O}(10^{-3})$. We observe that the computational solutions well approximate the exact solutions. For the Dirichlet problem, we monitor the displacement error by using (6.1). We observe that $e^{\mathtt{D}}(x_i,t_j) = \mathcal{O}(10^{-6}) = \mathcal{O}(\Delta t^2 + h^2)$. Due to scaling $1/\delta^3$, the error propagation is similar to the classical wave pattern free from dispersion. See Fig. 6.2.

For the Neumann problem, we monitor the displacement and stress error by using (6.1) and (6.2), respectively. We also observe that $e^{\mathtt{N}}(x_i,t_j) = e^{\mathtt{N}}_{\text{stress}}(x_i,t_j) = \mathcal{O}(10^{-6}) = \mathcal{O}(\Delta t^2 + h^2)$. See Fig. 6.3.

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