

# CONTROLLING EUTROPHICATION BY WATER ARTIFICIAL CIRCULATION

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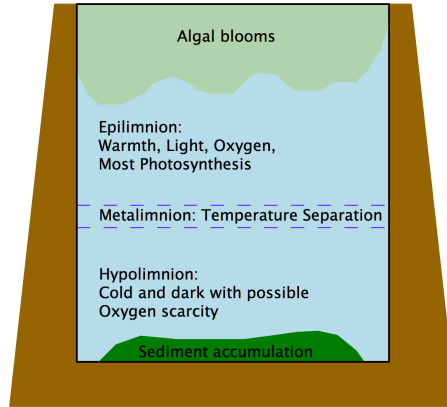
**Abstract.** This work deals with artificial circulation as a shallow water aeration technique. Large waterbodies (for instance, lakes or reservoirs) get much of their oxygen from the atmosphere through diffusion processes. Artificial circulation increases water's oxygen by forcefully circulating the water to expose more of it to the atmosphere. Two techniques are the most common: air injection and mechanical mixing. The former has been analyzed from an ecological viewpoint in several works (see, for instance, [3] and the references therein). However, in this work we will focus our attention on the latter that, as far as we know, has remained unaddressed in the mathematical literature.

In this work we will introduce a mathematical formulation of the environmental problem as a control/state constrained optimal control problem of partial differential equations. Then, we will analyze the optimal control problem and finally, we will deal with the numerical resolution of the problem, presenting a complete numerical algorithm and a realistic computational example [4].

## 1 Introduction

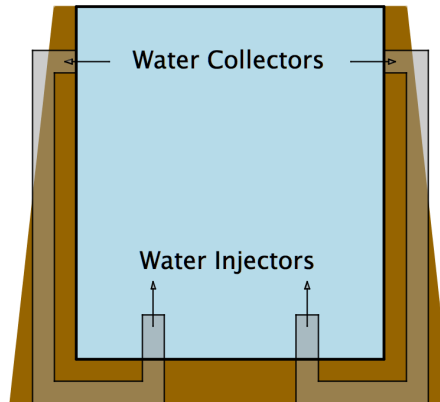
Eutrophication is one of the most important problems of large masses of water and it is caused by high levels of pollutants that reach the waters. These pollutants come mainly from human activities and can cause an excessive phytoplankton growth that leads to undesirable effects like algal blooms. Artificial circulation is a management technique for oxygenating eutrophic water bodies subject to quality problems, such as loss of oxygen, sediment accumulation and algal blooms. It disrupts stratification and minimizes the development of stagnant zones (see figure 1) that may be subject to water quality problems (low levels of oxygen). In our case we are interested in increasing the dissolved oxygen

content in the bottom layers. The main idea of artificial circulation technique consists of



**Figure 1:** Environmental problem

a flow pump takes water from the well aerated upper layers by means of a collector and injecting it into the poorly oxygenated bottom layers, setting up a circulation pattern that prevents stratification (see figure 1). That is, oxygen-poor water from the bottom is circulated to the surface, where oxygenation from the atmosphere and photosynthesis can occur. In this work we want to optimize the volume of pumped water in order to minimize the energy cost and ensure that the concentration level of dissolved oxygen in the bottom layer is over a threshold.



**Figure 2:** Artificial circulation technique

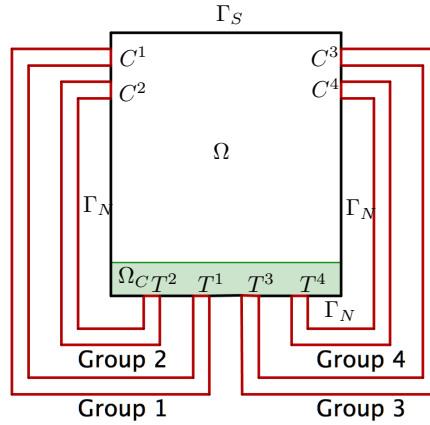
## 2 Mathematical formulation of the control problem

### 2.1 The physical domain

We consider a domain  $\Omega \subset \mathbb{R}^3$  (for instance, a lake) with boundary  $\Gamma = \partial\Omega$  smooth enough such that it can split in four disjoint subsets  $\Gamma = \Gamma_S \cup \Gamma_C \cup \Gamma_T \cup \Gamma_N$ , where:

- $\Gamma_S$  corresponds to the surface of the lake
- $\Gamma_C = \cup_{k=1}^{N_{CT}} C^k$  corresponds to the part of the boundary where the collectors  $C^k$ ,  $k = 1, \dots, N_{CT}$ , are located.
- $\Gamma_T = \cup_{k=1}^{N_{CT}} T^k$  corresponds to the part of the boundary where the injectors  $T^k$ ,  $k = 1, \dots, N_{CT}$ , are located.
- $\Gamma_N = \Gamma \setminus (\Gamma_S \cup \Gamma_C \cup \Gamma_T)$ .

Each collector  $C^k$  is joined to a injector  $T^k$  by a pipeline,  $k = 1, \dots, N_{CT}$ . In the next figure we can see an example of geometrical configuration with  $N_{CT} = 4$  pairs of injector-collector.



**Figure 3:** The physical domain

## 2.2 The state equations

We will consider a thermo-hydrodynamic model coupled with a Michaelis-Menten model for the species involved in the eutrophication processes.

### 2.2.1 The thermo-hydrodynamic model: Modified Navier-Stokes equations coupled with the heat equation with nonlinear Stefan-Boltzmann boundary conditions

In order to simulate the dynamics of water warmed by solar radiation we consider the modified Navier-Stokes equations following the Smagorinsky model of turbulence coupled

with the heat equation with nonlinear Stefan-Boltzmann boundary conditions:

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathbf{v} \mathbf{v} - \nabla \cdot \Xi(\mathbf{v}) + \nabla p = \alpha_0(\theta - \theta_0) \mathbf{a}_g, \text{ in } \Omega \times ]0, T[, \\ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \times ]0, T[, \\ \mathbf{v} = \mathbf{v}_S \text{ on } \Gamma_S \times ]0, T[, \\ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_N \times ]0, T[, \\ \mathbf{v} = +g^k(t)/\mu(C^k) \mathbf{n} \text{ on } C^k \times ]0, T[, \text{ for } k = 1, \dots, N_{CT}, \\ \mathbf{v} = -g^k(t)/\mu(T^k) \mathbf{n} \text{ on } T^k \times ]0, T[, \text{ for } k = 1, \dots, N_{CT}, \\ \mathbf{v}(0) = \mathbf{v}_0 \text{ in } \Omega, \end{array} \right. \quad (1)$$

where  $\mathbf{a}_g$  ( $m/s^2$ ) is the gravity acceleration,  $\theta$  ( $K$ ) is the water temperature,  $\theta_0$  is the reference temperature for the boussinesq simplification (we will suppose that it is equal to the initial temperature),  $\alpha_0 = -(\partial\rho/\partial\theta)/\rho$  ( $K^{-1}$ ) is the thermal expansion coefficient,  $\mathbf{v}_S$  ( $m/s$ ) is the water velocity on the surface (satisfies that  $\mathbf{v}_S \cdot \mathbf{n} = 0$ ),  $\mathbf{v}_0$  is the water velocity at the initial time ( $\mathbf{v}_0|_{\partial\Omega} = \mathbf{0}$ ) and, for  $k = 1, \dots, N_{CT}$ ,  $g^k(t) \in H^s(0, T)$  ( $s \geq 1$  is a suitable index) represents the volume of water displaced by pump  $k$  at each time  $t$  ( $g^k(t) > 0, \forall t \in (0, T)$ ). Finally, the term  $\Xi(\mathbf{v})$  is given by

$$\Xi(\mathbf{v}) = \frac{\partial D(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=\epsilon(\mathbf{v})}, \quad \text{with } \epsilon(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + \nabla \mathbf{v}^t), \quad (2)$$

where  $D$  is the following potential function:

$$D(\epsilon) = \nu [\epsilon : \epsilon] + \frac{2}{3} \nu_{tur} [\epsilon : \epsilon]^{3/2}, \quad (3)$$

where  $\mu_{tur}$  ( $m^2$ ) is eddy viscosity. So, if we define  $\beta(\epsilon(\mathbf{v})) = 2\nu + 2\nu_{tur} [\epsilon(\mathbf{v}) : \epsilon(\mathbf{v})]^{1/2}$ ,

$$\begin{aligned} \Xi(\mathbf{v}) &= \frac{\partial D(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=\epsilon(\mathbf{v})} = 2\nu \epsilon(\mathbf{v}) + 2\nu_{tur} [\epsilon(\mathbf{v}) : \epsilon(\mathbf{v})]^{1/2} \epsilon(\mathbf{v}) \\ &= \left( 2\nu + 2\nu_{tur} [\epsilon(\mathbf{v}) : \epsilon(\mathbf{v})]^{1/2} \right) \epsilon(\mathbf{v}) = \beta(\epsilon(\mathbf{v})) \epsilon(\mathbf{v}). \end{aligned} \quad (4)$$

Finally, the water temperature is given by the following nonlinear partial differential equation of parabolic type:

$$\left\{ \begin{array}{l} \frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla \theta - \nabla \cdot (K \nabla \theta) = 0, \text{ in } \Omega \times ]0, T[, \\ \theta = \int_{-T}^T \rho_\epsilon(t - \epsilon - s) \gamma_\theta^k(s) ds, \text{ on } T^k \times ]0, T[, \text{ for } k = 1, \dots, N_{CT}, \\ K \frac{\partial \theta}{\partial \mathbf{n}} = 0, \text{ on } C^k \times ]0, T[, \text{ for } k = 1, \dots, N_{CT}, \\ K \frac{\partial \theta}{\partial \mathbf{n}} = b_1^N (\theta_N - \theta), \text{ on } \Gamma_N \times ]0, T[, \\ K \frac{\partial \theta}{\partial \mathbf{n}} = b_1^S (\theta_S - \theta) + b_2^S (T_r^4 - |\theta|^3 \theta), \text{ on } \Gamma_S \times ]0, T[, \\ \theta(0) = \theta_0, \text{ in } \Omega, \end{array} \right. \quad (5)$$

where, for each  $k = 1, \dots, N_{CT}$ ,

$$\gamma_\theta^k(s) = \begin{cases} \frac{1}{\mu(C^k)} \int_{C^k} \theta_0 d\gamma, & \text{si } s \leq 0, \\ \frac{1}{\mu(C^k)} \int_{C^k} \theta(s) d\gamma, & \text{si } s > 0, \end{cases} \quad (6)$$

is the mean temperature in the collector  $C^k$  and  $\rho_\epsilon$  is the following function:

$$\rho_\epsilon(t) = \begin{cases} \frac{c}{\epsilon} \exp(t^2/(t^2 - \epsilon^2)) & \text{si } |t| < \epsilon, \\ 0 & \text{si } |t| \geq \epsilon, \end{cases} \quad (7)$$

where  $c \in \mathbb{R}$  is such that  $\int_{\mathbb{R}} \rho_1(t) dt = 1$ . We are assuming that the temperature in the injectors are a weighted average over time of the average temperatures in the collectors and the parameter  $0 < \epsilon < T$  represents the technical characteristics of the pipeline. Finally,  $K = \alpha/(\rho c_p)$  ( $m^2/s$ ) is the thermal diffusivity of the water,  $\alpha$  ( $W/(mK)$ ) is the thermal conductivity,  $c_p$  ( $Ws/(grK)$ ) is the specific heat capacity,  $b_1^N$  and  $b_1^S$  ( $m/s$ ) are the coefficients related to convective heat transfer, obtained from following relations:

$$\rho c_p b_1^N = h^N, \quad \rho c_p b_1^S = h^S, \quad (8)$$

where  $h^S$  and  $h^N$  ( $W/(m^2K)$ ) are the convective heat transfer coefficients between air/water and soil/water, respectively.  $b_2^S$  ( $m/(sK^2)$ ) is the coefficient related to radiative heat transfer for water:

$$\rho c_p b_2^S = \sigma_B \epsilon, \quad (9)$$

with  $\sigma_B$  ( $W/(m^2K^4)$ ) the Stefan-Boltzmann constant. In order to compute the radiation temperature  $T_r$  we use the following expression (involving the corresponding solar radiations, albedo and emissivitie):

$$\sigma_B \epsilon T_r^4 = (1 - a) R_{sw,net} + R_{lw,down}, \quad (10)$$

where  $R_{sw,net}(\mathbf{x}, t)$  ( $W/m^2$ ) denotes the net incident shortwave radiation on the surface, and  $R_{lw,down}$  ( $W/m^2$ ) denotes the downwelling longwave radiation.

## 2.2.2 The eutrophication model: Michaelis-Menten kinetics

We consider the following species concentrations:  $u^1$  ( $mg/l$ ) for a generic nutrient (usually, nitrogen and/or phosphorus),  $u^2$  ( $mg/l$ ) for phytoplankton,  $u^3$  ( $mg/l$ ) for zooplankton,  $u^4$  ( $mg/l$ ) for organic detritus, and  $u^5$  ( $mg/l$ ) for dissolved oxygen. Interactions between five previous species can be modelled by the following system of coupled partial differential equations [1]:

$$\begin{cases} \frac{\partial u^i}{\partial t} + \mathbf{v} \cdot \nabla u^i - \nabla \cdot (\mu^i \nabla u^i) = A^i(\mathbf{x}, t, \theta, \mathbf{u}), & \text{in } \Omega \times ]0, T[, \\ \mu^i \frac{\partial u^i}{\partial \mathbf{n}} = 0, & \text{on } (\Gamma_S \cup \Gamma_N \cup \Gamma_C) \times ]0, T[, \\ u^i = \int_{-T}^T \rho_\epsilon(t - \epsilon - s) \gamma_{u^i}^k(s) ds, & \text{on } T^k \times ]0, T[, \text{ for } k = 1, \dots, N_{CT}, \\ u^i(0) = u_0^i, & \text{in } \Omega, \quad i = 1, \dots, 5, \end{cases} \quad (11)$$

where, for each  $k = 1, \dots, N_{CT}$  and for  $i = 1, \dots, 5$ ,

$$\gamma_{u^i}^k(s) = \begin{cases} \frac{1}{\mu(C^k)} \int_{C^k} u_0^i d\gamma, & \text{si } s \leq 0, \\ \frac{1}{\mu(C^k)} \int_{C^k} u^i(s) d\gamma, & \text{si } s > 0, \end{cases} \quad (12)$$

is the Dirichlet boundary condition in the injectors. The reaction term  $\mathbf{A} = (A^i) : \Omega \times ]0, T[ \times [\mathbb{R}_+]^6 \rightarrow \mathbb{R}^5$  is defined by the following expression:

$$\mathbf{A}(\mathbf{x}, t, \theta, \mathbf{u}) = \begin{bmatrix} -\frac{C_{nc}L(\mathbf{x}, t, \theta)}{K_N + |u^1|} u^1 u^2 + C_{nc}K_r u^2 + C_{nc}K_{rd}D(\theta)u^4 \\ \frac{L(\mathbf{x}, t, \theta)}{K_N + |u^1|} u^1 u^2 - K_r u^2 - K_{mf} u^2 - \frac{K_z}{K_F + |u^2|} u^2 u^3 \\ \frac{C_{fz}K_z}{K_F + |u^2|} u^2 u^3 - K_{mz} u^3 \\ K_{mf} u^2 + K_{mz} u^3 - K_{rd}D(\theta)u^4 \\ \frac{C_{oc}L(\mathbf{x}, t, \theta)}{K_N + |u^1|} u^1 u^2 - C_{oc}K_r u^2 - C_{oc}K_{rd}D(\theta)u^4 \end{bmatrix}, \quad (13)$$

where  $C_{oc}$ ,  $C_{nc}$ ,  $C_{fz}$ ,  $K_{rd}$ ,  $K_r$ ,  $K_{mf}$ ,  $K_{mz}$ ,  $K_z$ ,  $K_F$ ,  $K_N$  are coefficients related to the interaction between species,  $\mu^i$  ( $m^2/s$ ),  $i = 1, \dots, 5$ , are the diffusion coefficients of each species,  $D$  is the thermic regeneration function for the organic detritus:

$$D(\theta) = 1 + \log(\Theta)(\theta - \theta_0), \quad (14)$$

where  $\Theta$  is the thermic regeneration constant.  $L$  is the luminosity function, given by:

$$L(\mathbf{x}, t, \theta) = \mu (1 + \log(C_t)(\theta - \theta_0)) \frac{I_0(t)}{I_s} e^{-\varphi_1 x_3}. \quad (15)$$

with  $I_0$  the incident light intensity,  $I_s$  the light saturation,  $C_t$  the phytoplankton growth thermic constant,  $\varphi_1$  the light attenuation due to depth, and  $\mu$  the maximum phytoplankton growth rate. All previous coefficients are supposed to be strictly positive.

### 2.3 The optimal control problem

Our main aim is related to mitigating the adverse effects of eutrophication by means of the circulation of water from upper layers (rich in oxygen due to photosynthesis realized by phytoplankton and the contact with atmosphere) to lower areas (poor in oxygen). So, we are interesting in solving the following optimal control problem:

$$(\mathcal{P}) \quad \min\{J(\mathbf{g}) : \mathbf{g} \in \mathcal{U}_{ad}, \frac{1}{\mu(\Omega_C)} \int_{\Omega_C} u^5(t) d\mathbf{x} \in [\lambda^m, \lambda^M]\}, \quad (16)$$

where

$$\mathcal{U}_{ad} \subset \{\mathbf{g} \in \mathcal{U} : c_1 \leq g^k(t) \leq c_2, \forall t \in [0, T], \forall k = 1, \dots, N_{CT}\}, \quad (17)$$

with  $\mathcal{U}$  a suitable functional space, the positive constants  $c_1, c_2$  are related to technological constraints on the pumps,  $\lambda^m$  and  $\lambda^M$  are the minimal and maximal mean concentration allowed for the dissolved oxygen in the control domain  $\Omega_C \subset \Omega$ ,  $J(\mathbf{g})$  is the cost functional given by:

$$J(\mathbf{g}) = \frac{1}{2} \sum_{k=1}^{N_{CT}} \int_0^T g^k(t)^2 dt. \quad (18)$$

Finally,  $(\mathbf{v}, \mathbf{u}, \theta)$  are the solutions of the state system (1), (5) and (11).

### 3 Numerical solution of the optimal control problem

We have used the Characteristics Method for the time discretization of the material derivatives and the Finite Element Method for the space discretizations. For the resolution of the Stokes type equations we have considered a standard mini-element and piecewise linear functions for the eutrophication and thermal models. All the above discretizations have been implemented in the scientific software FreeFem++ [2]. The program is also prepared for using higher order spaces (like Taylor-Hood for Navier Stokes), but the CPU time increases a lot. Finally, the numerical resolution of the constrained optimization problem (obtained from the space-time discretization of the continuous control problem) have been developed with the help of the interior point algorithm IPOPT [5] interfaced with Freefem++.

Next, we present some numerical results that we have obtained considering the computational domain showing in figure 3,  $T = 12$  hours and  $\Delta t = 1$  hour. In figure 4 we can see the optimal control for each pump, we observe that the pumps corresponding with the most superficial collectors are the predominating. In figure 5 we present the evolution of the mean concentration in the control domain of dissolved oxygen. Finally in figures 6 and 7 we show the behavior of dissolved oxygen at 6 and 12 hours.

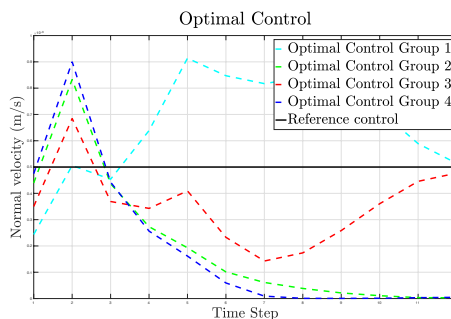


Figure 4: Optimal control

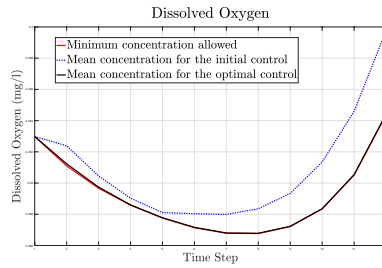


Figure 5: Dissolved oxygen in the control domain

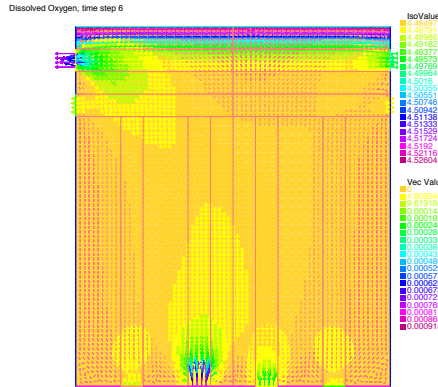


Figure 6: Dissolved oxygen (6 hours)

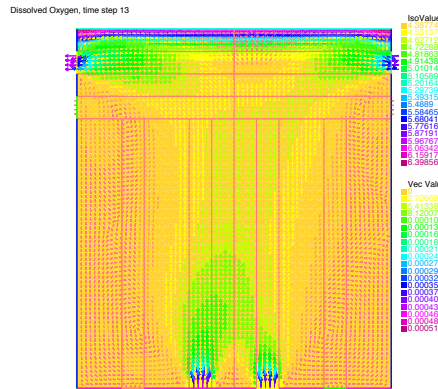


Figure 7: Dissolved oxygen (12 hours)

#### 4 Acknowledgments

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