

# SENSITIVITY ANALYSIS OF ELASTOPLASTIC STRUCTURAL RESPONSE REGARDING GEOMETRY AND EXTERNAL LOADS

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**Abstract.** This paper is concerned with the determination of design sensitivity information of structures with elastoplastic material behavior in the context of shape optimization. Incidentally, design means geometry and external load parameters. Sensitivity information is provided by a variational approach based on an enhanced kinematic concept that allows a strict separation of geometry and physics. Continuous equations are first derived and subsequently discretized to compute structural response and response sensitivities simultaneously within a finite element framework, which results in a very efficient algorithm to obtain gradient information compared to other numerical methods. The obtained gradients can be used to solve inverse problems utilizing gradient based mathematical optimization, such as Sequential Quadratic Programming (SQP).

## 1 INTRODUCTION

In structural shape optimization, the geometrical shape of a construction is considered and optimized with respect to certain properties. Additionally, in this paper not only the geometric shape but also the shape of the external loads is considered to define the design of a structure. In this context, we are interested in changes of state variables and the objective function caused by variations of geometry and external loads in the reference configuration. Design sensitivity analysis (DSA) provides these sensitivities required for gradient based mathematical optimization. Numerical, analytical or semi-analytical approaches can be used, an overview is given in [22] and [26]. Analytical gradients can either be derived from discretized quantities or from continuous quantities that are discretised subsequently. The material derivative approach (MDA), see [13] and [1], and the domain parametrization approach (DPA), see [10] and [23], are known from literature. In this work, we use a variational approach advocated in [6, 5, 4]. Here, an enhanced concept of continuum mechanics using intrinsic and local coordinates is introduced. As a consequence, geometry and physics are strictly separated and corresponding variations

can be easily computed on continuous level. A subsequent discretization yields the central ingredients of the DSA, i.e. the *pseudo load* and *sensitivity matrices*, see [3].

Sensitivity analysis regarding external loads has been incorporated in [27, 28] and references therein. Here, the position of the external load is chosen as design variable. DSA in the context of inelastic, path-dependent materials requires the additional computation of variations of internal variables that represent the deformation history. Since the late 1980s many publications concentrate on DSA for time-dependent problems. The authors in [7] presented a method to analyze sensitivities of structures with linear elastic, kinematic hardening material behavior, limited to monotonic loading. In [25] a general method of DSA for time-dependent problems is proposed. Here, especially the direct differentiation method (DDM) and the method of adjoint variables (ASM) are compared. The authors recommend using DDM for path-dependent problems, due to the numerical overhead arising in ASM. Similar works are [14], where explicit integrations are used, and [17], where the boundary element approach is used. Tangent operators for DSA of plasticity were firstly derived in [16], see [11] for more details on DPA. The authors in [19] formulated a DSA method for infinitesimal elastoplasticity without consideration of internal variables. In [12] a strategy of using an implicit integration algorithm in connection with a consistent DSA formulation considering internal variables is proposed. A formulation considering finite elastoplastic material behavior for stationary forming processes can be found in [2]. In [29] an effective algorithm based on the aforementioned direct variational approach has been developed. Here, infinitesimal as well as finite elastoplastic deformations and shakedown analysis is considered. The DSA also includes variations of internal variables. The latter work can be seen as basis for DSA in this paper.

The paper is organized as follows. In Sec. 2, we summarize some preliminaries about the notation, kinematics and fundamentals of variational sensitivity analysis. Sec. 3 gives a brief overview on the chosen constitutive model and integration algorithm. We focus on a large strain  $J_2$ -elastoplasticity as can be found in the relative literature, see e.g. [30], [31, 20, 21]. Next, variational sensitivity analysis considering inelastic deformations, as in [29], and further enhancements of the approach, that is considering external loads as variable design parameters are presented in Sec. 4. By rearranging the obtained equations, we end up with expressions only depending on design changes. Finally, we emphasize the findings with a selected numerical example in Sec. 5.

## 2 PRELIMINARIES

This section introduces the main concepts utilized in this paper and gives hints about the notation.

### 2.1 Classical kinematics

We assume an open bounded material body in an undeformed configuration  $\mathcal{K} \subset \mathbb{E}^3$  with Cartesian basis  $\mathbf{E}_i$ . Its boundary is considered piecewise smooth, polyhedral and Lipschitz-continuous  $\Gamma = \partial\mathcal{K}$ , with  $\Gamma = \Gamma_D \cup \Gamma_N$ , where  $\Gamma_D$  and  $\Gamma_N$  denote the Dirichlet and Neumann boundary, respectively, such that  $\Gamma_D \cap \Gamma_N = \emptyset$ . The deformation of the

material body from  $\mathcal{K}$  to a corresponding deformed configuration  $\mathcal{M} \subset \mathbb{E}^3$  with Cartesian basis  $\mathbf{e}_i$  is given by

$$\varphi : \begin{cases} \mathcal{K} \times I_t \rightarrow \mathcal{M} \subset \mathbb{E}^3 \\ (\mathbf{X}, t) \mapsto \mathbf{x} = \varphi(\mathbf{X}, t). \end{cases} \quad (1)$$

For any fixed time  $t \in I_t$ ,  $\varphi$  maps the reference particles  $\mathbf{X}$  from the reference configuration  $\mathcal{K}$  to the spatial coordinates  $\mathbf{x}$  in the deformed configuration  $\mathcal{M}$ .

The corresponding tangent mapping, that is the deformation gradient, and its Jacobian are given by

$$\mathbf{F} = \nabla_{\mathbf{X}} \varphi(\mathbf{X}, t) = \frac{\partial x_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{E}_j \quad \text{and} \quad J = \det \mathbf{F}. \quad (2)$$

## 2.2 Enhanced kinematics

In the context of shape optimization, we do not consider a material body with a fixed reference configuration. Thus, it is convenient to use an enhanced kinematic concept within a general continuum mechanical framework. The authors in [18] formulated an improved viewpoint on the material body using arguments from differential geometry, see [24] and [15] for a classical presentation. Motivated by this theoretical background, see [8] for further details, the idea is to rigorously separate physical quantities into geometry and displacement mappings, see [6, 5]. In detail, a fixed local parameter space  $\mathcal{B}$  with Cartesian basis  $\mathbf{Z}_i$  and local coordinates  $\Theta$  can be introduced.

This idea yields a decomposition of the deformation mapping, Eq. (1) into two independent mappings, i.e. the design dependent geometry mapping  $\kappa$  and the time dependent motion mapping  $\mu$

$$\kappa : \begin{cases} \mathcal{B} \times I_s \rightarrow \mathcal{K} \subset \mathbb{E}^3 \\ (\Theta, s) \mapsto \mathbf{X} = \kappa(\Theta, s) \end{cases}; \quad \mu : \begin{cases} \mathcal{B} \times I_t \rightarrow \mathcal{M} \subset \mathbb{E}^3 \\ (\Theta, t) \mapsto \mathbf{x} = \mu(\Theta, t). \end{cases} \quad (3)$$

for any fixed time  $t \in I_t$  and any design  $s \in I_s$ . The corresponding tangent mappings and their Jacobian are given by

$$\mathbf{K} = \nabla_{\Theta} \kappa(\Theta, s) = \frac{\partial X_i}{\partial \Theta_j} \mathbf{E}_i \otimes \mathbf{Z}_j \quad \text{and} \quad J_k = \det \mathbf{K}, \quad (4)$$

$$\mathbf{M} = \nabla_{\Theta} \mu(\Theta, t) = \frac{\partial x_i}{\partial \Theta_j} \mathbf{e}_i \otimes \mathbf{Z}_j \quad \text{and} \quad J_M = \det \mathbf{M}, \quad (5)$$

which denote the geometry gradient and the motion gradient and their Jacobian, respectively. With these mappings, the deformation mapping, Eq. (1), and the deformation gradient, Eq. (2), can be written as

$$\varphi = \mu \circ \kappa^{-1} \quad \text{and} \quad \mathbf{F} = \mathbf{M} \mathbf{K}^{-1}. \quad (6)$$

**Remark 1.** *The parameter  $s$  is used here as a scalar design variable, which parametrizes the material body in the reference configuration  $\mathcal{K} = \mathcal{K}(s)$ , as well as the material points  $\mathbf{X} = \mathbf{X}(s)$ . A valuable advantage of the aforementioned enhanced kinematic concept is the absence of implicit dependencies. These dependencies arise not until the definition of global equilibrium.*

### 2.3 Variations and derivatives

The variation of any quantity  $(\cdot)(\mathbf{s}; \mathbf{u})$  w.r.t.  $\mathbf{u}$  at fixed design  $\hat{\mathbf{s}}$  is defined as

$$\delta_u(\cdot) = \frac{d}{d\epsilon}(\cdot)(\hat{\mathbf{s}}; \mathbf{u} + \epsilon \delta \mathbf{u}) \Big|_{\epsilon=0} =: (\cdot)'_u \quad (7)$$

and the variation of any quantity  $(\cdot)(\mathbf{s}; \mathbf{u})$  w.r.t.  $\mathbf{s}$  at fixed  $\hat{\mathbf{u}}$  is given by

$$\delta_s(\cdot) = \frac{d}{d\epsilon}(\cdot)(\mathbf{s} + \epsilon \delta \mathbf{s}; \hat{\mathbf{u}}) \Big|_{\epsilon=0} =: (\cdot)'_s. \quad (8)$$

With this notation, the total variation of a quantity  $(\cdot)(\mathbf{u}, \mathbf{s})$ , depending on the deformation  $\mathbf{u}$  and the design  $\mathbf{s}$ , is given by the sum of the partial variations w.r.t.  $\mathbf{u}$  at fixed design  $\hat{\mathbf{s}}$  and w.r.t. to  $\mathbf{s}$  at fixed deformation  $\hat{\mathbf{u}}$

$$(\cdot)'(\mathbf{u}, \mathbf{s}) = (\cdot)'_u(\mathbf{u}, \hat{\mathbf{s}}) + (\cdot)'_s(\hat{\mathbf{u}}, \mathbf{s}) \quad \text{or} \quad \delta(\cdot) = \delta_u(\cdot) + \delta_s(\cdot). \quad (9)$$

### 2.4 Fundamentals of variational sensitivity analysis

To obtain the structural response  $\mathbf{u}$  for any given design  $\mathbf{s}$ , in structural analysis the weak form of equilibrium  $R(\mathbf{u}, \boldsymbol{\eta}) = 0$  is solved for any arbitrary test function  $\boldsymbol{\eta}$ . Any variation in design  $\delta \mathbf{s}$  must not violate equilibrium, i.e. the total variation  $\delta R$  has to vanish. In the elastic case,  $\delta R$  consists of the sum of the partial variations w.r.t. displacements  $\mathbf{u}$  and design  $\mathbf{s}$ , i.e.

$$\delta R = \delta_u R + \delta_s R = k(\boldsymbol{\eta}, \delta \mathbf{u}) + p(\boldsymbol{\eta}, \delta \mathbf{s}) \stackrel{!}{=} 0, \quad (10)$$

with the bilinear forms

$$k(\boldsymbol{\eta}, \delta \mathbf{u}) = \frac{\partial R}{\partial \mathbf{u}} \delta \mathbf{u}; \quad p(\boldsymbol{\eta}, \delta \mathbf{s}) = \frac{\partial R}{\partial \mathbf{s}} \delta \mathbf{s}, \quad (11)$$

representing the *tangent stiffness operator* and the general *tangent pseudo load operator*, respectively, see [3] for details. Finite element discretization leads the discrete residual vector  $\mathbf{R}$

$$R(\mathbf{u}, \boldsymbol{\eta}) \approx R(\mathbf{u}_h, \boldsymbol{\eta}_h) = \boldsymbol{\eta}^\top \mathbf{R}. \quad (12)$$

Thus, we obtain the discrete version of Eq. (10)

$$\delta R = \boldsymbol{\eta}^\top \delta \mathbf{R} = \boldsymbol{\eta}^\top [\mathbf{K} \delta \mathbf{u} + \mathbf{P} \delta \mathbf{s}] = 0 \quad (13)$$

with the discrete condition

$$\delta \mathbf{R} = \mathbf{K} \delta \mathbf{u} + \mathbf{P} \delta \mathbf{s} = \mathbf{0}. \quad (14)$$

Here,  $\mathbf{K}$  is the tangent stiffness matrix and  $\mathbf{P}$  is the pseudo load matrix.

Furthermore, by manipulating Eq. (14) we obtain the *sensitivity matrix*  $\mathbf{S}$  that describes the discrete reaction of the mechanical system to any design perturbation

$$\delta \mathbf{u} = -\mathbf{K}^{-1} [\mathbf{P} \delta \mathbf{s}] = \mathbf{S} \delta \mathbf{s}, \quad \text{with} \quad \mathbf{S} := -\mathbf{K}^{-1} \mathbf{P}. \quad (15)$$

With this, the variation of any arbitrary objective or constraint function  $f(\mathbf{u}, \mathbf{s})$  can be expressed in terms of design changes

$$\delta f = \delta_u f + \delta_s f = \frac{\partial f}{\partial \mathbf{u}} \delta \mathbf{u} + \frac{\partial f}{\partial \mathbf{s}} \delta \mathbf{s} = \left[ \frac{\partial f}{\partial \mathbf{u}} \mathbf{S} + \frac{\partial f}{\partial \mathbf{s}} \right] \delta \mathbf{s}. \quad (16)$$

**Remark 2.** *The enhancement of variational design sensitivity analysis for the elastoplastic case is shown in Sec. 4.*

### 3 CONSTITUTIVE MODEL

The general isotropic constitutive model used in this paper is a large strain extension of the classical  $J_2$ -elastoplasticity by means of a logarithmic strain measure. For details on constitutive equations and the integration method, see e.g. [31], [20] or [21].

#### 3.1 Model equations

The weak form of the equilibrium condition reads

$$R(\mathbf{u}, \boldsymbol{\eta}) = \int_{\mathcal{K}} \mathbf{S} : \delta \mathbf{E} \, dV - \int_{\mathcal{K}} \bar{\mathbf{b}}_o \cdot \boldsymbol{\eta} \, dV - \int_{\partial \mathcal{K}} \bar{\mathbf{t}}_o \cdot \boldsymbol{\eta} \, dA = 0 \quad (17)$$

with  $\mathbf{S}$  and  $\mathbf{E}$  denoting the second Piola-Kirchhoff stress tensor and the Green-Lagrange strain tensor, respectively. Assuming large deformations, the deformation gradient  $\mathbf{F}$  is split multiplicatively into an elastic and a plastic contribution

$$\mathbf{F} = \mathbf{F}^e \mathbf{F}^p. \quad (18)$$

Given a free energy potential of the general form

$$\psi = \psi(\boldsymbol{\varepsilon}^e, \mathbf{h}), \quad (19)$$

the Kirchhoff stress tensor can be calculated as

$$\boldsymbol{\tau} = \rho \frac{\partial \psi}{\partial \boldsymbol{\varepsilon}^e}, \quad (20)$$

with the elastic logarithmic strain  $\boldsymbol{\varepsilon}^e = \frac{1}{2} \ln \mathbf{B}^e = \frac{1}{2} \ln (\mathbf{F}^e \mathbf{F}^{eT})$ . A von Mises type yield function is incorporated

$$\Phi = \sqrt{J_2(\boldsymbol{\tau}_d)} - \sqrt{\frac{2}{3}} Y(\alpha) \leq 0, \quad (21)$$

where  $J_2(\boldsymbol{\tau}_d)$  denotes the second invariant of the deviatoric Kirchhoff stress tensor and  $Y(\alpha) = Y_0 + H \alpha$  defines the elastic threshold in terms of the linear hardening slope  $H$ . As the result of the maximum dissipation principle, we receive the Kuhn-Tucker conditions

$$\dot{\gamma} \geq 0, \quad \Phi \leq 0, \quad \dot{\gamma} \Phi = 0. \quad (22)$$

Here,  $\dot{\gamma}$  denotes the rate of the plastic Lagrange multiplier. The evolution equations for the plastic variables are given by Prandtl-Reuss flow rule and hardening laws

$$\mathbf{F}^e \mathbf{L}^p \mathbf{F}^{e-1} = \dot{\gamma} \frac{\partial \Phi}{\partial \boldsymbol{\tau}}; \quad \mathbf{L}^p = \dot{\mathbf{F}}^p \mathbf{F}^{p-1}; \quad \dot{\alpha} = \sqrt{\frac{2}{3}} \dot{\gamma}. \quad (23)$$

**Remark 3.** *Although the deformation process is quasi-static, we need to introduce a pseudo-time interval  $[t_o, T]$ , so as to capture the deformation history. Thus, all occurring quantities are dependent on pseudo-time  $t$ , e.g.  $\mathbf{F} = \mathbf{F}(t)$ . The pseudo-time dependency has been omitted here for notational convenience.*

### 3.2 Consistent tangent operator

The occurring evolution equations (23) have to be integrated over time. Hence, an implicit Return-Mapping scheme is incorporated. Corresponding to [21], the elastoplastic spatial tangent operator consistent with the Return-Mapping algorithm reads

$$a_{ijkl} = \frac{1}{J} \frac{\partial \tau_{ij}}{\partial F_{kl}} F_{lq} - \sigma_{il} \delta_{jk}. \quad (24)$$

The term  $\frac{\partial \tau_{ij}}{\partial F_{kl}}$  can be obtained utilizing the chain rule

$$\frac{\partial \boldsymbol{\tau}}{\partial \mathbf{F}} = \frac{\partial \boldsymbol{\tau}}{\partial \boldsymbol{\varepsilon}^{e,tr}} : \frac{\partial \boldsymbol{\varepsilon}^{e,tr}}{\partial \mathbf{B}^{e,tr}} : \frac{\partial \mathbf{B}^{e,tr}}{\partial \mathbf{F}}. \quad (25)$$

Note that the superscript *tr* denotes the trial values arising in the elastic predictor step of the Return-Mapping scheme.

**Remark 4.** *An implicit (backward) Euler method is used for time discretization, thus all occurring rate quantities are replaced by their incremental values within the pseudo-time interval  $[t_n, t_{n+1}]$ . The flow vector  $\mathbf{N}$  and the yield surface are replaced by their specific values at the end of the time increment  $t_{n+1}$ . In the following, the subscript  $n+1$  is omitted for notational convenience. Note that all quantities without subscript are quantities at pseudo-time  $t_{n+1}$  except those with subscript  $n$ .*

## 4 SENSITIVITY ANALYSIS

In order to use gradient based mathematical optimization methods, it is necessary to compute the gradients of the objective functional, as well as the gradients of the equality and inequality constraints. The design vector  $\mathbf{s} = [\mathbf{X}, \boldsymbol{\lambda}]^T$  contains the geometry parametrization  $\mathbf{X}$  and the load parametrization  $\boldsymbol{\lambda}$ . Thus, the equations in Sec. 2.4 must be developed further.

### 4.1 Variation of the equilibrium condition

In contrast to the purely elastic case, Eq. (10), we have to consider the partial variation of the weak equilibrium w.r.t. history terms additionally, cf. [29]. This yields

$$\begin{aligned} \delta R &= \delta_u R + \delta_s R + \delta_{h_n} R &= k(\boldsymbol{\eta}, \delta \mathbf{u}) + p(\boldsymbol{\eta}, \delta \mathbf{s}) + h(\boldsymbol{\eta}, \delta \mathbf{h}_n) \\ &= \delta_u R + \delta_X R + \delta_\lambda R + \delta_{h_n} R &= k(\boldsymbol{\eta}, \delta \mathbf{u}) + \hat{p}(\boldsymbol{\eta}, \delta \mathbf{X}) + l(\boldsymbol{\eta}, \delta \boldsymbol{\lambda}) + h(\boldsymbol{\eta}, \delta \mathbf{h}_n), \end{aligned} \quad (26)$$

with the already known tangent stiffness and pseudo load operators  $k(\boldsymbol{\eta}, \delta \mathbf{u})$  and  $\hat{p}(\boldsymbol{\eta}, \delta \mathbf{X})$  and the additional bilinear forms

$$l(\boldsymbol{\eta}, \delta \boldsymbol{\lambda}) = \frac{\partial R}{\partial \boldsymbol{\lambda}} \delta \boldsymbol{\lambda}; \quad h(\boldsymbol{\eta}, \delta \mathbf{h}_n) = \frac{\partial R}{\partial \mathbf{h}_n} \delta \mathbf{h}_n, \quad (27)$$

which represent the influence of the load parameters and the internal history variables and are therefore called the *load* and *history sensitivity operators*, respectively.

#### 4.1.1 Response sensitivity

Similar to Eq. (15) the variation of the structural response  $\delta \mathbf{u}$  can be calculated

$$\delta \mathbf{u} = - \left[ \frac{\partial \mathbf{R}}{\partial \mathbf{u}} \right]^{-1} \left[ \frac{\partial \mathbf{R}}{\partial \mathbf{X}} \delta \mathbf{X} + \frac{\partial \mathbf{R}}{\partial \boldsymbol{\lambda}} \delta \boldsymbol{\lambda} + \frac{\partial \mathbf{R}}{\partial \mathbf{h}_n} \delta \mathbf{h}_n \right]. \quad (28)$$

A matrix representation reads

$$\delta \mathbf{u} = -\mathbf{K}^{-1} [\mathbf{P}_X \delta \mathbf{X} + \mathbf{P}_\lambda \delta \boldsymbol{\lambda} + \mathbf{H} \delta \boldsymbol{\alpha}_n] = -\mathbf{K}^{-1} [\mathbf{P} \delta \mathbf{s} + \mathbf{H} \delta \boldsymbol{\alpha}_n], \quad (29)$$

with  $\mathbf{P} = [\mathbf{P}_X, \mathbf{P}_\lambda]$  and  $\mathbf{P}_\lambda$  and  $\mathbf{H}$  denoting the load and history sensitivity matrices.

**Remark 5.** *Since initially the total variations of the history terms  $\delta \mathbf{h}_0$  are zero at time  $t_0$ , in the first pseudo-time interval  $[t_0, t_1]$  Eq. (29) reduces to*

$$\delta \mathbf{u} = -\mathbf{K}^{-1} \mathbf{P} \delta \mathbf{s} = \mathbf{S} \delta \mathbf{s}, \quad (30)$$

*regardless of whether it is an elastic or a plastic step. The influence of the history terms only appears, if the prior load step has caused plastic deformations, cf. [29].*

#### 4.1.2 Total sensitivity matrix

For any load step, that follows a load step that has caused plastic deformations, we have to take the total variations of the history terms into account, see Eq. (29). Splitting the variations of the history terms into their partial variations w.r.t. displacements and design, we get

$$\delta \mathbf{h}_n = \frac{\partial \mathbf{h}_n}{\partial \mathbf{u}_n} \delta \mathbf{u}_n + \frac{\partial \mathbf{h}_n}{\partial \mathbf{s}} \delta \mathbf{s} = \left[ \frac{\partial \mathbf{h}_n}{\partial \mathbf{u}_n} \mathbf{S}_n + \frac{\partial \mathbf{h}_n}{\partial \mathbf{s}} \right] \delta \mathbf{s} = \mathbf{G}_n \delta \mathbf{s}, \quad (31)$$

where  $\mathbf{S}_n$  denotes the sensitivity matrix of the prior load step. Inserting Eq. (31) into Eq. (29), we receive the total sensitivity matrix  $\mathbf{S}$  including the influence of the plastic variables

$$\delta \mathbf{u} = -\mathbf{K}^{-1} [\mathbf{P} \delta \mathbf{s} + \mathbf{H} \mathbf{G}_n \delta \mathbf{s}] = \mathbf{S} \delta \mathbf{s}. \quad (32)$$

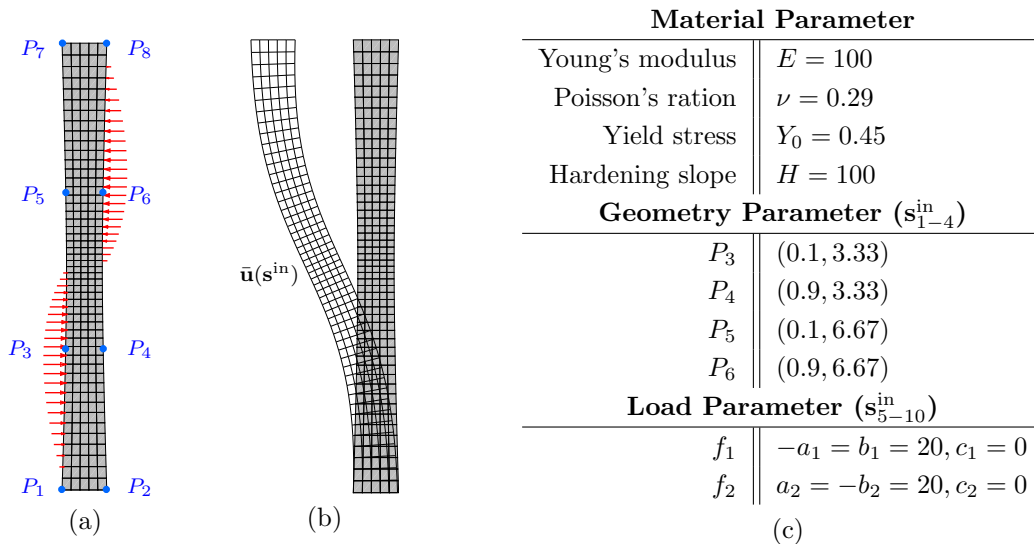


Figure 1: (a) Initial geometry and loading, (b) Deformation, (c) Parameter

## 4.2 Objective and constraints

The gradient of any objective functional  $J$  as well as any constraint function can be computed analogously to the variation of the equilibrium condition

$$\delta J = \delta_{\mathbf{u}} J + \delta_{\mathbf{s}} J + \delta_{\mathbf{h}_n} J = \frac{\partial J}{\partial \mathbf{u}} \delta \mathbf{u} + \frac{\partial J}{\partial \mathbf{s}} \delta \mathbf{s} + \frac{\partial J}{\partial \mathbf{h}_n} \delta \mathbf{h}_n. \quad (33)$$

With Eq. (31) and Eq. (32), we obtain an expression only depending on design changes

$$\delta J = \left[ \frac{\partial J}{\partial \mathbf{s}} + \frac{\partial J}{\partial \mathbf{u}} \mathbf{S} + \frac{\partial J}{\partial \mathbf{h}_n} \mathbf{G}_n \right] \delta \mathbf{s}. \quad (34)$$

## 5 NUMERICAL EXAMPLE

To verify the presented method, we consider the mechanical problem illustrated in Fig. 1 (a). The geometry is given by a NURBS surface with eight control points  $P_1 - P_8$  and the two knot vectors  $U = [0 \ 0 \ 1 \ 1]$  and  $V = [0 \ 0 \ 0 \ 0.5 \ 1 \ 1 \ 1]$ , thus the edges of the surface are linear in  $x$ -direction and cubic in  $y$ -direction. All weight factors are zero, thus the NURBS curves at the edges simplify to B-Splines. The geometry is discretized with  $\text{nel} = 250$  4-node quadrilateral elements. This results in a total amount of  $\text{dof} = 612$  degrees of freedom. The loads  $f_1$  and  $f_2$  are parametrized using a quadratic function of the general form  $f = \lambda (a x^2 + b x + c)$  and applied stepwise to their maximum value and then released ( $\lambda = 0, \dots, 1, \dots, 0$ ), so that only plastic deformations remain after the deformation process. The structural response, that is the deformation  $\bar{\mathbf{u}}(\mathbf{s}^{\text{in}})$  calculated with the initial values of the design variables  $\mathbf{s}^{\text{in}}$ , is displayed as wire frame in Fig. 1 (b). Chosen model parameters are summarized in Fig. 1 (c).



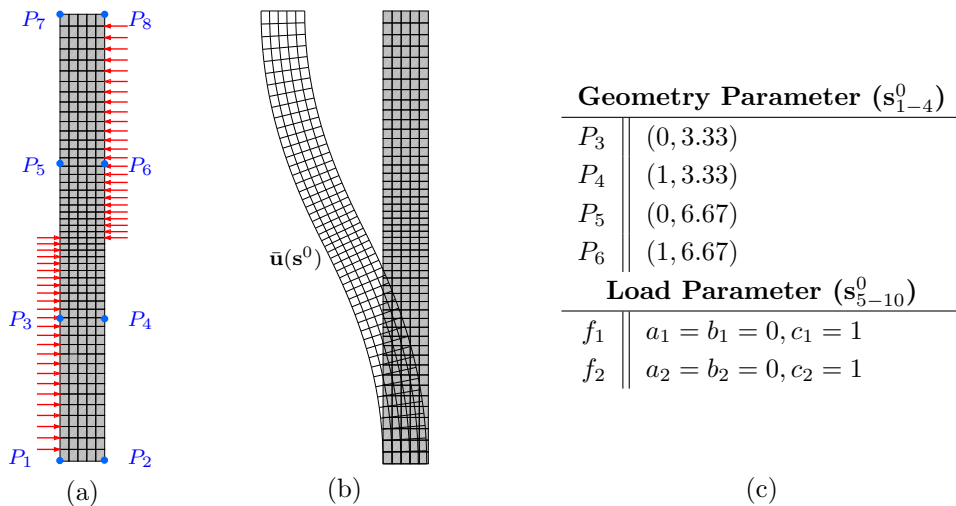


Figure 2: (a) Start geometry and loading, (b) Deformation, (c) Parameter

Starting with a different geometry and load setup, cf. Fig. 2, we want to identify the initial geometry and loading by minimizing the difference between the initial and the current structural response during the optimization process.

Thus, we state the following optimization problem

$$\begin{aligned}
 \min_{\mathbf{s} \in \mathbb{R}^{\text{ndv}}} J(\mathbf{u}(\mathbf{s})) &= \frac{1}{2} \mathbf{a}^\top \mathbf{a}, \quad \text{with } \mathbf{a} = \bar{\mathbf{u}}(\mathbf{s}^{\text{in}}) - \mathbf{u}(\mathbf{s}) \\
 \text{s.t. } s_i^l &\leq s_i \leq s_i^u.
 \end{aligned} \tag{35}$$

The geometric control points  $P_3, P_4, P_5$  and  $P_6$  are chosen as design variables  $s_{1-4}$  and allowed to move in horizontal direction within a relative range of  $[-0.2, 0.2]$ . Additionally, the load parameters  $a_1, b_1$  and  $c_1$ , as well as  $a_2, b_2$  and  $c_2$  are chosen as design variables  $s_{5-10}$  and are allowed to change within a range of  $[-24, 24]$ . Thus, in total we have  $\text{ndv} = 10$  design variables. The objective function  $J$  has a value of  $J^0 = 5.2976 \times 10^1$  at the beginning of the optimization process. Fig. 3 summarizes the optimization process. After 115 iterations, the truncation criterion, chosen as the relative difference of the objective function during the iterations  $\left| \frac{J_{it} - J_{it+1}}{J_{it}} \right| \leq t_{ol} = 1 \times 10^{-7}$ , is reached, where the value of the objective is  $J^{115} = 5.9512 \times 10^{-7}$ .

The history of the objective function is displayed in Fig. 3(a). Due to the precise gradient information obtained utilizing the variational approach, the objective is rapidly decreased within the first few iterations. In Tab. 1 all design variables identified in the optimization process are summarized and a relative error  $\epsilon = \left| \frac{s^{115} - s^{\text{in}}}{s^{\text{in}}} \right| \times 100$  is calculated.

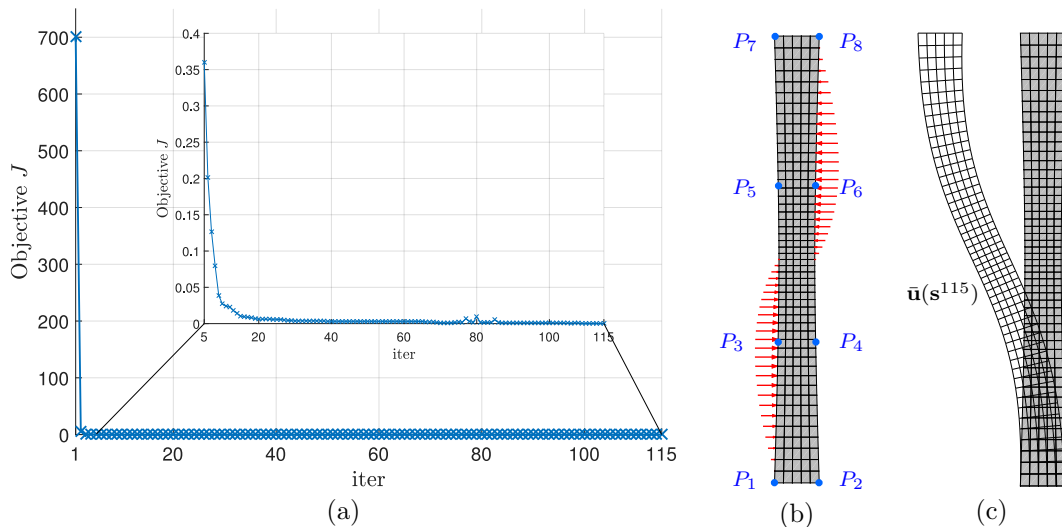


Figure 3: Optimization, (a) Objective history, (b) Optimized model (c) Deformation

$s_{1-4}^{115}$	value	ini. value	rel. error	$s_{5-10}^{115}$	value	init. value	rel. error
$P_3$	(0.1002, 3.33)	(0.1, 3.33)	$\epsilon = 0.20\%$	$a_1$	-19.895	-20	$\epsilon = 0.53\%$
$P_4$	(0.9002, 3.33)	(0.9, 3.33)	$\epsilon = 0.20\%$	$b_1$	19.913	20	$\epsilon = 0.44\%$
$P_5$	(0.1001, 6.67)	(0.1, 6.67)	$\epsilon = 0.10\%$	$c_1$	0.012	0	$\epsilon = 1.20\%$
$P_6$	(0.9001, 6.67)	(0.9, 6.67)	$\epsilon = 0.10\%$	$a_2$	19.886	20	$\epsilon = 0.57\%$
				$b_2$	-18.862	-20	$\epsilon = 5.69\%$
				$c_2$	0.0335	0	$\epsilon = 3.34\%$

Table 1: Optimization results and relative error

## 6 Conclusions and outlook

Design sensitivity information regarding geometry and external loads of elastoplastic structural response has been obtained using a variational approach based on an enhanced kinematic concept. This concept allows for a strict separation of design and physical quantities. After a subsequent discretization, the sensitivities can be calculated simultaneously to the structural response within a finite element framework. The numerical example shows reasonable results, as the method is able to identify a design that has caused a specific structural response.

Further investigations will address the topic of design exploration by means of singular value decomposition (SVD). The author in [9] introduced the term *internal structure of sensitivities* as an abbreviation for singular values and vectors of the sensitivity matrices. This additional information can be used to identify major and minor design changes and consequently redefine the optimization problem.

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