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A RIGOROUS DERIVATION OF THE INTERFACE CONDITIONS IN LINEAR POROELASTIC COMPOSITES

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Abstract. The paper describes the mechanical behavior of two linear isotropic poroelastic solids, bonded together by a thin plate-like layer, constituted by a linear isotropic poroelastic material, by means of an asymptotic analysis. After defining a small parameter ε, which will tend to zero, associated with the thickness and the constitutive coefficients of the intermediate layer, we characterize three different limit models and their associated limit problems, the so-called *soft*, *hard* and *rigid* poroelastic interface models, respectively. Moreover, we identify the non classical transmission conditions at the interface between the two three-dimensional bodies in terms of the jump of the stresses, increment of fluid, pressure and displacements.

1 INTRODUCTION

The modeling of complex structures obtained by joining simpler elements with highly contrasted geometric and/or material characteristics represents a source of a variety of problems of practical importance in all fields of engineering. The geometrical complexity of a multilayer structure requires an effort to deduce simplified mathematical models. In the present work we undertake a rigorous derivation of the interface conditions between two poroelastic solids separated by a thin poroelastic interphase layer by means of an asymptotic analysis. The poroelastic bodies are characterized by the simultaneous presence of the deformation and the filtration (flow). They are described by the quasi-static Biot's system of PDE's. It couples the Navier's equations of linearized elasticity, containing the pressure gradient, with the mass conservation equation involving the fluid content change and divergence of the filtration velocity. The filtration velocity is the relative velocity for the upscaled fluid-structure problem and obeys Darcy's law. The fluid content change is proportional to the pressure and the elastic body compression. For more modeling details, we refer to [1, 2].

The application of the asymptotic methods for the mathematical justification of thin structure models has met with success at both fields of elasticity and poroelasticity, (see, e.g., [3, 4, 5, 6, 7]): this has helped to tunnel the research toward a rational simplification of the modeling of complex structures constructed by joining elements presenting highly contrasted geometrical and mechanical properties. Within the theory of classical elasticity, the asymptotic analysis of a thin elastic interphase between two elastic materials has been deeply investigated through the years, by varying the rigidity ratios between the thin inclusion and the surrounding materials and by considering different geometry and mechanical features, see, e.g., [8, 9, 10, 11, 12, 13].

The goal of the present work is to identify the interface limit models of a linear poroelastic composite constituted by a thin poroelastic layer surrounded by two poroelastic bodies. By defining a small parameter ε, associated with the thickness and the constitutive properties of the middle layer, we perform an asymptotic analysis by letting ε tend to zero. We analyze different situations by varying the stiffnesses ratios between the middle layer and the adherents: namely, the *soft* poroelastic lowly permeable interface, where the intermediate material coefficients have order of magnitude ε with respect to those of the surrounding bodies; the *hard* poroelastic moderately permeable interface, where the costitutive parameters have the same order of magnitude; finally, the *rigid* poroelastic highly permeable interface, where the rigidities have order of magnitude $\frac{1}{\varepsilon}$. We characterize the limit transmission problem at order zero and we identify the ad hoc transmission conditions at the interface.

2 POSITION OF THE PROBLEM

In the sequel, Greek indices range in the set $\{1, 2\}$, Latin indices range in the set $\{1, 2, 3\}$, and the Einstein's summation convention with respect to the repeated indices is adopted. We introduce the following notations for the inner product: $\mathbf{a} \cdot \mathbf{b} := a_i b_i$.

Let Ω^+ and Ω^- be two disjoint open domains with smooth boundaries $\partial\Omega^+$ and $\partial\Omega^-$. Let $\omega := \{\partial\Omega^+ \cap \partial\Omega^-\}^\circ$ be the interior of the common part of the boundaries which is assumed to be a non empty domain in $\mathbb{R}^2$ having a positive two-dimensional measure. We consider the assembly constituted by two solids bonded together by an intermediate thin plate-like body $\Omega^{m,\varepsilon}$ of thickness $2h^\varepsilon$, where $0 < \varepsilon < 1$ is a dimensionless small real parameter which will tend to zero. We suppose that the thickness $2h^\varepsilon$ of the middle layer depends linearly on ε, so that $2h^\varepsilon = 2\varepsilon h$.

More precisely, we denote respectively with $\Omega^{\pm,\varepsilon} := \{x^\varepsilon := x \pm \varepsilon h \mathbf{e}_3;\ x \in \Omega^\pm\}$, the translation of Ω^+ (resp. Ω^-) along the direction $\mathbf{e}_3$ (resp. $-\mathbf{e}_3$) of the quantity εh, with $\Omega^{m,\varepsilon} := \omega \times (-\varepsilon h, \varepsilon h)$, the central plate-like domain, and with $\Omega^\varepsilon := \Omega^{+,\varepsilon} \cup \Omega^{m,\varepsilon} \cup \Omega^{-,\varepsilon}$, the reference configuration of the assembly. Moreover, we define with $S^{\pm,\varepsilon} := \omega \times \{\pm\varepsilon h\} = \Omega^{\pm,\varepsilon} \cap \Omega^{m,\varepsilon}$, the upper and lower faces of the intermediate plate-like domain, $\Gamma^{\pm,\varepsilon} := \partial\Omega^{\pm,\varepsilon}/S^{\pm,\varepsilon}$, and $\Gamma^{m,\varepsilon}_{lat} := \partial\omega \times (-\varepsilon h, \varepsilon h)$, its lateral surface, see Fig. 1.

A poroelastic medium consists of an elastic skeleton (the solid phase) and pores saturated by a viscous fluid (the fluid phase). At the pore scale, one deals with a complicated fluid-structure problem and in applications we model it using the effective medium approach, see [1]. We assume that $\Omega^{\pm,\varepsilon}$ and $\Omega^{m,\varepsilon}$ are constituted by three homogeneous

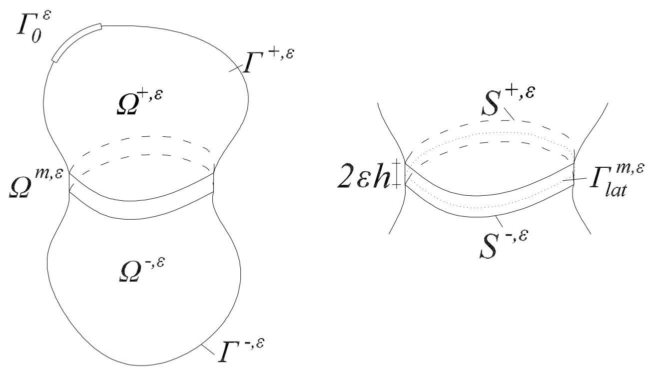

Figure 1: The reference configuration of the assembly and the geometry of the interphase.

linear isotropic poroelastic materials, whose constitutive laws are defined as follows:

$$\begin{cases} \sigma_{ij}^\varepsilon = \lambda^\varepsilon e_{pp}^\varepsilon \delta_{ij} + 2\mu^\varepsilon e_{ij}^\varepsilon - \alpha^\varepsilon p^\varepsilon \delta_{ij}, \\ \zeta^\varepsilon = \gamma_G^\varepsilon p^\varepsilon + \alpha^\varepsilon e_{pp}^\varepsilon, \\ q_i^\varepsilon = -\frac{k^\varepsilon}{\eta} \partial_i^\varepsilon p^\varepsilon, \end{cases}$$

where $(\sigma_{ij}^\varepsilon)$ is the Cauchy effective stress tensor, associated with the strain tensor $(e_{ij}^\varepsilon) := \frac{1}{2}(\partial_j^\varepsilon u_i^\varepsilon + \partial_i^\varepsilon u_j^\varepsilon)$, ζ^ε represents the increment of fluid content and (q_i^ε) is the specific discharge field. Constants λ^ε, μ^ε, α^ε, γ_G^ε, k^ε and η represent respectively the Lamé's constants, the effective stress coefficient, the Biot's inverse modulus, the permeability and the viscosity coefficients.

The deformable porous media, saturated by a fluid, are modelled using the Biot's diphasic equations for the effective solid displacement $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ and the effective pressure p^ε. The poroelastic state is then defined by the couple $s^\varepsilon := (\mathbf{u}^\varepsilon, p^\varepsilon)$. We assume that the poroelastic composite is subject to body forces $(f_i^\varepsilon)\colon \Omega^{\pm,\varepsilon} \times (0,T) \to \mathbb{R}^3$ applied on the top and bottom bodies, while the intermediate layer $\Omega^{m,\varepsilon}$ is considered to be free of charges. The poroelastic state s^ε solves the quasi-static Biot's system:

$$\begin{cases} -\partial_j^\varepsilon \sigma_{ij}^\varepsilon = f_i^\varepsilon & \text{in } \Omega^{\pm,\varepsilon} \times (0,T), \\ \partial_t \zeta^\varepsilon + \partial_i^\varepsilon q_i^\varepsilon = 0 & \text{in } \Omega^{\pm,\varepsilon} \times (0,T), \end{cases} \quad \begin{cases} -\partial_j^\varepsilon \sigma_{ij}^\varepsilon = 0 & \text{in } \Omega^{m,\varepsilon} \times (0,T), \\ \partial_t \zeta^\varepsilon + \partial_i^\varepsilon q_i^\varepsilon = 0 & \text{in } \Omega^{m,\varepsilon} \times (0,T), \end{cases}$$

where ∂_t denotes the time derivative. The transmission conditions across the interfaces $S^{+,\varepsilon}$ and $S^{-,\varepsilon}$ implies the continuity of the state s^ε and of its normal dual counterpart with respect to $S^{\pm,\varepsilon}$, meaning that $[\![u_i^\varepsilon]\!] = 0$, $[\![p^\varepsilon]\!] = 0$, $[\![\sigma_{i3}^\varepsilon]\!] = 0$, and $[\![q_3^\varepsilon]\!] = 0$ on $S^{\pm,\varepsilon} \times (0,T)$, where $[\![f]\!] := f^\pm - f^m$ denotes the jump function evaluated at the interface $S^{\pm,\varepsilon}$. The boundary conditions are posed on $\Gamma^\varepsilon \times (0,T)$, with $\Gamma^\varepsilon := \Gamma^{+,\varepsilon} \cup \Gamma^{-,\varepsilon}$; we recall that $\Gamma^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_0^\varepsilon$. For simplicity we consider homogeneous boundary conditions on $\Gamma_0^\varepsilon \times (0,T)$, concerning displacements and pressure, and non-homogeneous boundary conditions on $\Gamma_1^\varepsilon \times (0,T)$, concerning surface forces (g_i^ε) and fluid flux w^ε. Hence, one has

$$\begin{cases} \sigma_{ij}^\varepsilon n_j^\varepsilon = g_i^\varepsilon \quad \text{on } \Gamma_1^\varepsilon \times (0,T), \qquad u_i^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \times (0,T), \\ q_i^\varepsilon n_i^\varepsilon = w^\varepsilon \quad \text{on } \Gamma_1^\varepsilon \times (0,T), \qquad p^\varepsilon = 0 \quad \text{on } \Gamma_0^\varepsilon \times (0,T), \end{cases}$$

where (n_i^ε) is the outer unit normal vector to $\partial\Omega^\varepsilon$. The initial conditions are posed in Ω^ε. Let p_{in}^ε be the pressure at time $t = 0$, we have $p^\varepsilon(x^\varepsilon, 0) = p^\varepsilon(0) = p_{in}^\varepsilon$ in Ω^ε.

Let us introduce the functional spaces $V(\Omega^\varepsilon) := \{v^\varepsilon \in H^1(\Omega^\varepsilon);\ v^\varepsilon = 0 \ \text{ on } \Gamma_0^\varepsilon\}$ and $\mathbf{V}(\Omega^\varepsilon) := [V(\Omega^\varepsilon)]^3$. Given a certain state $s^\varepsilon := (\mathbf{u}^\varepsilon, p^\varepsilon) \in \mathbb{V}(\Omega^\varepsilon) := \mathbf{V}(\Omega^\varepsilon) \times V(\Omega^\varepsilon)$, for all test functions $r^\varepsilon = (\mathbf{v}^\varepsilon, \xi^\varepsilon) \in \mathbb{V}(\Omega^\varepsilon)$ and for any fixed $t \in (0,T)$, we introduce the following bilinear and linear forms:

$$A^\varepsilon(s^\varepsilon, r^\varepsilon) := \int_{\Omega^\varepsilon} \left\{\sigma_{ij}^\varepsilon e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) + \partial_t \zeta^\varepsilon \xi^\varepsilon + q_i^\varepsilon \partial_i^\varepsilon \xi^\varepsilon\right\} dx^\varepsilon,$$

$$L^\varepsilon(r^\varepsilon) := \int_{\Omega^{\pm,\varepsilon}} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_1^\varepsilon} \{g_i^\varepsilon v_i^\varepsilon + w^\varepsilon \xi^\varepsilon\}\, d\Gamma^\varepsilon.$$

The variational form of the Biot's system defined over the variable domain Ω^ε reads as follows:

$$\begin{cases} \text{Find } s^\varepsilon(t) \in \mathbb{V}(\Omega^\varepsilon),\ t \in (0,T), \ \text{ such that} \\ A^{-,\varepsilon}(s^\varepsilon(t), r^\varepsilon) + A^{+,\varepsilon}(s^\varepsilon(t), r^\varepsilon) + A^{m,\varepsilon}(s^\varepsilon(t), r^\varepsilon) = L^\varepsilon(r^\varepsilon), \end{cases} \tag{1}$$

for all $r^\varepsilon \in \mathbb{V}(\Omega^\varepsilon)$, with initial condition p_{in}^ε. In order to guarantee the well-posedness of problem (1), suitable regularity properties have to be assumed for the initial data, source and boundary values, and constitutive parameters, see [2].

3 THE ASYMPTOTIC EXPANSIONS METHOD

In order to study the asymptotic behavior of the solution of problem (1) when ε tends to zero, we rewrite the problem on a fixed domain Ω independent of ε. By using the approach of [3], we consider the bijection $\Pi^\varepsilon : x \in \overline{\Omega} \mapsto x^\varepsilon \in \overline{\Omega}^\varepsilon$ given by

$$\begin{cases} \Pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, x_3 - h(1-\varepsilon)), & \text{for all } x \in \overline{\Omega}_{tr}^+, \\ \Pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3), & \text{for all } x \in \overline{\Omega}^m, \\ \Pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, x_3 + h(1-\varepsilon)), & \text{for all } x \in \overline{\Omega}_{tr}^-, \end{cases}$$

where $\Omega_{tr}^\pm := \{x \pm h\mathbf{e}_3,\ x \in \Omega^\pm\}$, $\Omega^m := \omega \times (-h,h)$ and $S^\pm := \omega \times \{\pm h\}$. In order to simplify the notation, we identify $\Omega_{tr}^\pm$ with $\Omega^\pm$, and $\overline{\Omega}$ with $\overline{\Omega}^\pm \cup \overline{\Omega}^m$. Likewise, we note $\Gamma^\pm := \partial\Omega^\pm / S^\pm$, $\Gamma_{lat}^m := \partial\omega \times (-h,h)$, Γ_0 and Γ_1, the partitions of $\Gamma^+ \cup \Gamma^-$. Consequently, $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$ in Ω^m. In the sequel, only if necessary, we will note, respectively, with $r^\pm = (\mathbf{v}^\pm, \xi^\pm)$ and $r^m = (\mathbf{v}^m, \xi^m)$, the restrictions of functions $r = (\mathbf{v}, \xi)$ to $\Omega^\pm$ and Ω^m. With the unknown state $s^\varepsilon = (\mathbf{u}^\varepsilon, p^\varepsilon)$, we associate the scaled unknown state $s(\varepsilon) := (\mathbf{u}(\varepsilon), p(\varepsilon))$ defined by:

$$u_i^\varepsilon(x^\varepsilon) = u_i(\varepsilon)(x), \quad p^\varepsilon(x^\varepsilon) = p(\varepsilon)(x) \ \text{ for all } x^\varepsilon = \Pi^\varepsilon x \in \overline{\Omega}^\varepsilon.$$

We likewise associate with any test functions $r^\varepsilon = (\mathbf{v}^\varepsilon, \xi^\varepsilon)$, the scaled test functions $r = (\mathbf{v}, \xi)$, defined by the scalings:

$$v_i^\varepsilon(x^\varepsilon) = v_i(x), \quad \xi^\varepsilon(x^\varepsilon) = \xi(x) \quad \text{for all } x^\varepsilon = \Pi^\varepsilon x \in \overline{\Omega}^\varepsilon.$$

We assume that the constitutive coefficients of $\Omega^{\pm,\varepsilon}$ are independent of ε, while the constitutive coefficients of $\Omega^{m,\varepsilon}$ present the following dependences on ε: $\lambda^{m,\varepsilon} = \varepsilon^p \lambda^m$, $\mu^{m,\varepsilon} = \varepsilon^p \mu^m$, $\alpha^{m,\varepsilon} = \varepsilon^p \alpha^m$, $\gamma_G^{m,\varepsilon} = \varepsilon^p \gamma_G^m$ and $k^{m,\varepsilon} = \varepsilon^p k^m$, with $p \in \{-1, 0, 1\}$. Three different limit behaviors will be characterized according to the choice of the exponent p: by choosing $p = 1$, we deduce a model for a *soft* poroelastic interface with low permeability; when $p = 0$, we deduce a model for a *hard* poroelastic interface with moderate permeability; in the case of $p = -1$, we derive a model for a *rigid* poroelastic interface with high permeability. Finally we suppose that the data verify the following scaling assumptions:

$$\begin{aligned} &f_i^\varepsilon(x^\varepsilon) = f_i(x), \quad x \in \Omega^\pm, \\ &g_i^\varepsilon(x^\varepsilon) = g_i(x), \quad w^\varepsilon(x^\varepsilon) = w(x), \quad x \in \Gamma_1, \end{aligned}$$

so that $L^\varepsilon(r^\varepsilon) = L(r)$. According to the previous hypothesis, problem (1) can be reformulated on a fixed domain Ω independent of ε. Thus we obtain the following rescaled problem (in the sequel, we will omit the explicit dependences on time t of the unknowns and data):

$$\begin{cases} \text{Find } s(\varepsilon) \in \mathbb{V}(\Omega), \ t \in (0,T), \quad \text{such that} \\ A^-(s(\varepsilon), r) + A^+(s(\varepsilon), r) + \varepsilon^{p+1} A^m(s(\varepsilon), r) = L(r), \end{cases} \tag{2}$$

for all $r \in \mathbb{V}(\Omega)$, $p \in \{-1, 0, 1\}$, with initial condition $p_{in}(\varepsilon)$, where

$$\begin{aligned} A^\pm(s(\varepsilon), r) &:= \int_{\Omega^\pm} \left\{ \sigma_{ij}^\pm(\varepsilon) e_{ij}(\mathbf{v}) + \partial_t \zeta^\pm(\varepsilon) \xi + q_i^\pm(\varepsilon) \partial_i \xi \right\} dx, \\ A^m(s(\varepsilon), r) &:= \frac{1}{\varepsilon^2} a_0(s(\varepsilon), r) + \frac{1}{\varepsilon} a_1(s(\varepsilon), r) + a_2(s(\varepsilon), r), \end{aligned}$$

where

$$\begin{aligned} a_0(s(\varepsilon), r) &:= \int_{\Omega^m} \left\{ (\lambda^m + 2\mu^m) e_{33}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v}) + \mu^m \partial_3 u_\alpha(\varepsilon) \partial_3 v_\alpha + \frac{k^m}{\eta} \partial_3 p(\varepsilon) \partial_3 \xi \right\} dx, \\ a_1(s(\varepsilon), r) &:= \int_{\Omega^m} \{ \lambda^m (e_{\sigma\sigma}(\mathbf{u}(\varepsilon)) e_{33}(\mathbf{v}) + e_{33}(\mathbf{u}(\varepsilon)) e_{\sigma\sigma}(\mathbf{v})) + \mu^m \partial_\alpha u_3(\varepsilon) \partial_3 v_\alpha + \\ &\qquad + \mu^m \partial_3 u_\alpha(\varepsilon) \partial_\alpha v_3 - \alpha^m p(\varepsilon) e_{33}(\mathbf{v}) + \alpha^m \partial_t (\partial_3 u_3(\varepsilon)) \xi \} \, dx, \\ a_2(s(\varepsilon), r) &:= \int_{\Omega^m} \{ 2\mu^m e_{\alpha\beta}(\mathbf{u}(\varepsilon)) e_{\alpha\beta}(\mathbf{v}) + (\lambda^m e_{\sigma\sigma}(\mathbf{u}(\varepsilon)) - \alpha^m p(\varepsilon)) e_{\tau\tau}(\mathbf{v}) + \\ &\qquad + \mu^m \partial_\alpha u_3(\varepsilon) \partial_\alpha v_3 + \partial_t (\gamma_G^m p(\varepsilon) + \alpha^m \partial_\alpha u_\alpha(\varepsilon)) \xi + \tfrac{k^m}{\eta} \partial_\alpha p(\varepsilon) \partial_\alpha \xi \right\} dx. \end{aligned}$$

We can now apply the asymptotic expansion method to the rescaled problem (2) and distinguish the three cases of weak, hard and rigid poroelastic interfaces. Since the rescaled problem (2) has a polynomial structure with respect to the small parameter ε, we can look for the solution of the problem as a series of powers of ε:

$$s(\varepsilon) = s^0 + \varepsilon s^1 + \varepsilon^2 s^2 + \dots \;\Rightarrow\; \begin{cases} \mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon \mathbf{u}^1 + \varepsilon^2 \mathbf{u}^2 + \dots \\ p(\varepsilon) = p^0 + \varepsilon p^1 + \varepsilon^2 p^2 + \dots \end{cases} \tag{3}$$

Hence, by substituting expressions (3) in (2) and by identifying the terms with identical power of ε, we can finally characterize the limit problems for $p = 1$, $p = 0$ and $p = -1$.

4 THE SOFT POROELASTIC INTERFACE PROBLEM

In this section we derive the limit model for a soft poroelastic interface with low permeability. Let us define the following functional spaces $W(\Omega) := \{v \in L^2(\Omega);\ v^\pm \in H^1(\Omega^\pm),\ \partial_3 v^m \in L^2(\Omega^m),\ v^\pm = v^m \text{ on } S^\pm,\ v = 0 \text{ on } \Gamma_0\}$, $\mathbf{W}(\Omega) := [W(\Omega)]^3$. Let us choose $p = 1$ in (2), the formulation of the limit problem reads as follows:

$$\begin{cases} \text{Find } s^0 \in \mathbb{W}(\Omega) := \mathbf{W}(\Omega) \times W(\Omega),\ t \in (0,T), \quad \text{such that} \\ A^-(s^0, r) + A^+(s^0, r) + a_0(s^0, r) = L(r), \end{cases} \tag{4}$$

for all $r \in \mathbb{W}(\Omega)$, with

$$a_0(s^0, r) := \int_{\Omega^m} \left\{ \mathbf{K}\partial_3 \mathbf{u}^{m,0} \cdot \partial_3 \mathbf{v} + \frac{k^m}{\eta} \partial_3 p^{m,0} \partial_3 \xi \right\} dx,$$

and $\mathbf{K} := \mathrm{diag}\{\mu^m, \mu^m, 2\mu^m + \lambda^m\}$, being the diagonal interface stiffness matrix. The limit problem (4) can be simplified if one considers the structure of the bilinear form $a_0(\cdot,\cdot)$, which involves only the derivatives along the x_3-coordinates. Indeed, by choosing test functions v_i, $\xi \in \mathcal{D}(\Omega^m)$, with compact support in Ω^m, one has

$$\int_{\Omega^m} \left\{ \mathbf{K}\partial_3 \mathbf{u}^{m,0} \cdot \partial_3 \mathbf{v} + \frac{k^m}{\eta} \partial_3 p^{m,0} \partial_3 \xi \right\} dx = 0.$$

The previous variational equation implies the existence of two constant functions with respect to the x_3-coordinate, namely, $\mathbf{z}^u = \mathbf{z}^u(\tilde{x})$ and $z^p = z^p(\tilde{x})$, with $\tilde{x} = (x_\alpha)$, such that

$$\begin{cases} \mathbf{K}\partial_3 \mathbf{u}^{m,0} = \mathbf{z}^u, \\ \frac{k^m}{\eta} \partial_3 p^{m,0} = z^p. \end{cases}$$

We can now solve the above linear system and, by imposing the continuity conditions at the interfaces S^+ and S^- for the displacements $\mathbf{u}^0$ and pressure p^0, we find that $\mathbf{z}^u = \frac{1}{2h}\mathbf{K}[\![\mathbf{u}^0]\!]$ and $z^p = \frac{1}{2h}\frac{k^m}{\eta}[\![p^0]\!]$. Moreover, since $\partial_3 \mathbf{u}^{m,0} = \frac{[\![\mathbf{u}^0]\!]}{2h}$ and $\partial_3 p^{m,0} = \frac{[\![p^0]\!]}{2h}$, $\mathbf{u}^{m,0}$ and $p^{m,0}$ become affine functions of x_3. Indeed,

$$\begin{aligned} \mathbf{u}^{m,0}(\tilde{x}, x_3) &= \langle \mathbf{u}^0 \rangle(\tilde{x}) + \tfrac{x_3}{2h}[\![\mathbf{u}^0]\!](\tilde{x}), \\ p^{m,0}(\tilde{x}, x_3) &= \langle p^0 \rangle(\tilde{x}) + \tfrac{x_3}{2h}[\![p^0]\!](\tilde{x}), \end{aligned} \tag{5}$$

where $\langle f\rangle := \frac{1}{2}(f|_{S^+}+f|_{S^-})$ and $[\![f]\!] = f|_{S^+}-f|_{S^-}$ denote respectively the mean value and the jump of the restrictions of f on S^+ and S^-. Equations (5) represent a representation formula of the limit kinematics of the adhesive layer in the limit problem. By using the continuity conditions on S^+ and S^- of the displacement field and the pressure and after an integration by parts on x_3, we get

$$a_0(s^0, r) = \int_{S^+}(\mathbf{z}^u\cdot\mathbf{v}^+ + z^p\xi^+)d\Gamma - \int_{S^-}(\mathbf{z}^u\cdot\mathbf{v}^- + z^p\xi^-)d\Gamma.$$

Hence, using expressions (5) and by identifying S^+ and S^- with the interface ω, the limit problem can be reformulated in the following reduced form:

$$\begin{cases} \text{Find } s^0 \in \tilde{\mathbb{W}}(\Omega) := \tilde{\mathbf{W}}(\Omega)\times\tilde{W}(\Omega),\ t\in(0,T), \quad \text{such that} \\ A^-(s^0,r) + A^+(s^0,r) + \tilde{a}_0(s^0,r) = L(r), \end{cases} \tag{6}$$

for all $r \in \tilde{\mathbb{W}}(\Omega)$, where $\tilde{W}(\Omega) := \{v \in L^2(\Omega);\ v^\pm \in H^1(\Omega^\pm),\ v = 0 \text{ on } \Gamma_0\}$, with self-explanatory notation, and

$$\tilde{a}_0(s^0,r) := \frac{1}{2h}\int_\omega \left\{\mathbf{K}[\![\mathbf{u}^0]\!]\cdot[\![\mathbf{v}]\!] + \frac{k^m}{\eta}[\![p^0]\!][\![\xi]\!]\right\} d\tilde{x}.$$

Thanks to the asymptotic analysis, we transform the limit problem onto a coupled interface problem between Ω^+ and Ω^-, with non classical transmission conditions at the interface ω. This problem represents a poroelastic generalization of the soft linear elastic interface model obtained in [9]. We rewrite problem (6) in its differential form and we obtain:

Quasi-static Biot's system in $\Omega^\pm$

$$\begin{cases} -\partial_j\sigma_{ij}^{\pm,0} = f_i & \text{in } \Omega^\pm, \\ \partial_t\zeta^{\pm,0} + \partial_i q_i^{\pm,0} = 0 & \text{in } \Omega^\pm, \\ \sigma_{ij}^{\pm,0}n_j = g_i,\ q_i^{\pm,0}n_i = w & \text{on } \Gamma_1, \\ u_i^0 = p^0 = 0 & \text{on } \Gamma_0, \end{cases}$$

Transmission conditions

$$\begin{cases} \sigma_{i3}^{+,0} = \frac{1}{2h}K_{ij}[\![u_j^0]\!] & \text{on } \omega, \\ \sigma_{i3}^{-,0} = \frac{1}{2h}K_{ij}[\![u_j^0]\!] & \text{on } \omega, \\ q_3^{+,0} = \frac{1}{2h}\frac{k^m}{\eta}[\![p^0]\!] & \text{on } \omega, \\ q_3^{-,0} = \frac{1}{2h}\frac{k^m}{\eta}[\![p^0]\!] & \text{on } \omega, \end{cases}$$

The limit model for a soft poroelastic interface provides a discontinuity of the limit state $s^0 = (\mathbf{u}^0, p^0)$ at the interface between Ω^+ and Ω^-. The interphase behaves, from a mechanical point of view, as a distribution of extensional linear springs reacting to the gap of the displacements between the top and bottom faces. The low permeability still allows the saturated fluid to flow through the interface thanks to the difference of pressures between to top and bottom faces, identifying the so-called open pore interface, see [14]. Besides, subtracting two by two the transmission conditions above, we obtain that the jump of the stress vector and the jump of the normal flux referred to the plane of the interface ω, vanish, so that $[\![\sigma_{i3}^0]\!] = 0,\ [\![q_3^0]\!] = 0$.

5 THE HARD POROELASTIC INTERFACE PROBLEM

In this section we derive the limit model for a hard poroelastic interface, whose permeability has the same order of magnitude of the adherents. Let us define the following functional spaces $X(\tilde{\Omega}) := \{v \in H^1(\tilde{\Omega}),\ v|_\omega \in H^1(\omega),\ v = 0 \text{ on } \Gamma_0\}$, $\mathbf{X}(\tilde{\Omega}) := [\tilde{X}(\tilde{\Omega})]^3$, with $\tilde{\Omega} := \Omega^+ \cup \omega \cup \Omega^-$.

The limit problem can be obtained using classical tools of asymptotic analysis. Let $p = 0$ in (2). First, we analyze the variational problem corresponding to the order ε^{-1} and we obtain that $a_0(s^0, r) = 0$, for all $r \in \mathbb{V}(\Omega)$. This problem is satisfied when $\partial_3 u_i^{m,0} = 0$ and $\partial_3 p^{m,0} = 0$, meaning that $u_i^{m,0}(\tilde{x}, x_3) = u_i^{m,0}(\tilde{x})$ and $p^{m,0}(\tilde{x}, x_3) = p^{m,0}(\tilde{x})$ are independent of the x_3-coordinate. By virtue of the above results and by choosing test functions $r \in \mathbb{X}(\tilde{\Omega}) := \mathbf{X}(\tilde{\Omega}) \times X(\tilde{\Omega})$, we write the variational problem associated with the order ε^0:

$$\begin{cases} \text{Find } s^0 \in \mathbb{X}(\tilde{\Omega}),\ t \in (0,T), \quad \text{such that} \\ A^-(s^0, r) + A^+(s^0, r) = L(r), \end{cases} \tag{7}$$

for all $r \in \mathbb{X}(\tilde{\Omega})$. By rewriting problem (7) in its differential form, we have

Quasi-static Biot's system in $\Omega^\pm$

$$\begin{cases} -\partial_j \sigma_{ij}^{\pm,0} = f_i & \text{in } \Omega^\pm, \\ \partial_t \zeta^{\pm,0} + \partial_i q_i^{\pm,0} = 0 & \text{in } \Omega^\pm, \\ \sigma_{ij}^{\pm,0} n_j = g_i,\ q_i^{\pm,0} n_i = w & \text{on } \Gamma_1, \\ u_i^0 = p^0 = 0 & \text{on } \Gamma_0, \end{cases}$$

Transmission conditions

$$\begin{cases} [\![u_i^0]\!] = 0, \quad [\![p^0]\!] = 0 & \text{on } \omega, \\ [\![\sigma_{i3}^0]\!] = 0, \quad [\![q_3^0]\!] = 0 & \text{on } \omega. \end{cases}$$

As we can notice, at order 0, we do not perceive the presence of the thin layer, having the same rigidity and permeability of the surrounding bodies. The transmission conditions provide the continuity of the displacement field, pressure, stress vector and flux related to the plane of the interface, respectively. In this case, the two bodies are perfectly bonded together and the presence of the interlayer does not influence the mechanical behavior of the composite.

6 THE RIGID POROELASTIC INTERFACE PROBLEM

In this section we derive the limit model for a rigid and highly permeable poroelastic interface. The asymptotic procedure to obtain the rigid interface model follows the same steps adopted in Section 5. In the sequel, we will present just the expression of the limit problem in its variational and differential form. The limit poroelastic state s^0 satisfies the following limit problem

$$\begin{cases} \text{Find } s^0 \in \mathbb{X}(\tilde{\Omega}),\ t \in (0,T), \quad \text{such that} \\ A^-(s^0, r) + A^+(s^0, r) + \tilde{a}_2(s^0, r) = L(r), \end{cases} \tag{8}$$

for all $r \in \mathbb{X}(\tilde{\Omega})$, where the bilinear form $\tilde{a}_2(\cdot, \cdot)$ is defined as follows

$$\tilde{a}_2(s^0, r) := 2h \int_\omega \Big\{ 2\mu^m e_{\alpha\beta}(\mathbf{u}^0) e_{\alpha\beta}(\mathbf{v}) + (\tilde{\lambda}^m e_{\sigma\sigma}(\mathbf{u}^0) - \tilde{\alpha}^m p^0) e_{\tau\tau}(\mathbf{v}) + \\ + \partial_t(\tilde{\alpha}^m e_{\sigma\sigma}(\mathbf{u}^0) + \tilde{\gamma}_G^m p^0)\xi + \tfrac{k^m}{\eta} \partial_\alpha p^0 \partial_\alpha \xi \Big\}\, d\tilde{x},$$

with $\tilde{\lambda}^m := \frac{2\mu^m\lambda^m}{\lambda^m+2\mu^m}$, $\tilde{\alpha}^m := \frac{2\mu^m\alpha^m}{\lambda^m+2\mu^m}$ and $\tilde{\gamma}_G^m := \gamma_G^m - \frac{(\alpha^m)^2}{\lambda^m+2\mu^m}$. The limit problem can be formulated as a differential system as follows:

Quasi-static Biot's system in $\Omega^\pm$

$$\begin{cases} -\partial_j\sigma_{ij}^{\pm,0} = f_i & \text{in } \Omega^\pm, \\ \partial_t\zeta^{\pm,0} + \partial_i q_i^{\pm,0} = 0 & \text{in } \Omega^\pm, \\ \sigma_{ij}^{\pm,0} n_j = g_i,\ q_i^{\pm,0} n_i = w & \text{on } \Gamma_1, \\ u_i^0 = p^0 = 0 & \text{on } \Gamma_0, \end{cases}$$

Transmission conditions

$$\begin{cases} [\![u_i^0]\!] = 0, \quad [\![p^0]\!] = 0 & \text{on } \omega, \\ [\![\sigma_{\alpha 3}^0]\!] = -2h\partial_\beta t_{\alpha\beta}^0, \quad [\![\sigma_{33}^0]\!] = 0 & \text{on } \omega, \\ [\![q_3^0]\!] = -2h\left(\partial_t\Sigma^0 - \frac{k^m}{\eta}\Delta p^0\right) & \text{on } \omega, \end{cases}$$

where $\Delta := \partial_{\alpha\alpha}$ denotes the Laplacian operator, $t_{\alpha\beta}^0 := 2\mu^m e_{\alpha\beta}(\mathbf{u}^0) + (\tilde{\lambda}^m e_{\sigma\sigma}(\mathbf{u}^0) - \tilde{\alpha}^m p^0)\delta_{\alpha\beta}$ and $\Sigma^0 := \tilde{\alpha}^m e_{\sigma\sigma}(\mathbf{u}^0) + \tilde{\gamma}_G^m p^0$ represent the two-dimensional poroelastic membrane stress tensor and the membrane increment of fluid, respectively, and define the two-dimensional constitutive law of a poroelastic membrane. The careful reader can notice that the transmission conditions, associated with the jump of stresses and flux lead to a two-dimensional membrane quasi-static Biot's system at the interface ω. In this particular case, the continuity of displacements and pressure is verified. This problem represents a poroelastic generalization of the Ventcel-type transmission conditions obtained in [8].

7 Concluding remarks

In the present work we derive three interface models for a poroelastic composite presenting a thin poroelastic interphase in the framework of quasi-static Biot's diphasic system by means of the asymptotic expansions method. We analyze three particular cases: the first case, for $p = 1$, corresponding from a mechanical point of view to a soft lowly permeable interphase; the second case, for $p = 0$, corresponding from a mechanical point of view to a hard moderately permeable interphase; the latter, for $p = -1$, corresponding to a rigid highly permeable interphase into two poroelastic media. We identify the order 0 interface models, achieving a poroelastic generalization of the soft, hard and rigid interface models obtained in [8, 9, 11].

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