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MODELING AND SIMULATION OF FREE FLUID OVER A POROUS MEDIUM BY MESHLESS METHOD

MAGDALENA MIERZWICZAK[1]

[1]Poznan University of Technology
5 M. Skłodowska-Curie Square
magdalena.mierzwiczak@wp.pl, www.put.poznan.pl/en

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Abstract. The effective viscosity in the Brinkman filtration equation is determined by a numerical simulation. The porous medium is modelled by a bundle of parallel fibers arranged in a regular square array. In the first step, in order to determine the permeability, the flow driven by the pressure gradient in an unbounded porous medium is considered. The flow is longitudinal with respect to fibers. In the second step, to determine the effective viscosity, the Poisseulle flow in a flat layer of porous medium is considered. Comparison of the tangential stress allows to determine the effective viscosity in filtration equation. In the third step, to determine the length of the area where the Brinkman equation is applicable, the flow through a porous medium connected with a pure fluid region is considered The Trefftz method with the special purpose Trefftz functions is applied in numerical simulations.

1 INTRODUCTION

In the surrounding of a boundary of a porous region with a high porosity adjacent to the free flow area, two modeling methods are known. In the first method, the Darcy filtration equation is used to incompressible fluid flow in porous region

$$\nabla P = -\frac{\mu}{k}\mathbf{q} \tag{1}$$

where $\mathbf{q}$ is the macroscopic velocity, P is the pressure, μ is the dynamic viscosity of the fluid, k is the permeability of the porous medium. In the free flow area the Navier - Stokes equations are used. The Beavers - Joseph boundary condition is applied for the tangent component of the velocity vector between this regions [1]. In such modeling a slip constant in the boundary condition appears. In paper [1] this constant was determined in a physical experiment, however, in some papers this constant is determined in the numerical experiment.

In the second modeling method, the Brinkman filtration equation [2] is used for a porous medium with very high porosity in the presence of a free fluid region, or for a wall-bounded porous medium

$$\nabla P = -\frac{\mu}{k}\mathbf{q} + \tilde{\mu}\nabla^2\mathbf{q} \tag{2}$$

where $\tilde{\mu}$ is the effective viscosity of the porous medium. As a matter of fact, this is the Stokes equation for creeping flow with the Darcy resistant term. In contrast to Darcy's equation, Eq. (2) includes the term related to the viscous transfer of momentum. The continuity of the filtration and the free flow velocity is assumed at the contact boundary of the considered areas [2]. In this modeling case, an effective viscosity in the Brinkman equation appears. Almost all authors using the Brinkman equation have supposed that the effective viscosity $\tilde{\mu}$ is equal to the viscosity μ of a pure fluid. However, there are some papers in which this viscosity is determined by numerical simulation [3].

The Beavers-Joseph boundary condition appiers in the firs method can be deduced as a consequence of what is now commonly called the Brinkman equation. In the problem of flow in a channel bounded by a thick porous wall one gets the same solution with the Brinkman equation as with the Darcy equation together with the Beavers-Joseph condition, provided that one identifies the constant α in this condition as $\sqrt{\tilde{\mu}/\mu}$. In this way, experimental or computational data for the constant α can be used for determination of the effective viscosity.

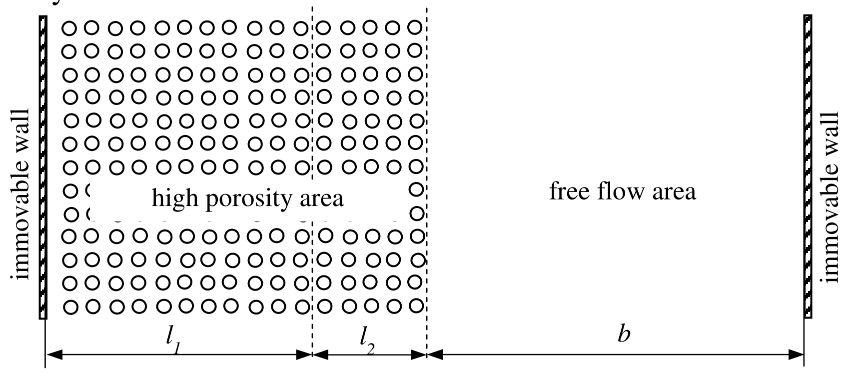

Figure 1. Porous medium with very high porosity region in the presence of free fluid region between two impermeable walls.

In this paper, a combination of these two modeling approaches is proposed. For this purpose, it is assumed that in the porous area with high porosity there is a layer near the boundary in which the flow is governed by the Brinkman filtration equation (2), while in the rest of the area the flow is governed by the Darcy filtration equation (1). Then we consider three subareas. At the boundaries the continuity of the velocity vector is assumed. The problem in such a modeling is connected with the width of this layer l_2 and with the unknown effective viscosity $\tilde{\mu}$. To demonstrating of the proposed new model the following numerical simulation is carried out.

The longitudinal laminar flow (the Poisseulle flow) in a parallel-plate conduit is considered (Fig. 1). The first half of the considered region is a porous medium and the second one pure fluid region. The porous medium is modeled as a bundle of parallel fibres arranged in a square array. The purpose of the present consideration is determination of the effective viscosity $\tilde{\mu}$ in the Brinkman equation and the width of the layer l_2 in which this equation is to be used. But for do this; the permeability of the porous medium is required. Then to determine the permeability the flow with the pressure gradient in unbounded porous medium was considered. Numerical simulations are conducted using the meshless methods: the special purpose Trefftz functions (SPTF) [4].

Let us consider a plane infinite channel adjoining with a plane infinite porous region in which flow is governed by the Darcy filtration equation

$$\frac{dp}{dz} = -\frac{\mu}{k} q_D \text{ for } 0 \le x \le l_1 \tag{3}$$

the Brinkman filtration equation

$$\frac{dp}{dz} = -\frac{\mu}{k} q_B + \tilde{\mu} \frac{d^2 q_B}{dx^2} \text{ for } l_1 \le x \le b \tag{4}$$

and the Stokes equation

$$\frac{dp}{dz} = \mu \frac{d^2 w}{dx^2} \text{ for } b \le x \le 2b \tag{5}$$

For the dimensionless variables

$$W = -\frac{w}{-\frac{b^2}{\mu}\frac{dp}{dz}}, \; X = \frac{x}{b}, \; Q_{B(D)} = -\frac{q_{B(D)}}{-\frac{b^2}{\mu}\frac{dp}{dz}}, \; K = \frac{k}{b^2}, \; \alpha = \frac{\tilde{\mu}}{\mu} \tag{6}$$

the governing Eq. (3-5) obtain the form

$$\frac{1}{K} Q_D = 1 \text{ for } 0 \le X \le L_1 \tag{7a}$$

$$\alpha \frac{d^2 Q_B}{dX^2} - \frac{1}{K} Q_B = -1 \text{ for } L_1 \le X \le 1 \tag{7b}$$

$$\frac{d^2 W}{dX^2} = -1 \text{ for } 1 \le X \le 2 \tag{7c}$$

which is solved with following boundary conditions

$$Q_D = Q_B \text{ for } X = L_1 \tag{8a}$$

$$\begin{cases} Q = W \\ \dfrac{dQ}{dX} = \dfrac{dW}{dX} \end{cases} \text{ for } X = 1 \tag{8b}$$

$$W = 0 \text{ for } X = 2 \tag{8c}$$

The exact solutions of the problem (7-8) takes the form

$$Q_B(X) = D_1 \exp\left(\frac{X}{\sqrt{\alpha K}}\right) + D_2 \exp\left(\frac{-X}{\sqrt{\alpha K}}\right) + K \tag{9a}$$

$$W(X) = -\frac{X^2}{2} + C_1 X + C_2 \tag{9b}$$

where $D_1 = \frac{1}{B}\exp\left(\frac{1}{\sqrt{\alpha K}}\right)(1-2K)$; $D_2 = \frac{1}{B}\exp\left(\frac{2L_1+1}{\sqrt{\alpha K}}\right)(2K-1)$;

$C_1 = \frac{1}{B}\left[\exp\left(\frac{2L_1}{\sqrt{\alpha K}}\right) + \exp\left(\frac{2}{\sqrt{\alpha K}}\right)\right]\frac{(1-2K)}{\sqrt{\alpha K}} + 1$; $C_2 = \frac{2}{B}\left[\exp\left(\frac{2L_1}{\sqrt{\alpha K}}\right) + \exp\left(\frac{2}{\sqrt{\alpha K}}\right)\right]\frac{(2K-1)}{\sqrt{\alpha K}}$;

$B = 2\left[\exp\left(\frac{2L_1}{\sqrt{\alpha K}}\right)\left(\frac{1}{\sqrt{\alpha K}} - 1\right) + \exp\left(\frac{2}{\sqrt{\alpha K}}\right)\left(\frac{1}{\sqrt{\alpha K}} + 1\right)\right]$.

2 DETERMINATION OF THE PERMEABILITY

One of the method for determination of the permeability of the porous medium based on the mesurment filtration velocity at the known pressure gradient. In the present paper the longitudinal flow problem is consider by using the special purpose Trefftz functions. The method is semi-analytical. Application of the method gives analytical form of the dimensionless permeability of the porous medium. Only the unknown coefficients of the solution are obtained numerically.

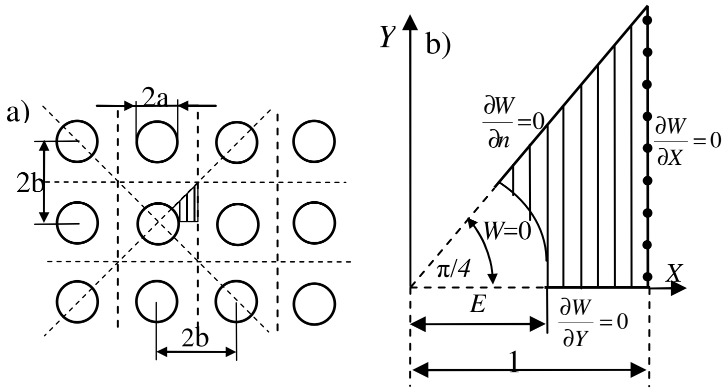

Figure 2: Unbounded porous medium (a), repeated element Ω_F (b).

Let us consider steady, fully developed, laminar, incompressible viscous fluid flow driven by a constant pressure. A porous medium is modelled by a regular array of parallel fibers. The flow is longitudinal with respect to fibers. which are arranged in a regular square (Fig. 2a) arrays. The equation of motion is given by

$$\nabla^2 w = \frac{1}{\mu}\frac{dp}{dz}, \text{ in } \Omega_F \tag{10}$$

where w is the axial velocity, p is the pressure, μ is the viscosity of the fluid, and Ω_F is the fluid domain.

For the dimensionless variables

$$W = -\frac{w}{\frac{b^2}{\mu}\frac{dp}{dz}}, \quad R = \frac{r}{b}, \quad E = \frac{a}{b}. \tag{11}$$

the governing Eq. (10) obtain the form

$$\nabla^2 W = -1. \tag{12}$$

Equation (12) is solvd with the boundary conditions (no slip and symmetry conditon) presented on Fig. 2b.

The exact solution of Eq. (12) can be expressed using the special purpose Trefftz functions

$$W(R,\theta) = -\frac{1}{4}\left(R^2 - E^2\right) + \sum_{k=1}^{N-1} B_k\left(R^{4k} - \frac{E^{8k}}{R^{4k}}\right)\cos(4k\theta) + B_N \ln\left(\frac{R}{E}\right), \tag{13}$$

The unknown coefficients B_k ($k = 1,\ldots, N$) are determined by solving the system of linear equations resulting from satisfying of the boundary condition $\partial W/\partial X = 0$ for $X = 1$, using the boundary collocation technique [5].

Using the Darcy law (1) the longitudinal component of the filtration velocity can be related to the average velocity through the repeated element of the fiber system

$$q_z = -\frac{b^2}{\mu} F(\varphi) \frac{dp}{dz}, \tag{14}$$

where

$$F(\varphi) = \frac{k}{\beta \cdot b^2} = \frac{\iint_{\Omega_F} W(X,Y)\,dXdY}{\beta \cdot \Omega_T} = \frac{1}{2}\tan\left(\frac{\pi}{4}\right)^{-2} \int_0^{\frac{\pi}{4}} \int_E^{\sec(\theta)} W \cdot R\,dR\,d\theta \tag{15}$$

$\beta = 4\tan(\pi/4)$ is parameter of the porous medium.

The dimensionless component of the permeability tensor in the direction parallel to the fibers is a function of the number of collocation points N and can be calculated from

$$F = \left(\frac{E^2}{4} - \frac{1}{12} - \frac{1}{24\cos\left(\frac{\pi}{4}\right)^2} \right) - \frac{\pi E^4}{32\tan\left(\frac{\pi}{4}\right)} + \frac{2}{\tan\left(\frac{\pi}{4}\right)} \sum_{k=1}^{N-1} B_k \left[\frac{H_k E^{8k}}{4k-2} + \frac{G_k}{4k+2} \right] + B_N \left[\ln\left(\frac{1}{E\cos\left(\frac{\pi}{4}\right)} \right) - \frac{3}{2} \right]$$

where $H_k = \dfrac{\sin\left[(1-4k)\frac{\pi}{4}\right]}{(1-4k)\left[\cos\left(\frac{\pi}{4}\right)\right]^{1-4k}}$, $G_k = \dfrac{\sin\left[(1+4k)\frac{\pi}{4}\right]}{(1+4k)\left[\cos\left(\frac{\pi}{4}\right)\right]^{1+4k}}$.

The value of the permeability parameter is calculated as

$$\sigma = \frac{b}{\sqrt{k}} = \frac{1}{\sqrt{\beta \cdot F}}. \tag{16}$$

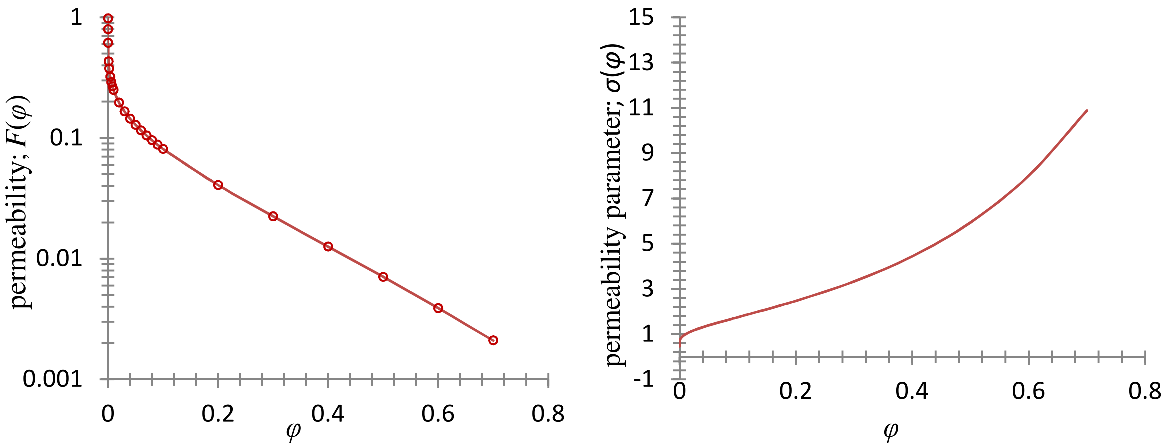

Figure 3. The non-dimensional component of the permeability tensor in the direction parallel to the fibers and the permeability parameter. The numerical results (marks) compared with the analytical solution (line)[19]

The value of the dimensionless permeability as a function of the fibers volume fraction is presented in Fig. 3. The problem was solved for $N = 7$ collocation points on the boundary.

The numerical results (marks) are compared with the analytical presented by Drummond and Tahir [6] (line). Compatibility between the results is great.

3 DETERMINATION OF THE EFFECTIVE VISCOSITY

In the rotational viscometer the fluid viscosity is determined by measurement of the torque of the rotating cylindrical surface. The effective viscosity in the Brinkman equation can be determined in a similar way. In this case the imaginary physical experiment is realized by a numerical simulation. The porous region is placed between two parallel plates. One plate is fixed ($q_z = 0$) whereas the other moves with a constant velocity ($q_z = u$). The determination of the effective viscosity is possible by calculation of the shear stress on the moveable plate.
The Brinkman equation for one-dimensional shear flow in the absence of pressure gradient through the porous layer has the following form

$$\tilde{\mu}\frac{d^2 q_z}{dx^2} - \frac{\mu}{k} q_z = 0, \text{ in } \Omega_F . \tag{17}$$

where q_z is the filtration velocity in direction of fiber axis, $\tilde{\mu}$ is the effective viscosity of the fluid.
The Eq. (17) can be written in the non-dimensional form

$$\frac{d^2 Q}{dX^2} - \frac{1}{\alpha\beta F} Q = 0 \text{ in } \Omega_F . \tag{18}$$

where $Q = \frac{q_z}{u}$, $X = \frac{x}{b}$, $\alpha = \frac{\tilde{\mu}}{\mu}$, $F = \frac{k}{4b^2}$, $H = \frac{h}{b}$.
The exact solution takes the form

$$Q = \frac{\exp\left(\frac{X}{\sqrt{\alpha\beta F}}\right) - \exp\left(-\frac{X}{\sqrt{\alpha\beta F}}\right)}{\exp\left(\frac{H}{\sqrt{\alpha\beta F}}\right) - \exp\left(\frac{-H}{\sqrt{\alpha\beta F}}\right)} = \frac{\sinh\left(\frac{X}{\sqrt{\alpha\beta F}}\right)}{\sinh\left(\frac{H}{\sqrt{\alpha\beta F}}\right)} \tag{19}$$

The tangential stress on the movable plate can be expressed as

$$\tau = \tilde{\mu}\left.\frac{dq_z}{dx}\right|_{x=H} = \tilde{\mu}\frac{u}{b}\frac{1}{\sqrt{\alpha\beta F}}\coth\left(\frac{H}{\sqrt{\alpha\beta F}}\right) \tag{20}$$

On the other hand, the microstructural shear flow problem between two plates

$$\frac{\partial^2 W}{\partial R^2} + \frac{1}{R}\frac{\partial W}{\partial R} + \frac{1}{R^2}\frac{\partial^2 W}{\partial \theta^2} = 0 \tag{21}$$

with no-slip boundary conditions: $W = 0$ on the immovable wall and fibers and $W = 1$ on the movable wall is solved by means of the Trefftz method. The velocity of the flow is approximated by a linear combination of special purpose Trefftz functions.

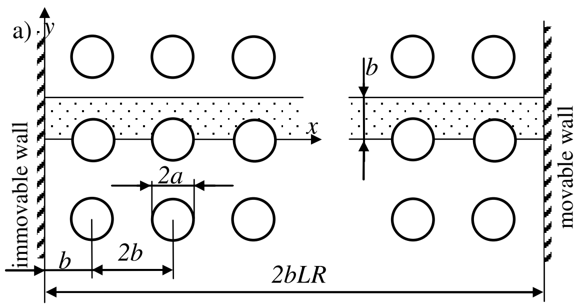

Figure. 4. Wall-bounded porous medium.

Since the array of fibers is streaked and periodic in one direction it is sufficient to consider the problem only in one repeated strip. The repeated strip is divided into smaller elements associated with each of the fibers which are called large finite elements.

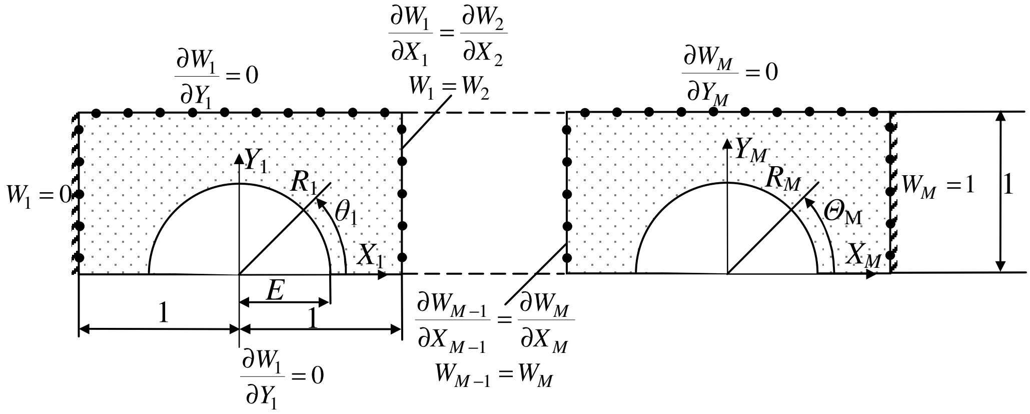

Figure. 5. The symmetry lane of porous medium

For each large finite element the approximate solution is express as

$$W(R,\theta)=C_1\ln\frac{R}{E}+\sum_{k=2}^{N}C_k\left[R^{(k-1)}-\frac{E^{2(k-1)}}{R^{(k-1)}}\right]\cos[(k-1)\theta] \tag{22}$$

which satisfies exactly the governing Eq. (21) and some of the boundary conditions. The unknown coefficients C_k (k = 1,2,…,N) are determined using the boundary collocation technique by satisfying the remaining boundary conditions (in particular the splitting boundary conditions between the large finite elements). The tangential stress on the movable wall can be determined from

$$\tau=\mu\frac{u}{b}\int_0^1\frac{dW}{dX}dY \tag{23}$$

Comparing Eqs. (20) and (23) the following relationship can be written

$$\alpha=\frac{\tilde{\mu}}{\mu}=\frac{c(\varphi,LR,N)\cdot\sqrt{\alpha\beta F}}{\coth\left(\dfrac{H}{\sqrt{\alpha\beta F}}\right)} \tag{24}$$

where $c(\varphi, LR, N) = \int_0^1 \frac{dW}{dX} dY$ and LR denotes the number of fibers rows.

The value of the constant c and the dimensionless effective viscosity as a function of the fiber volume fraction are presented in Fig. 6. The dimensionless effective viscosity was calculated for 5 collocation points on each boundary and 8 rows of fibers. The effective viscosity is smaller than the pure fluid viscosity. The ratio of the effective viscosity to the viscosity of the pure fluid; α at the beginning is decreasing, obtains minimum and then is growing.

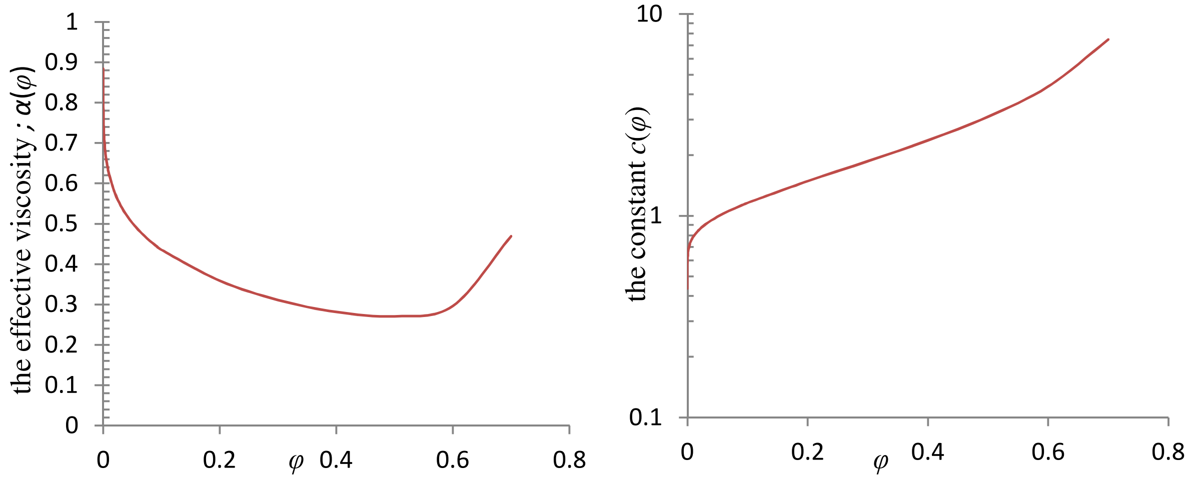

Figure 6. Dimensionless effective viscosity α, and the constant *c*.

2 THE MICROSTRUCTURAL SHEAR FLOW PROBLEM

Let us consider a layer of the porous medium located between two fixed plates (Fig. 1). The microstructural flow in the layer of porous medium is governed by the dimensionless Poisson equation

$$\nabla^2 W = -1 \tag{25}$$

with the no slip boundary conditions: $W = 0$ on the immovable wall.

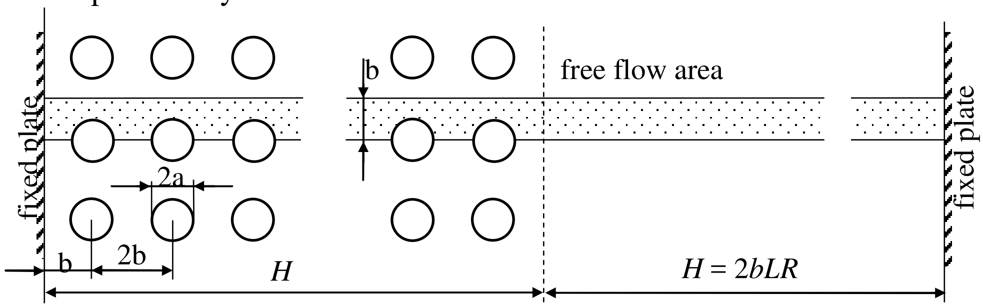

Figure 7. The repeating part of the considered channel

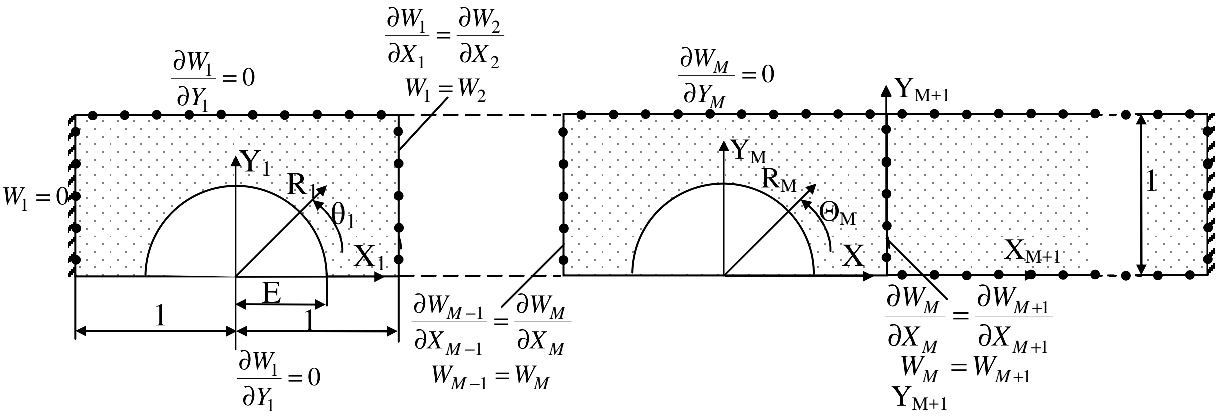

Figure 8. Large finite element with boundaty conditions

Since the array of fibers is streaked and periodic in one direction it is sufficient to consider the problem only in one repeated strip (Fig. 7). In all cases the repeated strip is divided into smaller elements associated with each of the fibers (Fig. 8).

The flow problem is solved by means of the Trefftz method. For each large finite element the special purpose Trefftz functions are used to express the approximate solution

$$W(R,\theta)=-\frac{1}{4}\left(R^2-E^2\right)+\sum_{k=1}^{N-1}B_k\left(R^k-\frac{E^{2k}}{R^k}\right)\cos(k\theta)+B_N\ln\left(\frac{R}{E}\right), \tag{26}$$

Eq. (26) satisfies exactly the governing Eq. (25) and the boundary conditions $W=0$ for $R=E$ and $\partial W/\partial\theta=0$ for $\theta=\{0,\pi\}$.

For the free flow area the approximate solution is express as

$$W^*(X,Y)=-\frac{X^2+Y^2}{4}+\sum_{n=0}^{M}c_nF_n(X,Y)+\sum_{n=1}^{M}d_nG_n(X,Y) \tag{27}$$

where $F_n(X,Y)$ and $G_n(X,Y)$ are the trial functions

$$F_n(X,Y)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\frac{X^{n-2k}Y^{2k}}{(n-2k)!(2k)!};\quad G_n(X,Y)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\frac{X^{n-2k-1}Y^{2k+1}}{(n-2k-1)!(2k+1)!}. \tag{28}$$

The unknown coefficients B_k (k = 1,2,…,N), c_n, d_k (k = 1,2,…,M) are determined using the boundary collocation technique by satisfying the boundary conditions (Fig. 8).

After solution of the microstructural boundary value problem the average value of the velocity derivative can be calculated

$$\omega(X)=\int_0^1\frac{dW}{dX}dY \tag{29}$$

The average velocity derivative ω(X) for the different value of the fiber volume fraction φ are presented in Fig. 9. The problem was calculated for 5 collocation points on each boundary and 8 rows of fibers. Thus, the width of the area H = 16. The width of the layer with the Brinkman equation can be determined on the basis of the obtained results, ω(X). It depends on the fiber volume fraction. The Darcy filtration equation is valid as long as ω(X) is close to zero. For greater porosity, this area is wider and approximately equals to half of the width of the porous area. The width of the Brinkman area decreases as the the fiber volume fraction increases.

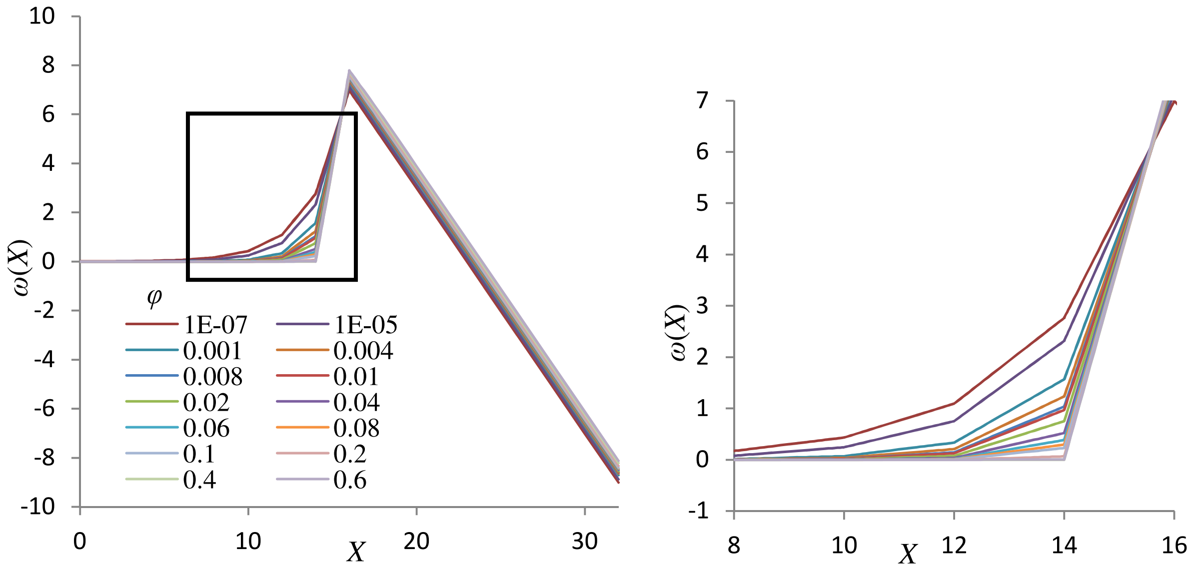

Figure 9. The average of velocity derivative for different value of the fiber volume fraction.

12 CONCLUSIONS

In this paper to avoid calculation of the slip constant in the Beavers - Joseph boundary condition the Brinkman filtration layer between porous medium and pure flow fluid area was introduced. In the Brinkman filtration equation there are two parameters of the porous medium: the permeability and the effective viscosity. These two parameters were determined by numerical simulation of an imaginary physical experiment.

The porous medium was modeled by a parallel bundle of straight fibers arranged in a regular square array. In order to determine the permeability, the flow driven by the pressure gradient in an unbounded porous medium was considered. To determine the effective viscosity, the shear flow in a flat layer of porous medium was considered. To determined the width of the Brinkman filtration area the microstructural shear flow was considered.

In numerical simulations, the Trefftz method with the special-purpose Trefftz functions was applied. The permeability of the porous medium decreases as the the fiber volume fraction increases.The effective viscosity is lower than the viscosity of the pure fluid. The ratio of the effective viscosity to the viscosity of the pure fluid decreases as the porosity decreases.The width of the Brinkman area decreases as the the fiber volume fraction increases.

ACLOWLEGMENTS

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