

# SEMI-ANALYTICAL INFLUENCE COMPUTATION FOR VORTEX SHEET WITH PIECEWISE CONSTANT INTENSITY DISTRIBUTION IN 3D VORTEX METHODS

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**Abstract.** The original numerical scheme is developed for vortex sheet intensity computation for 3D incompressible flow simulation using meshless Lagrangian vortex methods. It is based on boundary condition satisfaction for tangential components of the velocity on the body surface instead of widespread condition for normal components. For the body triangulated surface the corresponding integral equation is approximated by the system of linear algebraic equations, which dimension is doubled number of triangular panels. Vortex layer intensity on the panels assumed to be piecewise-constant.

The coefficients of the matrix are expressed through double integrals over the pairs of panels. When these panels have common edge or common vertex these integrals become improper. In order to compute them it is necessary to exclude the singularities, i. e., to split the integrals into regular and singular parts. Regular parts are expressed through integrals of smooth bounded functions, so they can be integrated numerically with high precision by using Gaussian quadrature formulae. For singular parts exact analytical integration formulae are derived.

The developed approach allows to raise significantly the accuracy of vortex layer intensity computation in vortex method for flow simulation around arbitrary 3D bodies and use arbitrary triangular mesh on body surface including mesh refinement near sharp edges, what is especially important for flow simulation around bodies with complicated geometry.

## 1 INTRODUCTION

The problem of 3D incompressible flow simulation around an immovable body is considered. The governing equations are the Navier — Stokes equations with no-slip boundary conditions on the body surface  $K$  and perturbation decay condition on infinity.

The immovable body is simulated by the influence of the vortex sheet with unknown intensity  $\boldsymbol{\gamma}(\boldsymbol{r}, t)$ , which is placed on the body surface,  $\boldsymbol{r} \in K$ . There are two fundamental

approaches to its intensity computation, which are based on the elimination of the limit values of the normal ( $N$ -scheme) or tangential ( $T$ -scheme) velocity components on the body surface [1].

The flow velocity in the flow domain can be computed by using the Biot — Savart law:

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{V}_\infty + \frac{1}{4\pi} \oint_{S(t)} \frac{\boldsymbol{\Omega}(\boldsymbol{\xi}, t) \times (\mathbf{r} - \boldsymbol{\xi})}{|\mathbf{r} - \boldsymbol{\xi}|^3} dS_\xi + \frac{1}{4\pi} \int_K \frac{\boldsymbol{\gamma}(\boldsymbol{\xi}, t) \times (\mathbf{r} - \boldsymbol{\xi})}{|\mathbf{r} - \boldsymbol{\xi}|^3} dS_\xi, \quad (1)$$

where  $\mathbf{V}_\infty$  is the incident flow velocity;  $S(t)$  is the vortex wake region;  $\boldsymbol{\Omega}(\boldsymbol{\xi}, t) = \text{curl } \mathbf{V}(\boldsymbol{\xi}, t)$  is the vorticity distribution in  $S(t)$ , which assumed to be known;  $\mathbf{n}(\mathbf{r})$  is unit outer normal vector to body surface  $K$ .

Due to the presence of the vortex sheet on the body surface, velocity field, which can be expressed by using (1), has jump discontinuity; its limit value from the body side is [2]

$$\mathbf{V}_-(\mathbf{r}, t) = \mathbf{V}(\mathbf{r}, t) - \frac{\boldsymbol{\gamma}(\mathbf{r}, t) \times \mathbf{n}(\mathbf{r})}{2}, \quad \mathbf{r} \in K.$$

According to the no-slip boundary condition on the body surface, the following equation should be satisfied

$$\mathbf{V}_-(\mathbf{r}, t) = \mathbf{0}, \quad \mathbf{r} \in K. \quad (2)$$

which corresponds to the boundary integral equation with respect to unknown vortex sheet intensity  $\boldsymbol{\gamma}(\mathbf{r}, t)$ .

It is shown in [1] that the sufficient condition for (2) is its satisfaction only for the tangent component of the limit value of velocity field:

$$\mathbf{n}(\mathbf{r}) \times (\mathbf{V}_-(\mathbf{r}, t) \times \mathbf{n}(\mathbf{r})) = \mathbf{0}.$$

It leads to the integral equation of the 2-nd kind

$$\frac{\mathbf{n}(\mathbf{r})}{4\pi} \times \left( \int_K \frac{\boldsymbol{\gamma}(\boldsymbol{\xi}, t) \times (\mathbf{r} - \boldsymbol{\xi})}{|\mathbf{r} - \boldsymbol{\xi}|^3} \times \mathbf{n}(\mathbf{r}) dS_\xi \right) - \frac{\boldsymbol{\gamma}(\mathbf{r}, t) \times \mathbf{n}(\mathbf{r})}{2} = \mathbf{f}(\mathbf{r}, t), \quad \mathbf{r} \in K, \quad (3)$$

where

$$\mathbf{f}(\mathbf{r}, t) = -\mathbf{n}(\mathbf{r}) \times \left( \left( \mathbf{V}_\infty + \frac{1}{4\pi} \int_{S(t)} \frac{\boldsymbol{\Omega}(\boldsymbol{\xi}, t) \times (\mathbf{r} - \boldsymbol{\xi})}{|\mathbf{r} - \boldsymbol{\xi}|^3} dS_\xi \right) \times \mathbf{n}(\mathbf{r}) \right)$$

is known vector function. It expresses the vortex wake and the incident flow influence. An efficient strategy for the vortex wake representation in vortex loop usage. The corresponding algorithms are described in [3].

Note, that the kernel of the equation (3) is unbounded when  $|\mathbf{r} - \boldsymbol{\xi}| \rightarrow 0$ , so in order to solve it numerically with rather high accuracy the following assumptions are introduced:

1. The body surface is discretized into  $N$  triangular cells  $K_i$  with areas  $A_i$  and unit normal vectors  $\mathbf{n}_i$ ,  $i = 1, \dots, N$ . Such cells are usually called ‘panels’ in vortex methods.
2. The unknown vortex sheet intensity on the  $i$ -th panel is constant vector  $\boldsymbol{\gamma}_i$ ,  $i = 1, \dots, N$ , which lies in the plane of the  $i$ -th panel, *i. e.*,  $\boldsymbol{\gamma}_i \cdot \mathbf{n}_i = 0$ .

3. The integral equation (3) is satisfied on average over the panel, or, the same, its residual is orthogonal to the basis function which is equal to the 1 on the  $j$ -th panel and equal to 0 on all other panels. Such approach is mentioned in [1], its implementation for 2D problems is described in [4], the necessary formulae for computations (for more general case when vortex sheet intensity along the panel in 2D case is linearly distributed) can be found in [5]. In case of complicated airfoil shape with essentially non-uniform discretization this approach shows rather high efficiency [6].

According to these assumptions the discrete analogue of the equation (3) can be derived:

$$\frac{1}{4\pi A_i} \sum_{j=1}^N \int_{K_i} \left( \int_{K_j} \mathbf{n}_i \times \left( \frac{\boldsymbol{\gamma}_j \times (\mathbf{r} - \boldsymbol{\xi})}{|\mathbf{r} - \boldsymbol{\xi}|^3} \times \mathbf{n}_i \right) dS_{\boldsymbol{\xi}} \right) dS_r - \frac{\boldsymbol{\gamma}_i \times \mathbf{n}_i}{2} = \frac{1}{A_i} \int_{K_i} \mathbf{f}(\mathbf{r}, t) dS_r, \quad i = 1, \dots, N. \quad (4)$$

To write down (4) in the form of a linear algebraic system we choose local orthonormal basis on every cell  $(\boldsymbol{\tau}_i^{(1)}, \boldsymbol{\tau}_i^{(2)}, \mathbf{n}_i)$ , where tangent vectors  $\boldsymbol{\tau}_i^{(1)}, \boldsymbol{\tau}_i^{(2)}$  can be chosen arbitrary (in the plane of the cell, orthogonal one to the other) and  $\boldsymbol{\tau}_i^{(1)} \times \boldsymbol{\tau}_i^{(2)} = \mathbf{n}_i$ , so

$$\boldsymbol{\gamma}_i = \gamma_i^{(1)} \boldsymbol{\tau}_i^{(1)} + \gamma_i^{(2)} \boldsymbol{\tau}_i^{(2)}$$

and we can project (4) for every  $i$ -th panel on directions  $\boldsymbol{\tau}_i^{(1)}$  and  $\boldsymbol{\tau}_i^{(2)}$  [7, 8].

Note, that the obtained algebraic system has infinite set of solutions; in order to select the unique solution the additional condition for the total vorticity (the integral from the vorticity over the body surface) should be satisfied:

$$\int_K \boldsymbol{\gamma}(\mathbf{r}, t) dS_r = \mathbf{0},$$

which also should be written down in the discretized form.

The resulting algebraic system now is overdetermined, it should be regularized, for example, similarly to [2] by introducing the regularization vector  $\mathbf{R} = (R_1, R_2, R_3)^T$ :

$$\begin{aligned} \frac{1}{4\pi A_i} \boldsymbol{\tau}_i^{(1)} \cdot \left( \sum_{j=1}^N \gamma_j^{(1)} \boldsymbol{\nu}_{ij}^{(1)} + \sum_{j=1}^N \gamma_j^{(2)} \boldsymbol{\nu}_{ij}^{(2)} \right) - \frac{\gamma_i^{(2)}}{2} + \mathbf{R} \cdot \boldsymbol{\tau}_i^{(2)} &= \frac{b_i^{(1)}}{A_i}, \\ \frac{1}{4\pi A_i} \boldsymbol{\tau}_i^{(2)} \cdot \left( \sum_{j=1}^N \gamma_j^{(1)} \boldsymbol{\nu}_{ij}^{(1)} + \sum_{j=1}^N \gamma_j^{(2)} \boldsymbol{\nu}_{ij}^{(2)} \right) + \frac{\gamma_i^{(1)}}{2} + \mathbf{R} \cdot \boldsymbol{\tau}_i^{(1)} &= \frac{b_i^{(2)}}{A_i}, \\ \sum_{j=1}^N A_j \left( \gamma_j^{(1)} \boldsymbol{\tau}_j^{(1)} + \gamma_j^{(2)} \boldsymbol{\tau}_j^{(2)} \right) &= \mathbf{0}, \quad i = 1, \dots, N. \end{aligned} \quad (5)$$

Here

$$\boldsymbol{\nu}_{ij}^{(k)} = \int_{K_i} \left( \int_{K_j} \frac{\boldsymbol{\tau}_j^{(k)} \times (\mathbf{r} - \boldsymbol{\xi})}{|\mathbf{r} - \boldsymbol{\xi}|^3} dS_{\boldsymbol{\xi}} \right) dS_r, \quad b_i^{(k)} = \int_{K_i} \boldsymbol{\tau}_i^{(k)} \cdot \mathbf{f}(\mathbf{r}, t) dS_r, \quad k = 1, 2; \quad i, j = 1, \dots, N.$$

The aim of the paper is to develop the numerical algorithm for the integrals in coefficients  $\boldsymbol{\nu}_{ij}^{(k)}$  computation.

## 2 THE COEFFICIENTS CALCULATION IN THE GENERAL CASE

Coefficients  $\boldsymbol{\nu}_{ij}^{(k)}$  in formula (5) should be computed for all the pairs of the cells; note that  $\boldsymbol{\nu}_{ii}^{(k)} = 0$ , because the self-influence of cells is already taken into account through the non-integral term in (4) which corresponds to  $\pm \frac{\gamma_i^{(k)}}{2}$  terms in (5).

For  $i \neq j$  let us denote

$$\boldsymbol{\nu}_{ij}^{(k)} = \boldsymbol{\tau}_j^{(k)} \times \int_{K_i} \left( \int_{K_j} \frac{\mathbf{r} - \boldsymbol{\xi}}{|\mathbf{r} - \boldsymbol{\xi}|^3} dS_\xi \right) dS_r = \mathbf{I}_{ij} \times \boldsymbol{\tau}_j^{(k)}, \quad k = 1, 2, \quad i, j = 1, \dots, N. \quad (6)$$

Integral  $\mathbf{I}_{ij}$  is calculated over the triangular cells  $K_i$  and  $K_j$ , where the  $i$ -th cell we call ‘control’, the  $j$ -th cell – ‘influence’ cell. Note, that this problem in the whole is similar to the evaluation of Galerkin integrals which arise in the Boundary Element Method (BEM, [9]). There are some known approaches for numerical computation of such integrals, for example, Taylor — Duffy method [10]. However, the generality of the mentioned method leads to high computational cost of the numerical algorithm. In the present paper efficient numerical procedures are developed for the equation (4), which make it possible to compute the integrals (6).

Integral  $\mathbf{I}_{ij}$  is ‘skew-symmetric’, *i. e.*,  $\mathbf{I}_{ij} = -\mathbf{I}_{ji}$ . This property on the one hand makes it possible to compute only one half of such coefficients, and on the other hand can be used for error estimation of the integral numerical computation.

The inner integral in (6) over the influence cell  $K_j$

$$\mathbf{J}_j(\mathbf{r}) = - \int_{K_j} \frac{\mathbf{r} - \boldsymbol{\xi}}{|\mathbf{r} - \boldsymbol{\xi}|^3} dS_\xi \quad (7)$$

can be calculated exactly using the computational software of symbolic mathematics *Wolfram Mathematica* and Handbook of integrals [11]. It can be expressed through the known vectors  $\mathbf{s}_k = \mathbf{r}_k^{(j)} - \mathbf{r}$ ,  $k = 1, 2, 3$ , where  $\mathbf{r}$  is the point for which integral (7) is calculated,  $\mathbf{r}_k^{(j)}$  are the vertices of  $K_j$  triangular cell. Denoting unit vectors

$$\mathbf{e}_k^{(j)} = \frac{\mathbf{s}_{k+1} - \mathbf{s}_k}{|\mathbf{s}_{k+1} - \mathbf{s}_k|} = \frac{\mathbf{r}_{k+1}^{(j)} - \mathbf{r}_k^{(j)}}{|\mathbf{r}_{k+1}^{(j)} - \mathbf{r}_k^{(j)}|}, \quad \boldsymbol{\sigma}_k = \frac{\mathbf{s}_k}{|\mathbf{s}_k|}, \quad k = 1, 2, 3,$$

and assuming all the indices to be calculated using a modulus of 3, we obtain

$$\mathbf{J}_j(\mathbf{r}) = \Theta_j \mathbf{n}_j + \boldsymbol{\Psi}_j \times \mathbf{n}_j, \quad j = 1, \dots, N,$$

where  $\Theta_j$  is the solid angle subtended by triangular cell  $K_j$  which can be calculated by using the formula [12]

$$\Theta_j = 2 \arctan \left( \frac{\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_3}{|\mathbf{s}_1| \cdot |\mathbf{s}_2| \cdot |\mathbf{s}_3| + (\mathbf{s}_1 \cdot \mathbf{s}_2) |\mathbf{s}_3| + (\mathbf{s}_2 \cdot \mathbf{s}_3) |\mathbf{s}_1| + (\mathbf{s}_3 \cdot \mathbf{s}_1) |\mathbf{s}_2|} \right),$$

here  $\mathbf{s}_1 \mathbf{s}_2 \mathbf{s}_3$  denotes the scalar triple product of the vectors;

function  $\Psi_j$  is expressed as the following:

$$\Psi_j = \sum_{k=1}^3 \ln \psi_k^{(j)} \mathbf{e}_k,$$

where

$$\psi_k^{(j)} = \begin{cases} \frac{|\mathbf{s}_k|}{|\mathbf{s}_{k+1}|} \frac{1 + \mathbf{e}_k^{(j)} \cdot \boldsymbol{\sigma}_k}{1 + \mathbf{e}_k^{(j)} \cdot \boldsymbol{\sigma}_{k+1}}, & \mathbf{e}_k^{(j)} \cdot \boldsymbol{\sigma}_k \neq -1, \quad \mathbf{e}_k^{(j)} \cdot \boldsymbol{\sigma}_{k+1} \neq -1, \\ \frac{|\mathbf{s}_{k+1}|}{|\mathbf{s}_k|}, & \mathbf{e}_k^{(j)} \cdot \boldsymbol{\sigma}_k = \mathbf{e}_k^{(j)} \cdot \boldsymbol{\sigma}_{k+1} = -1. \end{cases}$$

The outer integral in (6)

$$\mathbf{I}_{ij} = \int_{K_i} \mathbf{J}_j(\mathbf{r}) dS_r \quad (8)$$

can't be expressed exactly in elementary functions, so it is computed by using Gaussian quadrature formulae:

$$\mathbf{I}_{ij} = \int_{K_i} \mathbf{J}_j(\mathbf{r}) dS_r \approx A_i \sum_{p=1}^{N_{GP}} \omega_p \mathbf{J}_j(\boldsymbol{\eta}_p),$$

where  $N_{GP}$  is the number of Gaussian points;  $\omega_p$  are weight coefficients;  $\boldsymbol{\eta}_p$  are the positions of the Gaussian points. Values of  $\omega_p$  and  $\boldsymbol{\eta}_p$  for different values of  $N_{GP}$  for triangular cells can be found, for example, in [13].

General approach, however, has a significant restriction: it can be implemented only for the influence and control cells which are far one from the other. For the cells which have common edge or common vertex ('neighboring cells') the corresponding integral is improper, so Gaussian quadratures become unsuitable. Direct numerical computation of improper integral is a non-trivial problem, so for such cases an original semi-analytical approach is developed.

If the cells have common edge or common vertex, the singularity should be excluded from the  $\mathbf{J}_j(\mathbf{r})$ :

$$\mathbf{J}_j(\mathbf{r}) = \mathbf{J}_j^{\text{reg}}(\mathbf{r}) + \mathbf{J}_j^{\text{sing}}(\mathbf{r}).$$

Here  $\mathbf{J}_j^{\text{reg}}(\mathbf{r})$ , which has the form

$$\mathbf{J}_j^{\text{reg}}(\mathbf{r}) = (\Theta_j(\mathbf{r}) - \Theta_j^{\text{sing}}(\mathbf{r})) \mathbf{n}_j + (\Psi_j(\mathbf{r}) - \Psi_j^{\text{sing}}(\mathbf{r})) \times \mathbf{n}_j,$$

has no singularities and can be easily integrated numerically with high accuracy by using Gaussian quadrature formulae

$$\int_{K_i} \mathbf{J}_j^{\text{reg}}(\mathbf{r}) dS_r \approx A_i \sum_{p=1}^{N_{GP}} \omega_p \mathbf{J}_j^{\text{reg}}(\boldsymbol{\eta}_p).$$

For the improper (singular) integral

$$\int_{K_i} \mathbf{J}_j^{\text{sing}}(\mathbf{r}) dS_r = \left( \int_{K_i} \Theta_j^{\text{sing}}(\mathbf{r}) dS_r \right) \mathbf{n}_j + \left( \int_{K_i} \Psi_j^{\text{sing}}(\mathbf{r}) dS_r \right) \times \mathbf{n}_j$$

the exact analytical quadrature formulae are derived, which are shown below.

### 3 SINGULARITY EXCLUSION IN CASE OF NEIGHBORING CELLS WITH COMMON EDGE

If the cells  $K_i$  and  $K_j$  have common edge with directing unit vector  $\mathbf{e}_3$ , as it is shown in fig. 1, singular terms have the following form (hereinafter the upper index  $(j)$  in unit vectors  $\mathbf{e}_1^{(j)}$ ,  $\mathbf{e}_2^{(j)}$  and  $\mathbf{e}_3^{(j)}$  is omitted):

$$\begin{aligned} \Theta_j^{\text{sing}}(\mathbf{r}) &= 2 \sum_{k=1}^2 \text{atan}(\mathbf{a}_k \mathbf{e}_k \mathbf{e}_3, (\mathbf{e}_3 - \mathbf{e}_k) \cdot (\mathbf{a}_k - \mathbf{e}_k)), \\ \Psi_j^{\text{sing}}(\mathbf{r}) &= \mathbf{e}_3 \ln \left( \frac{m_1}{m_2} \frac{\mathbf{e}_3 \cdot (\mathbf{e}_3 + \mathbf{a}_1)}{\mathbf{e}_3 \cdot (\mathbf{e}_3 - \mathbf{a}_2)} \right) + \sum_{k=1}^2 \mathbf{e}_k \ln \left( \frac{m_k}{m_3} \mathbf{e}_k \cdot (\mathbf{e}_k - \mathbf{a}_k) \right), \end{aligned}$$

where

$$\begin{aligned} m_1 &= |\mathbf{r} - \mathbf{r}_1^{(j)}|, & m_2 &= |\mathbf{r} - \mathbf{r}_3^{(j)}|, & m_3 &= |\mathbf{r}_1^{(j)} - \mathbf{r}_3^{(j)}|, \\ \mathbf{a}_1 &= \frac{\mathbf{r} - \mathbf{r}_1^{(j)}}{m_1}, & \mathbf{a}_2 &= -\frac{\mathbf{r} - \mathbf{r}_3^{(j)}}{m_2}, \end{aligned}$$

function  $\phi = \text{atan}(y, x)$  means the angle  $\phi \in (-\pi, \pi]$ , for which

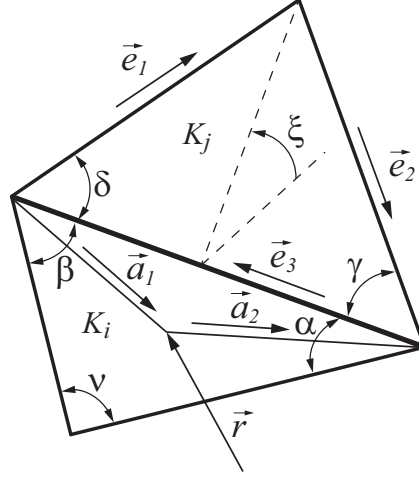
$$\sin \phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}};$$

this function corresponds to ‘traditional’ programming language function  $\text{atan2}(y, x)$ .

Expression for  $\Theta_j^{\text{sing}}$ , as well as all scalar multipliers of  $\Psi_j^{\text{sing}}$ , expressed via logarithms, can be integrated analytically over the cell  $K_i$ , and the resulting formulae are the following:

$$\begin{aligned} \int_{K_i} \Theta_j^{\text{sing}}(\mathbf{r}) dS_r &= -2A_i (q_0(\xi, \alpha, \beta, \mu, \gamma, \lambda) + q_0(\xi, \beta, \alpha, \sigma, \delta, \theta)), \\ \int_{K_i} \Psi_j^{\text{sing}}(\mathbf{r}) dS_r &= A_i (q_{12}(\xi, \alpha, \beta, \mu, \gamma, \lambda) \mathbf{e}_1 + q_{12}(\xi, \beta, \alpha, \sigma, \delta, \theta) \mathbf{e}_2 + q_3(\alpha, \beta) \mathbf{e}_3). \end{aligned}$$

Here auxiliary functions  $q_0$ ,  $q_{12}$  and  $q_3$  are introduced and there are the following expressions for them:


**Figure 1:** Cells  $K_i$  and  $K_j$  in case of having common edge

$$q_0(\xi, \alpha, \beta, \mu, \gamma, \lambda) = \begin{cases} \phi^* + \frac{\sin \gamma \sin \nu}{\sin^2 \mu \sin \alpha} \left( (\cos \beta \sin \gamma - \sin \beta \cos \gamma \cos \xi) \phi^{**} + \right. \\ \left. + \sin \xi \sin \beta \left( \cos^2 \frac{\mu}{2} \ln \frac{\cos \beta/2}{\sin \nu/2} + \sin^2 \frac{\mu}{2} \ln \frac{\sin \beta/2}{\cos \nu/2} + \ln \frac{\cos \lambda/2}{\sin \gamma/2} \right) \right), \\ \quad |\xi| + |\beta - \gamma| > 0, \\ \phi^*, \quad \xi = \beta - \gamma = 0, \end{cases}$$

$$q_{12}(\xi, \alpha, \beta, \mu, \gamma, \lambda) = \begin{cases} -\frac{3}{2} + \frac{1}{\sin \alpha \sin^2 \mu} \left( \ln(1 + \cos \lambda) \sin \beta (\cos \nu + \cos \mu \cos \lambda) + \right. \\ \left. + \ln(1 - \cos \gamma) \sin \nu (\cos \beta + \cos \gamma \cos \mu) + \right. \\ \left. + \ln \frac{\sin \beta}{\sin \nu} \sin \beta (1 - \cos \mu) (\cos \nu - \cos \lambda) - \right. \\ \left. - \sin \nu \sin \beta \left( -2\phi^{**} \sin \xi \sin \gamma + \right. \right. \\ \left. \left. + (\sin \gamma \cos \beta \cos \xi - \sin \beta \cos \gamma) \ln \frac{1 - \cos \nu}{1 + \cos \beta} \right) \right), \quad |\xi| + |\beta - \gamma| > 0, \\ -\frac{3}{2} + \frac{1}{2 \sin \alpha} \left( \cos \nu \sin \gamma \ln(1 + \cos \nu) + \sin \nu \cos \gamma \ln(1 - \cos \gamma) \right), \\ \quad \xi = \beta - \gamma = 0. \end{cases}$$

$$q_3(\alpha, \beta) = \frac{\sin \nu}{\sin \beta} \ln \left( \tan \frac{\alpha}{2} \tan \frac{\nu}{2} \right) + \frac{\sin \nu}{\sin \alpha} \ln \left( \tan \frac{\beta}{2} \tan \frac{\nu}{2} \right) + \ln \left( \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right).$$

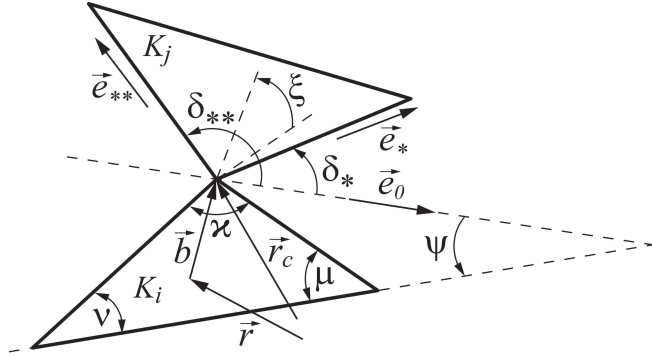
Here it is denoted:

$$\begin{aligned}
 \sigma &= \pi - \arccos(\cos \alpha \cos \delta + \cos \xi \sin \alpha \sin \delta), \\
 \mu &= \pi - \arccos(\cos \beta \cos \gamma + \cos \xi \sin \beta \sin \gamma), \\
 \lambda &= \pi - \arccos(\cos \alpha \cos \gamma - \cos \xi \sin \alpha \sin \gamma), \\
 \theta &= \pi - \arccos(\cos \beta \cos \delta - \cos \xi \sin \beta \sin \delta), \\
 \phi^* &= \text{atan}(\sin \xi \sin \alpha \sin \gamma, 1 - \cos \alpha + \cos \gamma + \cos \lambda), \\
 \phi^{**} &= \text{atan}(\sin \xi \sin \alpha \sin \gamma, 1 + \cos \alpha - \cos \gamma + \cos \lambda).
 \end{aligned}$$

Here  $\alpha$  and  $\beta$  are the angles of the triangle  $K_i$ , which adjoin the common edge of the cells  $K_i$  and  $K_j$ ,  $\nu = \pi - \alpha - \beta$ ;  $\gamma$  and  $\delta$  are the angles of the triangle  $K_j$ , which adjoin the common edge;  $\xi$  is the angle between the planes of the cells  $K_i$  and  $K_j$ .

#### 4 SINGULARITY EXCLUSION IN CASE OF NEIGHBORING CELLS WITH COMMON VERTEX

The scheme for the case when the cells  $K_i$  and  $K_j$  have a common vertex, is shown in fig. 2. Triangle cells  $K_i$  and  $K_j$  determine two planes, which intersection line has directing unit vector  $\mathbf{e}_0$ , which is collinear to vector  $\mathbf{n}_i \times \mathbf{n}_j$  and its direction is chosen such as triple vector product  $\mathbf{e}_0 \mathbf{c} \mathbf{n}_i > 0$ , where  $\mathbf{c}$  connects the center of the cell  $K_i$  with the common vertex.



**Figure 2:** Cells  $K_i$  and  $K_j$  in case of having common vertex

If the cells  $K_i$  and  $K_j$  are coplanar, *i. e.*,  $\mathbf{n}_i \times \mathbf{n}_j = \mathbf{0}$ , then the direction of  $\mathbf{e}_0$  can be chosen arbitrary, but in order to write down the formulae which are suitable for general case it should be equal to

$$\mathbf{e}_0 = \begin{cases} -\mathbf{e}_1, & \mathbf{e}_2 \cdot \mathbf{e}_3 \leq 0, \\ \mathbf{e}_3, & \mathbf{e}_2 \cdot \mathbf{e}_3 > 0, \end{cases}$$

where  $\mathbf{e}_k$  are unit vectors of the sides of the control cell  $K_i$ , when it is assumed that the common vertex is denoted by  $\mathbf{r}_1^{(i)}$ :

$$\mathbf{e}_k = \frac{\mathbf{r}_{k+1}^{(i)} - \mathbf{r}_k^{(i)}}{|\mathbf{r}_{k+1}^{(i)} - \mathbf{r}_k^{(i)}|}, \quad k = 1, 2, 3.$$



As it is shown in fig. 2, the shape of the influence cell  $K_j$  is defined by two unit vectors

$$\mathbf{e}_1^* = \frac{\mathbf{r}_2^{(j)} - \mathbf{r}_1^{(j)}}{|\mathbf{r}_2^{(j)} - \mathbf{r}_1^{(j)}|}, \quad \mathbf{e}_2^* = \frac{\mathbf{r}_3^{(j)} - \mathbf{r}_1^{(j)}}{|\mathbf{r}_3^{(j)} - \mathbf{r}_1^{(j)}|}$$

and two angles

$$\delta_k^* = \angle(\mathbf{e}_0, \mathbf{e}_k^*) = \text{atan}(\mathbf{e}_0 \mathbf{e}_k^* \mathbf{n}_j, \mathbf{e}_0 \cdot \mathbf{e}_k^*), \quad k = 1, 2.$$

If  $\delta_1^* = \pi$  or  $\delta_2^* = \pi$  then vector  $\mathbf{e}_0$  should be reversed, and angles  $\delta_k^*$  should be recomputed once again for new value of  $\mathbf{e}_0$ .

In addition, if  $\delta_1^* \delta_2^* < 0$  and  $|\delta_1^* - \delta_2^*| > \pi$ , vector  $\mathbf{e}_0$  also should be reversed, and angles  $\delta_k^*$  should be recomputed.

Finally, when direction of  $\mathbf{e}_0$  is determined, the angle between planes of  $K_i$  and  $K_j$  can be computed as the following:

$$\xi = \text{atan}(\mathbf{n}_i \mathbf{n}_j \mathbf{e}_0, \mathbf{n}_i \cdot \mathbf{n}_j).$$

In order to compute the regular part  $\mathbf{J}_j^{\text{reg}}(\mathbf{r})$ , the auxiliary 2D basis should be introduced:

$$\begin{aligned} \mathbf{u}_k^* &= \mathbf{e}_0 + \mathbf{e}_k^*, \\ \mathbf{v}_k^* &= (\mathbf{e}_0 + \mathbf{e}_k^*) \times \mathbf{e}_0, \end{aligned}$$

Then

$$\Theta_j^{\text{sing}}(\mathbf{r}) = -2(\Theta_2^* - \Theta_1^* + \tilde{\Theta})$$

can be expressed via vectors  $\mathbf{u}_k^*$ ,  $\mathbf{v}_k^*$  and

$$\mathbf{p}_k = ((\mathbf{b} + \mathbf{e}_0) \cdot \mathbf{u}_k^*) \mathbf{u}_k^* + ((\mathbf{b} + \mathbf{e}_0) \cdot \mathbf{v}_k^*) \mathbf{v}_k^*, \quad m_b = |\mathbf{r}_1^{(i)} - \mathbf{r}|, \quad \mathbf{b} = \frac{\mathbf{r}_1^{(i)} - \mathbf{r}}{m_b},$$

where

$$\Theta_k^* = \begin{cases} \text{atan}(\mathbf{p}_k \cdot \mathbf{v}_k^*, \mathbf{p}_k \cdot \mathbf{u}_k^*), & |\mathbf{p}_k| > 0, \\ \text{atan}(\mathbf{b} \cdot \mathbf{n}_j, \mathbf{b} \mathbf{n}_j \mathbf{e}_0), & |\mathbf{p}_k| = 0, \end{cases}$$

$\tilde{\Theta} = 0$  except the case when  $|\mathbf{p}_1| = 0$ ; in this case  $\tilde{\Theta} = \pi \text{sign } \xi$ .

Then,

$$\Psi_j^{\text{sing}}(\mathbf{r}) = -(\Psi_2^* - \Psi_1^*),$$

where

$$\Psi_k^* = \mathbf{e}_k^* \ln \left( \frac{m_b}{\sqrt{A_j}} \mathbf{e}_k^* \cdot (\mathbf{b} + \mathbf{e}_k^*) \right).$$

These expressions also can be integrated analytically over the cell  $K_i$ :

$$\int_{K_i} \Theta_j^{\text{sing}}(\mathbf{r}) dS_r = -2A_i(q_4(\delta_{**}) - q_4(\delta_*) + \tilde{q}q_4(0)),$$

$$\int_{K_i} \Psi_j^{\text{sing}}(\mathbf{r}) dS_r = -A_i(q_5(\delta_{**})\mathbf{e}_{**} - q_5(\delta_*)\mathbf{e}_*),$$

where

$$\tilde{q} = \begin{cases} 0, & \delta_1^* \delta_2^* \geq 0, \\ -2 \operatorname{sign} \xi, & \delta_1^* \delta_2^* < 0. \end{cases}$$

The auxiliary functions  $q_4$  and  $q_5$  are the following:

$$q_4(\delta) = \frac{q_{41}(\delta) - q_{42}(\delta)}{\sin \psi \sin \kappa},$$

where

$$q_{41}(\delta) = \sin \mu \sin(\nu + \psi) \operatorname{atan}\left(\sin \xi \sin \frac{\delta}{2}, \cos \xi \sin \frac{\delta}{2} + \cos \frac{\delta}{2} \cot \frac{\nu + \psi}{2}\right) -$$

$$- \sin \nu \sin(\mu - \psi) \operatorname{atan}\left(\sin \xi \sin \frac{\delta}{2}, \cos \xi \sin \frac{\delta}{2} + \cos \frac{\delta}{2} \tan \frac{\mu - \psi}{2}\right),$$

$$q_{42}(\delta) = \begin{cases} \frac{\sin \mu \sin \nu \sin \delta}{D} \left( \omega \cos \eta + \frac{\sin \psi \sin \xi}{2} \left( \ln \left( \frac{1 - \cos \lambda \sin \nu}{1 + \cos \theta \sin \mu} \right) - \cos \sigma \ln \left( \tan \frac{\nu}{2} \tan \frac{\mu}{2} \right) \right) \right), & \xi \neq 0 \text{ and } \sin(\delta - \psi) \neq 0, \\ 0, & \xi = \sin(\delta - \psi) = 0; \end{cases}$$

$$q_5(\delta) = q_{51}(\delta) - q_{52}(\delta),$$

where

$$q_{51}(\delta) = -\frac{3 - \ln 2}{2} + \frac{\sin \mu \sin(\nu + \psi) \ln(1 + \cos \theta) - \sin \nu \sin(\mu - \psi) \ln(1 - \cos \lambda)}{\sin \kappa \sin \psi} +$$

$$+ \frac{1}{2} \ln \frac{\sin \nu \sin \mu}{\sin \kappa} - \frac{\cos \nu \sin \mu}{\sin \kappa} \ln \sin \nu - \frac{\cos \mu \sin \nu}{\sin \kappa} \ln \sin \mu,$$

$$q_{52}(\delta) = \begin{cases} \frac{\sin \nu \sin \mu}{D \sin \kappa} \left( \frac{\sin \delta \cos \eta}{\sin \psi} \ln \frac{1 + \cos \theta}{1 - \cos \lambda} - \cos \chi \ln \left( \tan \frac{\nu}{2} \tan \frac{\mu}{2} \right) + \right. \\ \left. + 2\omega \sin \delta \sin \xi + \frac{1}{2} G \ln \frac{\sin \nu}{\sin \mu} \right), & \xi \neq 0 \text{ and } \sin(\delta - \psi) \neq 0, \\ \frac{1}{2} \left( \sin \psi \left( \frac{1 + \cos \nu}{\sin \nu} - \frac{\sin \mu}{1 + \cos \mu} \right) - \cos \psi \ln \frac{1 - \cos \nu}{1 + \cos \mu} \right), & \xi = \sin(\delta - \psi) = 0. \end{cases}$$

Here we denote for simplicity

$$\begin{aligned}
 D &= \sin^2(\delta - \psi) + 4 \sin \delta \sin \psi \left( \sin \delta \sin \psi \cos^2 \frac{\xi}{2} + \cos \delta \cos \psi \right) \sin^2 \frac{\xi}{2}, \\
 G &= \sin 2(\delta - \psi) - 4 \sin \delta \left( \sin \delta \sin 2\psi \cos^2 \frac{\xi}{2} + \cos \delta \cos 2\psi \right) \sin^2 \frac{\xi}{2}, \\
 \omega &= \text{atan} \left( \sin \xi \sin \delta \sin \frac{\kappa}{2}, \cos \xi \sin \delta \sin \left( \nu + \psi + \frac{\kappa}{2} \right) + \cos \frac{\kappa}{2} - \cos \delta \cos \left( \mu - \psi + \frac{\kappa}{2} \right) \right).
 \end{aligned}$$

Here  $\kappa$  is the angles of the triangle  $K_i$ , which adjoins the common vertex of the cells  $K_i$  and  $K_j$ ;  $\mu$  and  $\nu$  are the other angles of the cell  $K_i$ ;  $\xi$  is the angle between the planes of the cells  $K_i$  and  $K_j$ ;  $\psi$  is the angle between  $\mathbf{e}_0$  and the side of the triangle  $K_i$ , which is opposite to the common vertex;  $\delta_*$  and  $\delta_{**}$  are the angles between  $\mathbf{e}_0$  and vectors  $\mathbf{e}_*$  and  $\mathbf{e}_{**}$ , respectively;  $\cos \sigma$ ,  $\cos \lambda$ ,  $\cos \theta$ ,  $\cos \eta$  and  $\cos \chi$  can be calculated by using formulae

$$\begin{aligned}
 \cos \sigma &= \cos \psi \cos \delta + \cos \xi \sin \psi \sin \delta, \\
 \cos \eta &= \cos \delta \sin \psi \cos \xi - \sin \delta \cos \psi, \\
 \cos \chi &= \sin \delta \cos \psi \cos \xi - \cos \delta \sin \psi, \\
 \cos \lambda &= \cos \delta \cos(\mu - \psi) - \cos \xi \sin \delta \sin(\mu - \psi), \\
 \cos \theta &= \cos \delta \cos(\nu + \psi) + \cos \xi \sin \delta \sin(\nu + \psi).
 \end{aligned}$$

## 5 CONCLUSIONS

The derived formulae for  $\mathbf{I}_{ij}$  makes it possible to construct numerical procedure for solving of the discrete analogue of the integral equation for the vortex sheet intensity calculation in the framework of ‘tangent’ approach. It allows to use arbitrary triangular mesh on the body surface and to refine mesh near sharp edges, that is especially important for the flow around 3D wings simulation. Despite the fact that the dimension of the linear system in the developed numerical scheme is twice as large then in traditional implementations of vortex methods, its accuracy is much higher.

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## REFERENCES

- [1] Kempka, S.N., Glass, M.W., Peery, J.S. and Strickland, J.H. Accuracy Considerations for Implementing Velocity Boundary Conditions in Vorticity Formulations, *SANDIA Report* (1996) **SAND96-0583, UC-700**.
- [2] Lifanov, I.K, Poltavskii, L.N. and Vainikko, G.M. *Hypersingular Integral Equations and Their Applications* Taylor & Francis, 2004.
- [3] Shcheglov, G.A. and Dergachev, S.A. Hydrodynamic loads simulation for 3D bluff bodies by using the vortex loops based modification of the vortex particle method. *5th International Conference on Particle-Based Methods — Fundamentals and Applications, PARTICLES 2017* (2017), 725–731.

- [4] Kuzmina, K.S., Marchevsky, I.K. and Moreva, V.S. Vortex sheet intensity computation in incompressible flow simulation around an airfoil by using vortex methods. *Math. Models Comput. Simul.* (2018) **10:3**, 276–287.
- [5] Kuzmina, K.S., Marchevsky, I.K. and Ryatina, E.P. Exact analytical formulae for linearly distributed vortex and source sheets influence computation in 2D vortex methods. *J. Phys.: Conf. Ser.* (2017) **918**, art. 012013.
- [6] Kuzmina, K.S., Marchevsky, I.K. and Moreva, V.S. On vortex sheet intensity computation for airfoils with angle point in vortex methods. *International Journal of Mechanical Engineering and Technology.* (2018) **9:2**, 799–809.
- [7] Marchevsky, I.K. and Shcheglov, G.A. Efficient semi-analytical integration of vortex sheet influence in 3D vortex method. *5th International Conference on Particle-Based Methods — Fundamentals and Applications, PARTICLES 2017* (2017), 703–714.
- [8] Marchevsky, I.K. and Shcheglov, G.A. On integration of vortex sheet influence in 3D flow simulation by using vortex methods. *J. Phys.: Conf. Ser.* (2018). In press.
- [9] Becker, A. *The Boundary Element Method in Engineering: A Complete Course*. New York, McGraw-Hill, 1992.
- [10] Homer Reid, M.T., White, J.K. and Johnson, S.G. Generalized Taylor–Duffy method for efficient evaluation of galerkin integrals in boundary-element method computations. *IEEE Trans. on Antennas and Propagation* (2015) **63:1**, 195–209.
- [11] Gradshteyn, I.S., Ryzhik, I.M., Jeray, A. (Ed) and Zwillinger, D. (Ed) *Gradshteyn and Ryzhik’s Table of Integrals, Series, and Products*. Academic Press, 2007.
- [12] Van Oosterom, A. and Strackee, J. The Solid Angle of a Plane Triangle. *IEEE Trans. Biom. Eng.* (1983) **30:2**, 125–126.
- [13] Zienkiewicz, O.C., Taylor, R.L. and Zhu, J.Z. *The Finite Element Method: Its Basis and Fundamentals*. Oxford, Butterworth-Heinemann, 2013.