6th European Conference on Computational Mechanics (ECCM 6)
7th European Conference on Computational Fluid Dynamics (ECFD 7)
11-15 June 2018, Glasgow, UK

PERTURBATION ENERGY CONCEPT FOR STIFFENED SHELLS USING A MIXED-HYBRID FINITE ELEMENT FORMULATION

SIMON KERN[1] AND DIETER DINKLER[2]

[1] Institut für Statik
Beethovenstrasse 51, 38106 Brunswick, Germany
s.kern@tu-braunschweig.de, https://www.tu-braunschweig.de/statik

[2] Institut für Statik
Beethovenstrasse 51, 38106 Brunswick, Germany
d.dinkler@tu-braunschweig.de, https://www.tu-braunschweig.de/statik

Key words: Stability of Shells, Buckling, Stiffened Shells, Perturbation Energy

Abstract. The sensitivity to imperfections of stiffened shells will be evaluated using the perturbation energy concept introduced in [3]. The concept, which is among others described in [5] for a mixed formulation, cannot be used with common displacement based finite elements because the identification of critical states requires a free variation of stresses in the elastic potential. Whereas it is shown that the perturbation energy concept is transferable to a mixed-hybrid formulation and thus can be used in conjunction with elements, which contain only displacement and rotational degrees of freedom on system level. For modelling stiffened structures mixed-hybrid elements provide the advantage that discontinuities of stresses can be depicted because of their local definition. Furthermore balanced shape functions are used to avoid shear and membrane locking. All global degrees of freedom are approximated using bilinear shape functions. Due to the provision of non-linear terms in the kinematic equations, the geometric non-linear behaviour is considered by a theory of moderate rotations. On this basis critical perturbations, which cause a snap through of the ideal structure from a stable pre-buckling to an unstable post-buckling state, are calculated. Subsequently the influence of stiffeners on critical perturbation shapes is studied for stiffened cylindrical shells.

1 INTRODUCTION

Due to their complex buckling behaviour shell structures have been a subject of experimental and theoretical exploration for decades. Therefore many publications concerning stability of thin shell structures are available in the literature. Because of the significant influence on the load bearing capacity, geometrical and physical imperfections are of particular interest. Since the distribution of imperfections of real shell structures is unknown in general and in most civil engineering applications it is not possible or at least not economically viable to determine all imperfections, the Eurocode as major European standard specification in civil engineering defines tolerance quality classes on the basis of a limited number of measurable imperfections. Thereby the semi-empirical concept allows a buckling analysis under consideration of imperfections without any numerical treatment. Of course this approach contains an appropriate safety against collapse and might be too uneconomical for certain applications. Thus, the Eurocode also proposes different approaches using numerical analysis in which equivalent geometrical imperfections are directly part of a FEM model or their influence is estimated by an appropriate reduction factor in conjunction with calculations using a perfect model geometry. Material imperfections e.g. residual stresses or inhomogeneities are also considered by the equivalent geometrical imperfection. However all concepts are based on a classification of the shell quality regarding measurable geometrical imperfections.

Since stiffened shells are widely used in mechanical and civil engineering, these type of structures has also been research objects for many years. A collection of conducted theoretical and experimental studies on stiffened shells can be found in [11]. In comparison to unstiffened shells the complexity of buckling phenomenons increases considerably. As the Eurocode does not contain rules for stiffened shells at least the ECCS-Recommendations [9] offer approaches for particular cases. The design suggestions are partly based on the concepts for unstiffened shells and buckling of plates and widely intersect with the Eurocode. Though theoretical studies on buckling of stiffened shells are considered, e.g. [2] and [12]. Anyway the common opinion among experts is, that due to the advanced development of finite element programs and available computational capacity, guidelines for numerical analysis are more meaningful than design concepts based on manual calculations [10]. Therefore the problem of identifying a reasonable geometrical imperfection, which leads to the greatest decrease in the buckling load of the ideal structure, is even more encountered for stiffened shells.

The perturbation energy concept as described by Dinkler [3] was used by various authors in conjunction with mixed finite elements. Within the concept a non-linear eigenvalue analysis is used to find critical perturbations for a given pre-buckling state. Once the perturbation shape is identified the associated deformation energy describes the imperfection sensitivity as a scalar value. Since this value is dedicated to the analysed shell geometry it has to be normalised to express a general buckling criterion. Given that the perturbation is dominated by bending energy a normalisation can be performed using the bending stiffness of the shell [4].

Without special considerations the analysis of branched structures with mixed finite elements is not readily possible because of discontinuities in the stress gradients. To enable an analysis of stiffened shells, the perturbation energy concept is adopted to a mixed-hybrid formulation. Other advantages are avoidance of locking and lower numerical costs through balanced shape functions and less degrees of freedom respectively.

2 MIXED-HYBRID FINITE ELEMENT FORMULATION

The basis for the used finite element formulation is a non-linear shell theory of moderate rotations. Applying the Kirchhoff-Love theory leads to a description of the strain state using the three displacements of the shell mid-surface u_1, u_2 and u_3. In conjunction with neglecting the non-linear parts of the strain gradient in through-thickness direction a first-approximation theory is developed. The relations for the membrane and bending strains $\alpha_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ using symmetric tensor components apply to

$$\alpha_{\alpha\beta} = \frac{1}{2}\left(u_\alpha|_\beta + u_\beta|_\alpha - 2u_3 b_{\alpha\beta}\right) + \frac{1}{2}\left(-u_{3,\alpha} - b^\lambda_\alpha u_\lambda\right)\left(-u_{3,\beta} - b^\rho_\beta u_\rho\right), \tag{1}$$

$$\kappa_{\alpha\beta} = \frac{1}{2}\left(w_\alpha|_\beta + w_\beta|_\alpha\right). \tag{2}$$

A geometrical non-linearity is only considered in the membrane strains of equation 1. Using the principle of minimum complementary energy a mixed-hybrid finite element can be developed. By introducing an inter element boundary on which only displacements are defined equilibrium constraints are relaxed while continuity requirements for displacements are maintained [8]. For the stated assumptions this leads to the following functional describing the boundary value problem of thin elastic shells

$$\Pi = \{\int_{A_e} \left(n^{\alpha\beta}\alpha_{\beta\alpha} + m^{\alpha\beta}|_\alpha \kappa_{\beta\alpha} - \frac{1}{2}\left(n^{\alpha\beta}F^D_{\alpha\beta\rho\lambda}n^{\rho\lambda} + m^{\alpha\beta}F^B_{\alpha\beta\rho\lambda}m^{\rho\lambda}\right)\right)\sqrt{a}\,\mathrm{d}\Theta^1\,\mathrm{d}\Theta^2 \tag{3}$$
$$- \int_{A_e} \bar{p}^i u_i \sqrt{a}\,\mathrm{d}\Theta^1\,\mathrm{d}\Theta^2 - \int_{S_e}\left(m^{(\tau)}w_{(\tau)} - m^{(\nu)}\tilde{\varphi}_{(\nu)}\right)\mathrm{d}S - \int_{S_{eS}}\left(\bar{n}^i u_i + \bar{m}^{(\nu)}\tilde{\varphi}_{(\nu)}\right)\mathrm{d}S\}$$

Note that indices in parentheses denote physical components and that $\tilde{\varphi}_{(\nu)}$ is the only variable which is merely defined on the inter element boundary resulting from the mixed-hybrid concept. After substituting equations 1 and 2 into 3 and by describing the tangential and normal moments as well as the normal rotation of the element boundary using the corresponding tangential and normal vectors $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ with [1]

$$w_{(\tau)} = \tau^\alpha w_\alpha, \qquad m_{(\nu)} = m^{\alpha\beta}\nu_\alpha\nu_\beta, \qquad m_{(\tau)} = -m^{\alpha\beta}\nu_\alpha\tau_\beta \tag{4}$$

only $n^{\alpha\beta}$ and $m^{\alpha\beta}$ as well as u_α, u_3 and $\tilde{\varphi}_{(\nu)}$ remain as unknowns. Through a transformation of the functional into an incremental form $\Delta\Pi$ using a taylor series and the condition of a vanishing variation

$$\delta\Delta\Pi = 0, \tag{5}$$

a non-linear system of equations is obtained describing the problem in a mixed form. Since equation 3 contains only derivatives of first order the isoparametric concept with quadrilateral elements can be used. Therefore unknown displacement and rotations are approximated using linear shape functions. To avoid the widely known phenomenons of shear and membrane locking for plane stress states and curved elements respectively the normal stresses are described with balanced shape functions as suggested by Pian et al. [7]. Excluding boundary forces this leads to

$$\left[\mathbf{A}(\mathbf{z}_0) + \mathbf{A}^{NL}\left(\frac{1}{2}\Delta\mathbf{z}\right)\right]\Delta\mathbf{z} - \Delta\mathbf{p} = \mathbf{0} \tag{6}$$

with

$$\mathbf{z}^T = [\mathbf{u}_i, \tilde{\boldsymbol{\varphi}}_{(\nu)}, \mathbf{n}^{\alpha\beta}, \mathbf{m}^{\alpha\beta}], \qquad \mathbf{p}^T = [\bar{\mathbf{p}}^i, \mathbf{0}, \bar{\boldsymbol{\alpha}}^{\alpha\beta}, \bar{\boldsymbol{\beta}}^{\alpha\beta}].$$

By linearisation of equation 6 in the framework of the Newton-Raphson method and static condensation of the stress DOF the mixed-hybrid element stiffness matrix is obtained. In a mixed functional $\tilde{\varphi}_{(\nu)}$ is absent since continuity requirements are hold by the conjugated stresses as global DOF satisfying the essential boundary condition.

3 PERTURBATION ENERGY CONCEPT

To point out the changes in the perturbation energy concept that result from the application of the described mixed-hybrid formulation, the mixed form of the concept will be briefly introduced.

Figure 1 shows a load displacement path of an ideal axially compressed cylinder with a bifurcation point B. If a perturbation p_p is applied simultaneously with the axial load, point B transforms into a state M where the structure snaps through. This state is called critical state and the deformation energy of p_p perturbation energy. The main objective is to identify a perturbation associated with the minimum of deformation energy which is necessary to transform a stable pre-buckling state F, called basic state, of an ideal but imperfection sensitive structure into a critical state M.

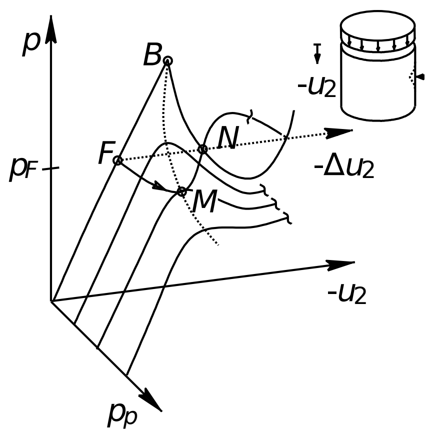


Figure 1: Load displacement behaviour of an axially compressed cylinder [5]

As already mentioned the perturbation energy of a specific structure, denoted by Π_p, depends on geometry and material parameters. To allow a general quantification of the imperfection sensitivity, the energy is normalized by the bending stiffness [4]

$$B = \frac{Et^3}{12\,(1-\nu^2)}, \tag{7}$$

resulting in the dimension less quantity

$$\pi_p = \frac{\Pi_p}{B}. \tag{8}$$

3.1 Identification of critical states using mixed elements

A critical state is defined by a vanishing second variation of the incremental functional $\Delta\Pi$. This is equivalent to a vanishing tangent stiffness and leads, under the assumption of M as snap through point and $\mathbf{z}_F$ as basic state, to the non-linear eigenvalue problem [5]

$$\left[\mathbf{A}(\mathbf{z}_F) + \lambda \mathbf{A}^{NL}(\mathbf{\Phi})\right] \mathbf{\Phi} = \mathbf{0} \tag{9}$$

with

$$\Delta \mathbf{z}_M = \lambda \mathbf{\Phi}. \tag{10}$$

With the critical state $\Delta \mathbf{z}_M$ the perturbation energy with respect to the basic state F follows from the incremental form of the elastic potential

$$\Pi_p^m \quad = \Delta \mathbf{z}_M^T \left[\tfrac{1}{2}\mathbf{A}\left(\mathbf{z}_F\right) + \tfrac{1}{6}\mathbf{A}^{NL}\left(\Delta \mathbf{z}_M\right)\right] \Delta \mathbf{z}_M. \tag{11}$$

The solution of equation 9 can be obtained iteratively by evaluating $\mathbf{A}^{NL}(\mathbf{\Phi})$ with a user specified initial eigenvector and usage of appropriate algorithms for linear eigenvalue problems. An important issue is that the eigenvector generally has to be normalised in every iteration loop. Otherwise the solution will not necessarily converge to the perturbation associated with the minimum of deformation energy. It can be shown that if the inverse vector iteration method is used, the normalisation rule, which is part of the method, leads to an eigenvalue that is directly associated with the deformation energy by

$$\Pi_p^m = \frac{1}{3}\lambda_m^3. \tag{12}$$

The index m denotes that this relation is valid for the mixed formulation. By virtue of equation 12 it is obvious that the smallest eigenvalue is related to the minimum of perturbation energy.

3.2 Application to mixed-hybrid elements

The eigenvalue problem for the mixed-hybrid formulation simply results from the mixed form with a static condensation of the stress DOF. With the vectors $\mathbf{z}_F$ and $\mathbf{\Phi}$ separated into displacement and stress parts equation 9 yields

$$\left[\mathbf{K}_0\left(\mathbf{v}_F, \mathbf{s}_F\right) + \lambda \mathbf{K}_1\left(\mathbf{v}_F, \mathbf{\Phi}_v, \mathbf{\Phi}_s\right) + \lambda^2 \mathbf{K}_2\left(\mathbf{\Phi}_v \cdot \mathbf{\Phi}_v\right)\right] \mathbf{\Phi}_v = \mathbf{0}. \tag{13}$$

In contrast to the mixed form the condensation leads to an quadratic eigenvalue problem. The perturbation energy follows from

$$\Pi_p^h = \Delta \mathbf{v}_M^T \left[\frac{1}{2}\mathbf{K}_0\left(\mathbf{v}_F, \mathbf{s}_F\right) + \frac{1}{6}\mathbf{K}_1\left(\mathbf{v}_F, \Delta\mathbf{v}_M, \Delta\mathbf{s}_M\right) - \frac{1}{6}\mathbf{K}_2\left(\Delta\mathbf{v}_M \cdot \Delta\mathbf{v}_M\right)\right] \Delta\mathbf{v}_M. \tag{14}$$

Essential for an application of the perturbation energy concept to the mixed-hybrid form is the normalisation in every iteration loop. By ensuring

$$\mathbf{\Phi}_v^T \left[\frac{1}{2}\mathbf{K}_0\left(\mathbf{v}_F,\ \mathbf{s}_F\right) + \lambda_h \frac{1}{6}\mathbf{K}_1\left(\mathbf{v}_F,\ \mathbf{\Phi}_v,\ \mathbf{\Phi}_s\right) - \lambda_h^2 \frac{1}{6}\mathbf{K}_2\left(\mathbf{\Phi}_v \cdot \mathbf{\Phi}_v\right)\right] \mathbf{\Phi}_v = 1.0 \tag{15}$$

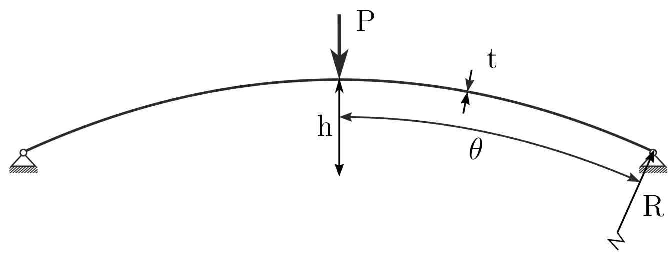
the perturbation energy is again associated to the smallest eigenvalue with

$$\Pi_p^h = \lambda_h^2. \tag{16}$$

4 NUMERICAL RESULTS

4.1 Verification of the mixed-hybrid formulation

To verify the mixed-hybrid approach the arc depicted in figure 2 is calculated using 20 elements (l denotes the length in meridional direction). The material constants are given with $E = 1,035 \cdot 10^6$ N/cm^2 and $\nu = 0$. Comparative results for mixed elements are taken from [6].



$R = 2540$ cm $\qquad l = 2.54$ cm $\qquad t = 5.08$ cm $\qquad h = 38.59$ cm $\qquad \theta = 10°$

Figure 2: Arc under concentrated load

Figure 3 depict the load-displacement path and the perturbation energy respectively. It is shown that both approaches are identical and the mixed-hybrid formulation is able to identify critical states. A classical displacement based formulation fails at this point, because the free variation of stresses as for mixed or mixed-hybrid elements is mandatory to express a possible perturbation of the kinematic relations.

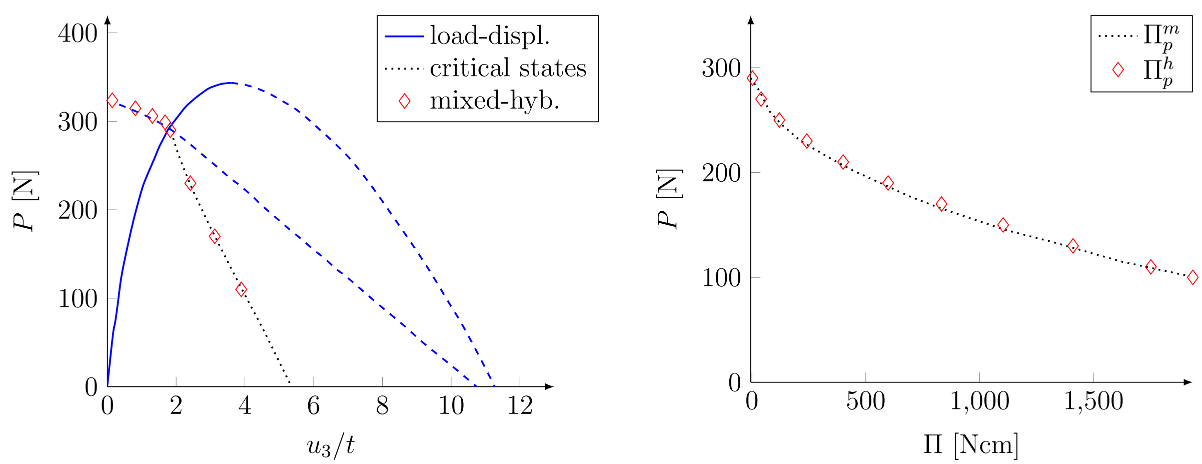
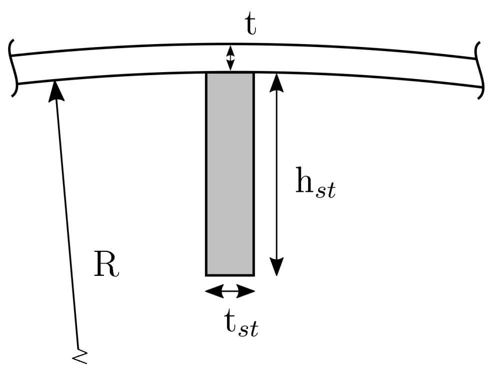


Figure 3: Load-displacement path and perturbation energy for mixed and mixed-hybrid formulation

4.2 Meridional stiffened cylindrical shell

To study the imperfection sensitivity of stiffened shells the perturbation energy for a set of stringer stiffened cylinders is calculated. The geometry and material specifications are listed in Figure 4 (l denotes the meridional length of the cylinder) and are identical for all analysed shells. Only the number of stiffeners is varied.



$$\begin{aligned} R &= 150 \text{ mm} \\ l &= 150 \text{ mm} \\ t &= 1.0 \text{ mm} \\ h_{st} &= 7.5 \text{ mm} \\ t_{st} &= 2.0 \text{ mm} \\ E &= 206000 \text{ N/mm}^2 \\ \nu &= 0.3 \end{aligned}$$

Figure 4: Geometry of stiffener and shell panel

Figure 5 shows the results for 48, 10, 8 and 6 stiffeners which corresponds to angles of 7.5°, 36°, 45° and 60° between two stiffeners. The applied load is normalized by the buckling load p_{cr} which is dedicated to the first bifurcation point. It can be seen that the slope of the paths decrease as the shell gets more stiffened. For relatively large angles the change in perturbation energy for a specific load level is small but still observable while there is a large increase for the cylinder with closely spaced stiffeners. This is equivalent to a reduced sensitivity to imperfections. As the typical perturbation shape known from unstiffened cylinders shows a single bulge near the loaded boundary which fades out in circumferential direction (see [4]), the influence of the stiffeners vanish if the spacing is too large. Figure 6 show the shape denoted by $\Delta \mathbf{z}_M$ in equation 10 for stiffener spacings of 60°, 18° and 7.5° and $p/p_{cr} \approx 0.9$. As the deformation pattern is marginal effected in the first case, in the second one torsional deformation of the stiffeners is more pronounced. Yet radial deformation is still small at the stiffener locations. Whereas in the last case it is clearly visible that the critical shape includes excessive bending of the stiffeners.

A direct comparison with critical loads defined through the ECCS-Recommendation [9] is not possible for linear elastic material models. The design process combines the concepts for unstiffened shells and for the buckling of plates. Therefore critical stresses are always dependent on the yield stress of the material. For this purpose further calculations under consideration of plasticity have to be carried out.

5 CONCLUSION

The perturbation energy concept, as shown in [3] for a mixed finite element formulation, is applicable to a mixed-hybrid formulation. The major difference lies in the generalised eigenvalue problem which transforms into a quadratic one due to static condensation of the stress DOF. Beside an appropriate solution algorithm, this requires a different normalisation to ensure that the smallest eigenvalue corresponds to the minimum of perturbation

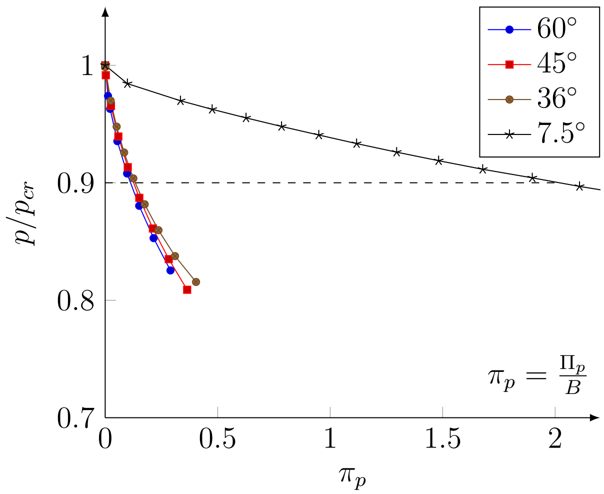


Figure 5: Perturbation energy for different cylinders

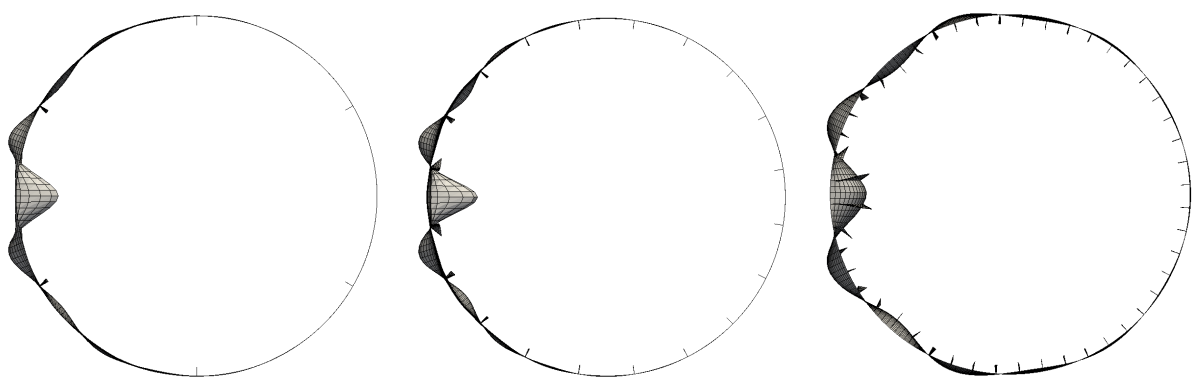


Figure 6: Perturbation shapes for stiffened cylinders at $p \approx 0.9 \cdot p_{cr}$

energy. It is shown that both approaches are equivalent for simple geometries. The benefit of using mixed-hybrid elements is, that branched structures like stiffened shells can be easily analysed. An additional advantage is the general behaviour of mixed-hybrid elements which avoid locking through balanced shape functions and generate less computational costs since the dimension of the global system of equations reduces. However a linearisation in order to solve the quadratic eigenvalue problem doubles the matrix dimensions which oppose the comparatively small system of equations resulting from mixed-hybrid elements. Furthermore an increased occurrence of complex eigenvalues, which are without physical meaning in terms of the perturbation energy concept, may arise. Therefore it can be necessary to compute several eigenvalues till the smallest real one is found.

The application of meridional stiffeners to cylindrical shells reduces the sensitivity to imperfections in general. When the spacing is small enough, so that the perturbation shape include considerably radial displacements of the stiffeners, a distinct increase of the perturbation energy is noticed.

REFERENCES

[1] Başar, Y. and Krätzig, W. B. *Mechanik der Flächentragwerke: Theorie, Berechnungsmethoden, Anwendungsbeispiele*, Friedr. Vieweg & Sohn, (1985).

[2] Block, D. L., Card, M. F. and Mikulas Jr, M. M. Buckling of eccentrically stiffened orthotropic cylinders. *No. NASA-TN-D-2960*, NASA Langley Research Center (1965).

[3] Dinkler, D. Phenomena in nonlinear dynamic buckling behaviour of elastic structures. *33rd Structures, Structural Dynamics and Materials Conference* (1992), 2399-2406.

[4] Dinkler, D. and Knoke, O. Elasto-Plastic Limit Loads of Cylinder-Cone Configurations. *Journal of Theoretical and Applied Mechanics* (2003), **41**(3):443-457.

[5] Dinkler, D. and Pontow, J. Evaluation of the perturbation sensitivity of composite laminated shells. *International Journal of Structural Stability and Dynamics* (2010), **10**(4):779-790.

[6] Kröplin, B., Dinkler, D. and Hillmann, J. An Energy Perturbation Applied to Nonlinear Structural Analysis. *Comp. Meth. Appl. Mech. Eng.* (1985) **52**:885-897.

[7] Pian, T. H. H. and Sumihara, K. Rational approach for assumed stress finite elements. *Int. J. Num. Meth. Eng.* (1984), **20**(9):1685-1695.

[8] Pian, T. H. and Tong, P. Basis of finite element methods for solid continua. *Int. J. Num. Meth. Eng.* (1969), **1**(1):3-28.

[9] Rotter, J. M. and Schmidt, H. *Buckling of Steel Shells - European Design Recommendations.* European Convention for Constructional Steelwork (ECCS), 5th Edition, (2013).

[10] Schmidt, H. Stabilität stählerner Schalentragwerke. *Stahlbau-Kalender* (2009), Ernst & Sohn, **11**:529-612.

[11] Singer, J., Arbocz, J., Weller, T. *Buckling experiments: experimental methods in buckling of thin-walled structures. Shells, built-up structures, composites and additional topics.*, Wiley, Vol. 2, (2002).

[12] Walker, A. C. and Sridharan, S. Analysis of the behaviour of axially compressed stringer-stiffened cylindrical shells. *Proceedings of the Institution of Civil Engineers* (1980) **69**(2):447-472.