6th European Conference on Computational Mechanics (ECCM 6)
7th European Conference on Computational Fluid Dynamics (ECFD 7)
11 – 15 June 2018, Glasgow, UK

DIRECT PLASTIC STRUCTURAL DESIGN BY CHANCE CONSTRAINED PROGRAMMING

NGỌC-TRÌNH TRẦN[1], HERMANN G. MATTHIES[2], GEORGIOS E. STAVROULAKIS[3] AND MANFRED STAAT[1]*

[1]Aachen University of Applied Sciences, Institute of Bioengineering
Heinrich-Mußmann-Str. 1, 52428 Jülich, Germany
m.staat@fh-aachen.de , http://www.ifb.fh-aachen.de/

[2]Technische Universität Braunschweig
38092 Braunschweig, Germany
wire@tu-braunschweig.de

[3] Technical University of Crete
73100 Chania, Greece
gestavr@dpem.tuc.gr

Key words: Limit and shakedown analysis, Primal dual programming, Chance constraints.

Abstract. We propose a stochastic programming method to analyse limit and shakedown of structures under random strength with lognormal distribution. In this investigation a dual chance constrained programming algorithm is developed to calculate simultaneously both the upper and lower bounds of the plastic collapse limit or the shakedown limit. The edge-based smoothed finite element method (ES-FEM) using three-node linear triangular elements is used.

1 INTRODUCTION

In course no. 299 at the Centro Internazionale Scienze Meccaniche (CISM) in Udine in Italy direct plastic design of structures has been proposed with probabilistic and fuzzy modelling of uncertainties [1], [2]. Both use the Tresca yield function and limit analysis which determines the plastic collapse load under monotonic loading by linear programming. The probabilistic model can be modelled by chance constrained programming. Only a normal distribution has been assumed so that an equivalent deterministic problem could be formulated. The methods found little attention because of the numerical difficulties of chance constrained programs for realistic distributions.

We have generalized the approach to shakedown analysis for plastic design under time-variant loading and could show how it relates to the methods of structural reliability. The more realistic von Mises yield function is assumed which leads to nonlinear programming. First we have considered only uncertain material data [3]–[5].The approach has been still restricted to normal distributions which are not well suited to material strength which is non-negative.

The present contribution investigates the more realistic lognormal distribution for uncertain strength data [6]. The duality of the primal and dual program is used to derive deterministic equivalents. An outlook to open problems, further developments and alternative approaches is given. Design codes and probabilistic design are compared in [6].

2 RANDOM STRENGTH WITH LOGNORMAL DISTRIBUTION

The problem of shakedown analysis of structures under random strength with normal distribution was solved successfully in [5]. However the strength of material is a positive quantity. Therefore the normal distribution is not so suitable to model the random strength variable. In this section, a lognormal distribution is chosen as model of the random strength. We employ a smoothed FEM discretisation as described in more detail in [5], [7], [11].

2.1 Lower bound approach to chance constrained programming

Starting from the discretized form of the deterministic formulation the lower bound load factor α^- is the maximum of all safe load factors α:

$$\alpha^- = \max \alpha \quad \text{s.t.:} \begin{cases} \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{0} \\ f\left[\alpha \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right] - r_i \le 0, \quad \forall k = \overline{1,m}, \quad \forall i = \overline{1,Ne} \end{cases} \tag{1}$$

in which $\hat{\mathbf{B}}_i$ denotes the smoothed FEM deformation matrix, Ne is the total number of edge in the problem domain, k is the number of vertices of the load domain, r_i is the strength of the material in sub-domain sharing the edge i. The first constraint of **Fehler! Verweisquelle konnte nicht gefunden werden.** describes the self-equilibrium condition of time independent residual stresses $\overline{\boldsymbol{\rho}}_i$ and $\boldsymbol{\sigma}_{ik}^E$ denotes the vector of elastic stress in an infinitely elastic material. The second constraint describes the von Mises yield condition.

Consider the situation that the strength of the material is not given but must be modelled through random variables $r = r(\omega)$ in a certain probability space. Under uncertainty, the inequalities of are not always satisfied, the probability that the i^{th} yield condition is satisfied is required to be greater than some reliability level ψ_i. Problem (1) becomes an individually chance constrained programming problem:

$$\alpha^- = \max \alpha \quad \text{s.t.:} \begin{cases} \sum_{i=1}^{Ne} \hat{\mathbf{B}}_i^T \overline{\boldsymbol{\rho}}_i = \mathbf{0} \\ \text{Prob}\left[f\left(\alpha \boldsymbol{\sigma}_{ik}^E + \overline{\boldsymbol{\rho}}_i\right) - r_i(\omega) \le 0\right] \ge \psi_i \end{cases} \tag{2}$$

In this work, random strength is assumed to follow a lognormal distribution. Random variables r_i are said to be lognormally distributed if their logarithm is normally distributed. We write $\ln(r_i) \sim \mathcal{N}(\mu_i, \sigma_i^2)$ or $r_i \sim \mathcal{LN}(\mu_i, \sigma_i)$. Here μ, σ are called parameter of lognormal distribution, they relate with mean m and variance v as follows

$$\mu = \ln\left(\frac{m^2}{\sqrt{v+m^2}}\right), \qquad \sigma = \sqrt{\ln\left(\frac{v}{m^2}+1\right)} \tag{3}$$

Let us consider the i^{th} individual chance constraint of (2):

$$\text{Prob}\left[f\left(\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right)-r_i(\omega)\le 0\right]=\text{Prob}\left[f_i-r_i(\omega)\le 0\right]\ge \psi_i \tag{4}$$

After some transformations we can write (4) as follows:

$$1-\Phi\left[\frac{\ln\left(f_i\right)-\mu_i}{\sigma_i}\right]=\Phi\left[\frac{\mu_i-\ln\left(f_i\right)}{\sigma_i}\right]\ge\psi_i\,. \tag{5}$$

Introducing a new variable $\kappa_i=\Phi^{-1}(\psi_i)$ so that $\psi_i=\Phi(\kappa_i)$, inequality (5) becomes:

$$\Phi\left[\frac{\mu_i-\ln\left(f_i\right)}{\sigma_i}\right]\ge\Phi(\kappa_i) \tag{6}$$

Because Φ is monotonic thus

$$\kappa_i\le\frac{\mu_i-\ln\left(f_i\right)}{\sigma_i} \tag{7}$$

From (8) we have:

$$f_i\le e^{\mu_i-\kappa_i\sigma_i}\,. \tag{8}$$

Finally we get an equivalent deterministic formulation of the static approach for lognormally distributed strength:

$$\begin{aligned}&\alpha^{-}=\max\alpha\\ &\text{s.t.:}\ \begin{cases}\sum_{i=1}^{Ne}\hat{\mathbf{B}}_i^T\overline{\boldsymbol{\rho}}_i=\mathbf{0}\\ f\left[\alpha\boldsymbol{\sigma}_{ik}^{E}+\overline{\boldsymbol{\rho}}_i\right]\le e^{\mu_i-\kappa_i\sigma_i},\quad\forall k=\overline{1,m},\quad\forall i=\overline{1,Ne}\end{cases}\end{aligned} \tag{9}$$

2.2 Upper bound approach to chance constrained programming

The deterministic shakedown problem can be formulated based on Koiter's theorem. The ES-FEM formulation is written in the following normalized form with the von Mises plastic dissipation rate:

$$\begin{aligned}&\alpha^{+}=\min\sum_{k=1}^{m}\sum_{i=1}^{Ne}\sqrt{\frac{2}{3}}r_i\sqrt{\dot{\mathbf{e}}_{ik}^T\dot{\mathbf{e}}_{ik}+\varepsilon_0^2}\\ &\text{s.t.:}\ \begin{cases}\sum_{k=1}^{m}\dot{\mathbf{e}}_{ik}-\hat{\mathbf{B}}_i\dot{\mathbf{u}}=\mathbf{0} & \forall i=\overline{1,Ne}\\ \mathbf{D}_v\dot{\mathbf{e}}_{ik}\qquad=\mathbf{0} & \forall i=\overline{1,Ne},\ \ \forall k=\overline{1,m}\\ \sum_{k=1}^{m}\sum_{i=1}^{Ne}\dot{\mathbf{e}}_{ik}^T\mathbf{t}_{ik}-1=0\end{cases}\end{aligned} \tag{10}$$

If the strength follows log-normal distribution, $r_i\sim\mathcal{LN}(\mu_i,\sigma_i)$, the objective function of the kinematic problem is a stochastic variable. We can state the problem in such a way that one looks for a minimum lower bound η of the objective function under the constraint that the probability ψ of violation of that bound is prescribed ([8], [9])

$$\alpha^{+} = \min \eta$$
$$\text{s.t.}: \begin{cases} \text{Prob}\left(\sum_{k=1}^{m} \sum_{i=1}^{Ne} \sqrt{\frac{2}{3}} r_i \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}} \geq \eta \right) = \psi \\ \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} \qquad = \mathbf{0} \\ \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 \end{cases} \tag{11}$$

For the case of lognormally distributed random strength $r_i(\omega)$ there is no existence of closed form probability distribution for the sum

$$\theta(\omega) = \sum_{i=1}^{Ne} \sum_{k=1}^{m} \sqrt{\frac{2}{3}} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik} + \varepsilon_0^2} \cdot r_i(\omega) = \sum_{k=1}^{m} D_p(\omega) \tag{12}$$

Either an approximate probability distribution is derived mathematically or the assumption that a sum of independent lognormal random variables is also lognormally distributed is used and the sum is approximated by a single lognormal random variable [10].

The probability distribution of the plastic dissipation $D_p(\omega)$ in (13) and thus the transformation of (12) into an equivalent deterministic form can only be obtained as an approximation. Nevertheless, there is a duality between lower bound and upper bound formulation. Consequently, one can assume the equivalent deterministic of (12) as (14)

$$\alpha^{+} = \min \sum_{k=1}^{m} \sum_{i=1}^{Ne} e^{\mu_i - \kappa_i \sigma_i} \sqrt{\dot{\mathbf{e}}_{ik}^T \dot{\mathbf{e}}_{ik}}$$
$$\text{s.t.}: \begin{cases} \sum_{k=1}^{m} \dot{\mathbf{e}}_{ik} - \hat{\mathbf{B}}_i \dot{\mathbf{u}} = \mathbf{0} & \forall i = \overline{1, N_e} \\ \mathbf{D}_v \dot{\mathbf{e}}_{ik} \qquad = \mathbf{0} & \forall i = \overline{1, Ne}, \quad \forall k = \overline{1, m} \\ \sum_{k=1}^{m} \sum_{i=1}^{Ne} \dot{\mathbf{e}}_{ik}^T \mathbf{t}_{ik} - 1 = 0 \end{cases} \tag{13}$$

By duality we can prove that the maximum problem (10) and the minimum problem (14) are dual to each other. This mean (14) is the equivalent deterministic of (12).The primal and dual problem can be written in a unified for normally distributed or Primal problem (14) and dual problem (10) can be solved simultaneously by dual algorithm which was presented in [5], [7], [11].

3 NUMERICAL APPLICATIONS

3.1 Two span continuous beam

We first consider the two span continuous beam with rectangular cross-section. The beam is subjected to two point forces as shown in figure 1. This test is investigated analytically by Sikorski and Borkowski in [1] for the deterministic problem and for normal distributions. The numerical solution for the case of random strength with normal distribution was obtained in [5]. Let us determine the limit load factor in situation: Loads are deterministic with $P_1 = \alpha 3\text{kN}$, $P_2 = \alpha 2\text{kN}$. The strength is lognormally distributed with the mean values $M_{0,1} = 2.0\text{kNm}$, $M_{0,2} = 3.0\text{kNm}$ which correspond to the first and the second span. The given partial reliability levels are $\psi_s = \psi_p = 0.9999$ so that $\kappa_r = \Phi^{-1}(\psi_r) = \kappa_p = \Phi^{-1}(\psi_p) = \Phi^{-1}(0.9999) = 3.719$.

The analytical solution was investigated in [5] for the deterministic plastic moment and normally distributed plastic moment, limit load factor

$$\alpha^{+}_{\lim} = \frac{3M_{0,1}}{P_1 L} = \frac{3 \cdot 2\text{kNm}}{3\text{kN} \cdot 1\text{m}} = 2 \tag{14}$$

If the plastic moment $M_{0,1}$ is lognormally distributed with $\mathrm{E}\left[M_{0,1}\right] = 2\,\text{kNm}$ and $\mathrm{Var}\left(M_{0,1}\right) = (0.1\text{m})^2 = \left(0.2\,\text{kNm}\right)^2$, respectively. Then the parameters of the lognormal distribution are computed using (3): $\mu = 0.6882$; $\sigma = 0.0998$.

For the chosen reliability level ($\kappa = 3.719$) the limit load factor is:

$$\alpha = \frac{3e^{\mu - \kappa\sigma}}{P_1 L} = \frac{3e^{0.6882 - 3.719 \cdot 0.0998}}{3 \cdot 1} = 1.373 \tag{15}$$

In table 1 our results are shown in comparison with the results of Sikorski and Borkowski [1]. The limit loads in [1] and the analytical limit loads are based on beam theory and are therefore different from the numerical limit loads which are based on plane stress FEM discretization.

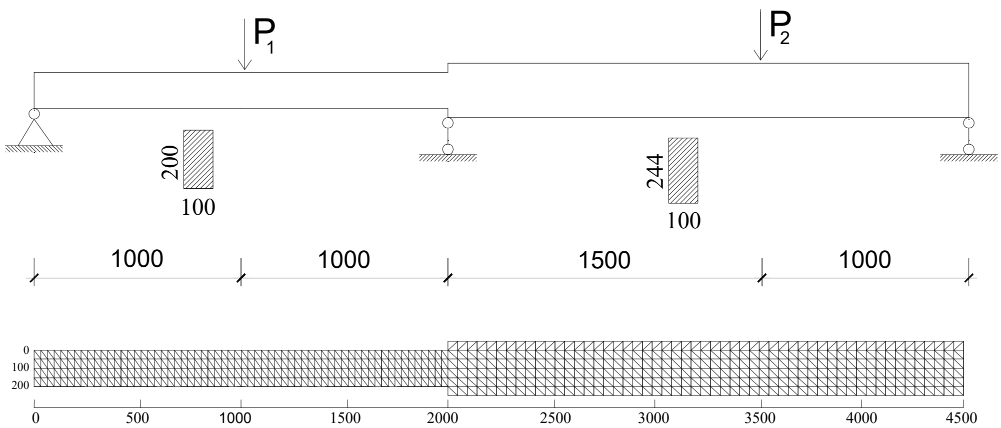

Figure 1: Two-span beam and FE mesh with T3 elements

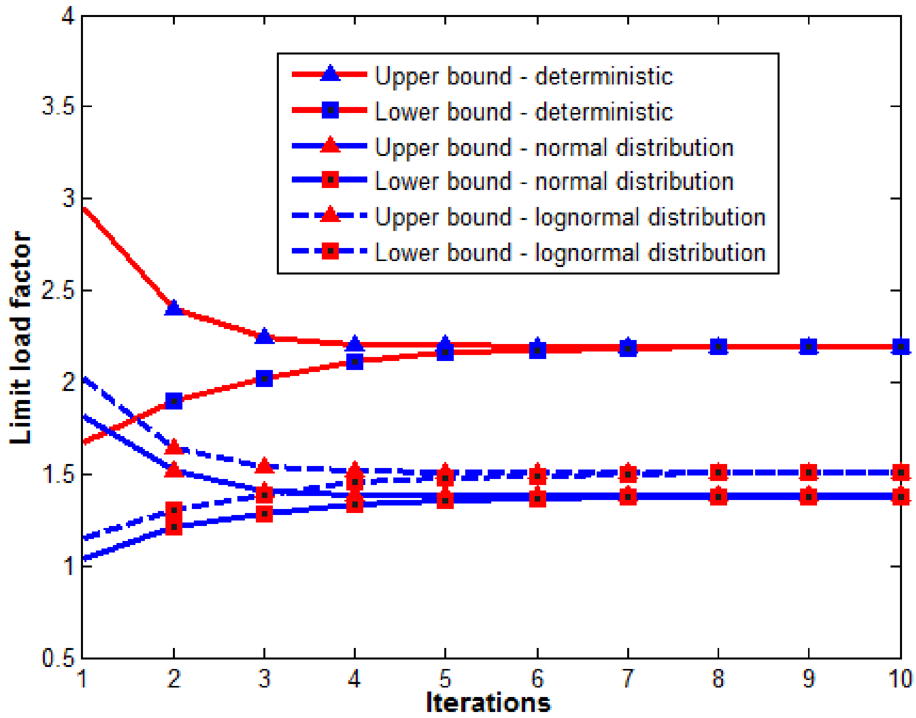

Figure 2: Convergence of the limit load factor in case of deterministic and random strength.

The figure 1 shows the two-span beam with its FE mesh. The beam is modelled by 1350 T3 elements. In figure 2 the convergence of the limit load factors is shown for some cases of random strength. The convergent numerical solutions are 2.19, 1.51, 1.38 for deterministic, lognormal distribution and normal distribution of strength, respectively.

The dependence of load factors on the coefficient of variation ξ and on failure probability are presented in figures 3, 4.

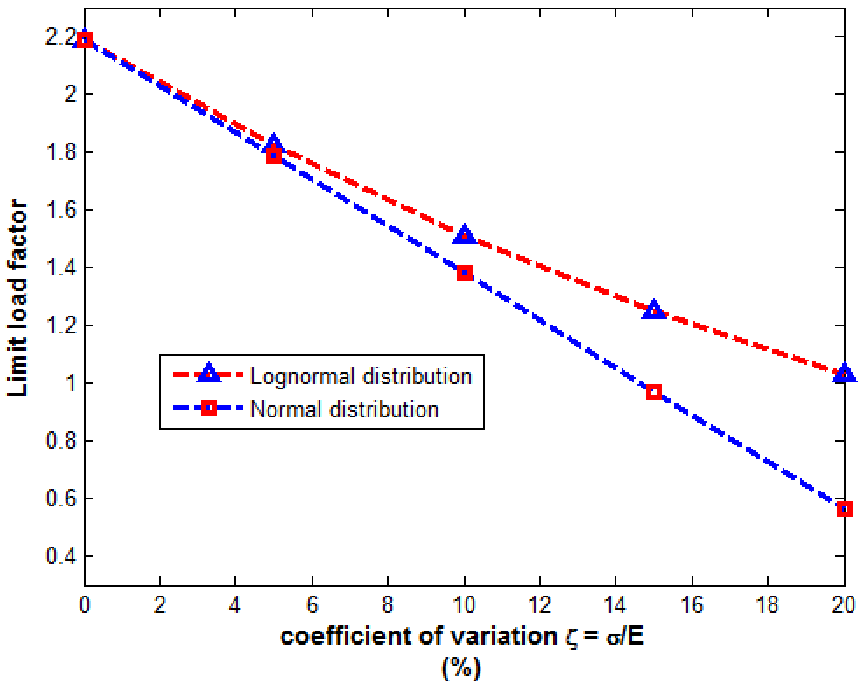

Figure 3: Dependence of load the factor on the coefficient of variation ξ

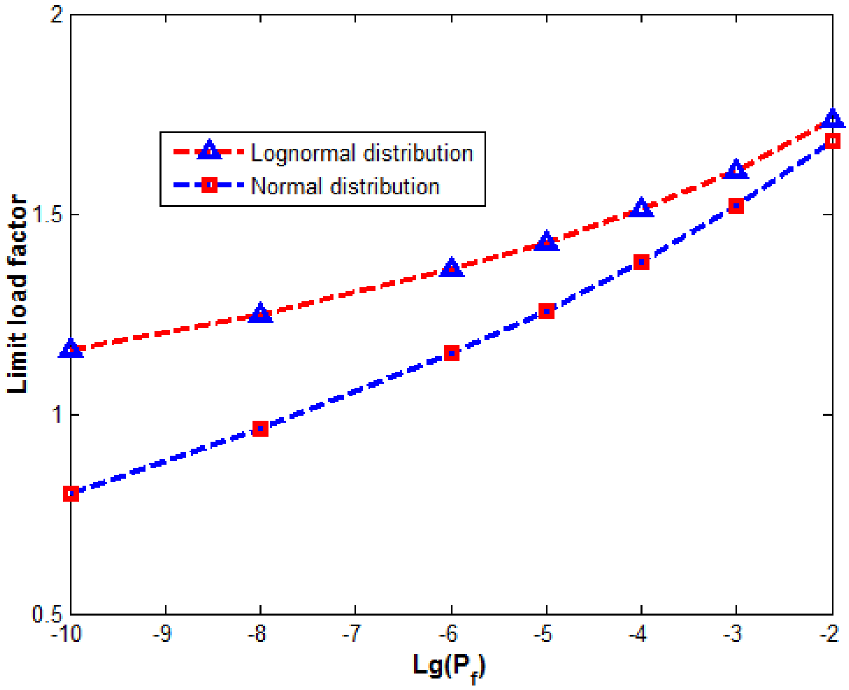

Figure 4: Dependence of the load factor on the failure probability

Table 1: Limit load factor of the two-span beam

Lower bound -	Upper bound -	Method, reference
2.19 (deterministic)	2.19 (deterministic)	numerically [5]
2 (deterministic)	2 (deterministic)	analytically [5]
1.15 (normal)	1.36 (normal)	[1]
1.38 (normal)	1.38 (normal)	numerically [5]
1.256 (normal)	1.256 (normal)	analytically [5]
1.509 (lognormal)	1.509 (lognormal)	numerically
1.373 (lognormal)	1.373 (lognormal)	analytically

3.2. Simple frame

In the second example, we investigate a simple frame which is depicted in Figure 7. The left side of beam component can move only in horizontal direction. The frame carries uniormly distributed loads which can vary independently in the load domain as shown in figure 4.11b. The loads are considered as random variables which are considered to be distributed normally. The geometrical and material data are chosen as in [12], i.e. $E = 2 \cdot 10^5\,\text{MPa}$, $\nu = 0.3$, and $\sigma_y = 10\text{MPa}$. $p_1 \in [1.2, 3.0]$ and $p_2 \in [0.4, 1.0]$. The frame is discritized by 1600 smoothed T3 elements as shown in Fig. 7.

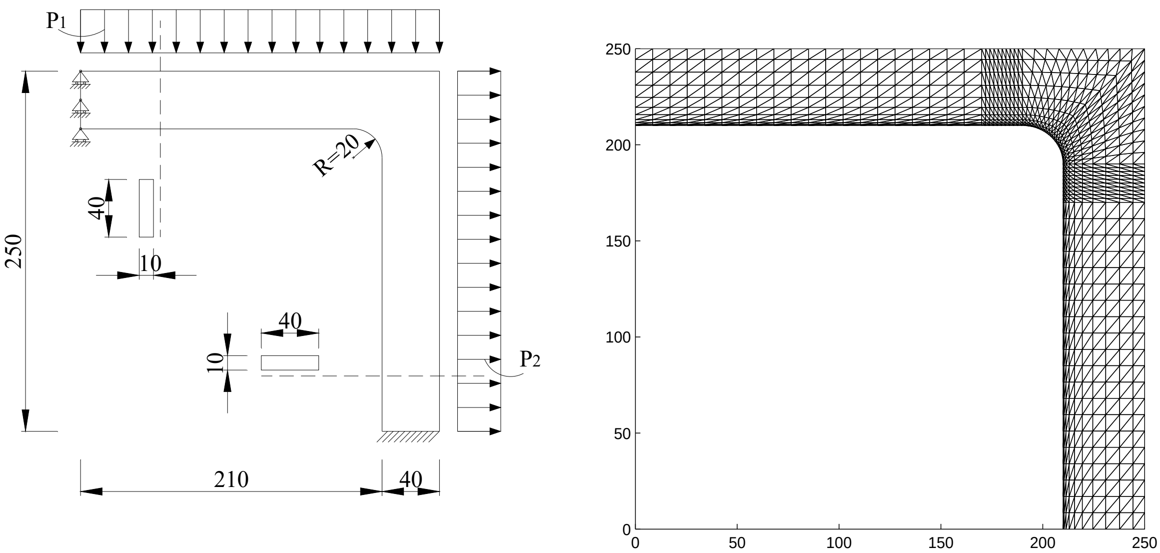

Figure 7: The geometrical dimensions and FE-mesh

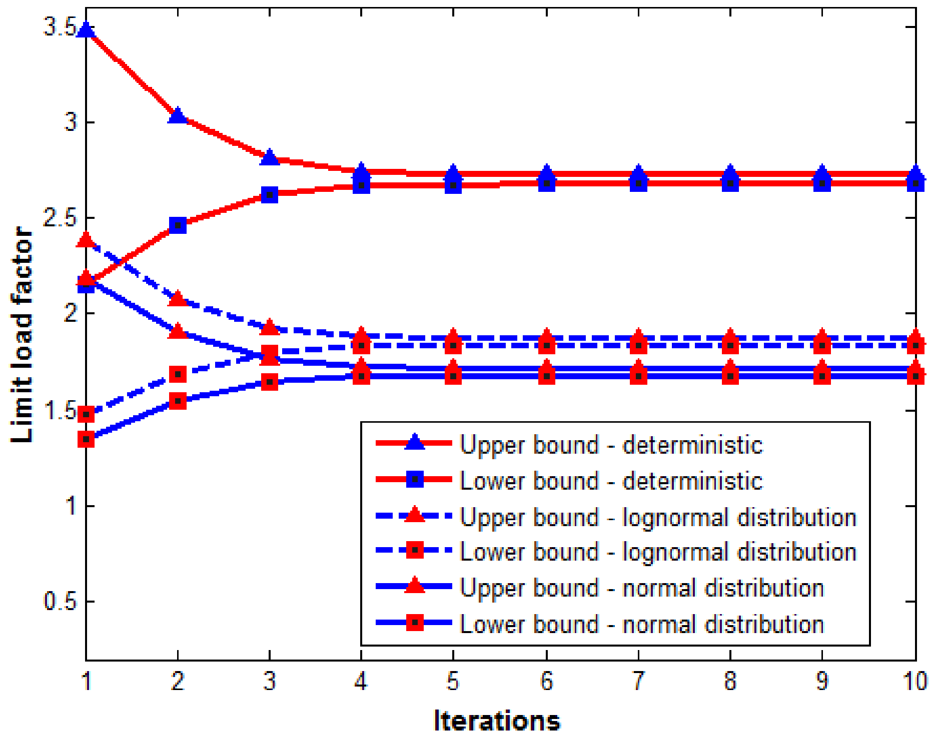

Figure 8: Limit load factor with random strength, deterministic loads.

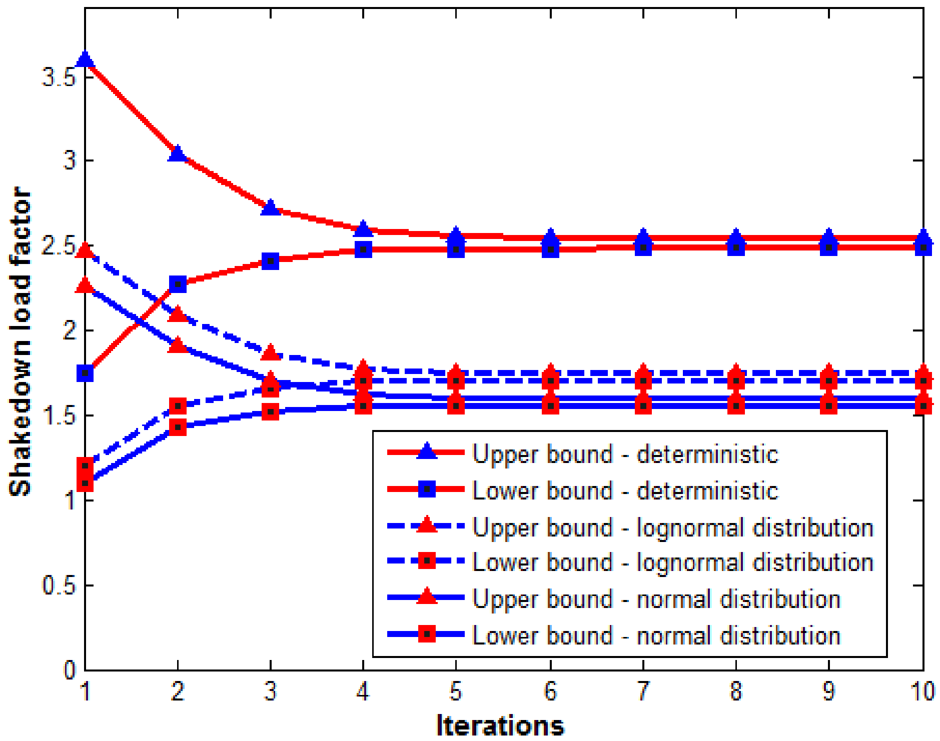

Figure 9: Shakedown load factor with random strength, deterministic loads.

Table 2: Limit analysis: comparison

	Garcea *et al.* [12]	Present		
(p_1, p_2)	Deterministic	Deterministic	Normal	Lognormal
(1.2, 1.0)	2.975	2.930	1.793	1.963
(3.0, 0.4)	2.831	2.985	1.856	2.045
(3.0, 1.0)	2.645	2.705	1.697 [5]	1.856

Table 3: Shakedown analysis: comparison

	Garcea *et al.* [12]	Present		
Limits	Deterministic	Deterministic	Normal	Lognormal
Elastic	1.203	1.192	0.749	0.819
Alternating	2.940	2.922	1.835	2.006
Ratcheting	2.473	2.521	1.582 [5]	1.730

Figure 8 and 9 show the evolutions of limit and shakedown load factors for case (a) for both situations: deterministic and random strength. For limit analysis with $p_1 = 3.0,\ p_2 = 1.0$, all the two bounds converge to the solutions $\alpha_{\lim} = 2.705$ in case of deterministic strength and 1.856 in case of lognormally distributed strength. For the shakedown analysis, the results give the shakedown load factors $\alpha = 2.521$ and $\alpha = 1.730$ corresponding to deterministic and lognormally distributed random strength, respectively. Tables 2-3 present results in comparison with deterministic results of Garcea *et al.* [12]

4 RELIABILITY ANALYSIS OF STRUCTURE BY FORM

In the stochastic programming approach we have prescribed a reliability level and calculated the load factor. In structural reliability the failure probability is calculated for a given load factor. In order to find the relation between both approaches we consider briefly the First Order Reliability Method (FORM), which has been used in [5], [13] to calculate failure probabilities in limit and shakedown analysis. For more detail, see the given references.

We discuss the reliability of the two-span beam with log-normal distributions. Let the plastic moment $M_{0,1}$ be lognormally distributed with the above mean value m and variance v and let the load P_1 be deterministic. The parameters (μ, σ) of $M_{0,1} \sim \mathcal{LN}(\mu, \sigma)$ can be computed from the mean value m and the variance v using eq.(3).
The limit state function with the property

$$g(\mathbf{X}) \begin{cases} < 0 & \text{for failure,} \\ = 0 & \text{for limit state,} \\ > 0 & \text{for safe structure.} \end{cases} \tag{16}$$

of the beam is $g\left(M_{0,1}\right)=M_{0,1}-\frac{\alpha P_1 \cdot L}{3}=0$. Its natural logarithm has also the propterty (16):

$$g\left(M_{0,1}\right)=\ln(M_{0,1})-\ln\left(\frac{\alpha P_1 \cdot L}{3}\right)=0\,. \tag{17}$$

The transformation $\ln M_{0,1}=u_y\sigma+\mu$ yields

$$g\left(u_y\right)=u_y\sigma+\mu-\ln\left(\frac{\alpha P_1 \cdot L}{3}\right)=0\,. \tag{18}$$

With realizations $\mathbf{u}=\left(u_y\right)$ of the new standard normal random variable $\mathbf{U}$ it may be written

$$g_L\left(\mathbf{u}\right)=\frac{\sigma}{\sqrt{\sigma^2}}\mathbf{u}+\frac{\mu-\ln\left(\frac{\alpha P_1 \cdot L}{3}\right)}{\sqrt{\sigma^2}}=\boldsymbol{\alpha}^T\mathbf{u}+\beta=0 \tag{19}$$

Using $\left(\mu,\sigma\right)$ from eq. (3), we have the reliability index β which is the distance of the plane $g_L\left(\mathbf{u}\right)=0$ from the origin in the standard normal space:

$$\beta=\frac{\mu-\ln\left(\frac{\alpha P_1 \cdot L}{3}\right)}{\sigma}=\frac{0.6882\text{kNm}-\ln\left(\frac{1.373\cdot 3\text{kN}\cdot 1\text{m}}{3}\right)}{0.0998\text{kNm}}=3.719 \tag{20}$$

and the failure probability is $P_f=\Phi(-\beta)=\Phi(-3.719)=1\cdot 10^{-4}$. In this case comparing with the reliability $1-P_f=\psi=\Phi(\kappa)=0.9999$ we have $\kappa=\beta$.

5 CONCLUSIONS

- For engineering design, structural reliability is a post-design problem while stochastic programming is a pre-design problem. In the simple case of only one uncertain strength variable or only one random load variable, reliability analysis is “invers” to chance constrained programming and can be used to check the latter. In the same way numerical reliability analysis can be used to check any constrained programming solution for normally or lognormally distributed variables. It is found that the load factor decreases quickly with increasing coefficient of variation of the strength and load.
- One result is that the load factors for normally distributed strength is always larger than for lognormally distributed strength. Therefore, working with the simpler normal distribution will give safe results which is most important for engineering applications. This makes the method more transparent to many engineers and it is easily extended to the case that the strengths in different points of the structure are correlated (stochastic field).

REFERENCES

[1] Sikorski K.A. and A. Borkowski, A. Ultimate load analysis by stochastic programming. In D.L. Smith (ed.) M*athematical programming methods in structural plasticity*. Springer Wien, New York, (1990) pp. 403–424.

[2] Chuang, P.-H. and Munro, J. Fuzzy linear programming in plastic limit design. In D.L. Smith (ed.) M*athematical programming methods in structural plasticity*. Springer Wien, New York, (1990), pp. 425–435.

[3] Tran, N.T., Tran, T.N., Matthies, H.G., Stavroulakis, G.E., and Staat, M. Shakedown analysis of plate bending under stochastic uncertainty by chance constrained programming. In M. Papadrakakis, V. Papadopoulos, G. Stefanou, V. Plevris (eds.): *ECCOMAS Congress 2016, VII European Congress on Computational Methods in Applied Sciences and Engineering. Crete Island, Greece, 5–10 June 2016, Vol. 2, pp. 3007-3019.*

[4] Tran, N.T., Tran, T.N., Matthies, H.G., Stavroulakis, G.E., and Staat, M. FEM shakedown of uncertain structures by chance constrained programming. *PAMM* (2016) **16**(1):715–716, 2016.

[5] Tran, N.T., Tran, T.N., Matthies, H.G., Stavroulakis, G.E., and Staat, M. Shakedown analysis under stochastic uncertainty by chance constrained programming. In O. Barrera, A. Cocks, and A. Ponter (eds.) *Advances in direct methods for materials and structures*. Springer, Cham, (2018), 85-103.

[6] Nadolksi, V. and M. Sykora, M. Uncertainty in resistance models for steel members," *Trans. VŠB – Tech. Univ. Ostrava, Civ. Eng. Ser.*, (2014) **14**(2):26–37.

[7] Tran, T.N., Nguyen-Xuan, H., Nguyen-Thoi, T., and Liu, G.R. An edge-based smoothed finite element method for primal-dual shakedown analysis of structures. *Int. J. Numer. Methods Eng.* (2010) **82**:917–938, 2010.

[8] Charnes A. and Cooper, W.W. Chance-constrained programming. *Manage. Sci.* (1959) **6**(1):73–79.

[9] Charnes A. and Cooper, W.W. Chance constraints and normal deviates. *J. Am. Stat. Assoc.*, (1962) **57**(297): 134–148.

[10] Rook, C. and Kerman, M. Approximating the sum of correlated lognormals: An implementation. (2015) *Available SSRN* http://dx.doi.org/10.2139/ssrn.2653337

[11] Trân T.N. and Staat, M., An edge-based smoothed finite element method for primal-dual shakedown analysis of structures under uncertainties. In G. de Saxcé, A. Oueslati, E. Charkaluk, J.-B. Tritsch (eds.) *Limit state of materials and structures*, vol. 2, Springer, Dordrecht (2013) pp. 89–102.

[12] Garcea, G., Armentano, G., Petrolo, S., and Casciaro, R. Finite element shakedown analysis of two-dimensional structures. *Int. J. Numer. Methods Eng.* (2005) **63**(8): 1174–1202.

[13] Heitzer. M. and Staat, M. Probabilistic limit and shakedown problems. In M. Staat and M. Heitzer (eds.) *Numerical methods for limit and shakedown analysis. Deterministic and probabilistic approach.* NIC Series Vol. 15, John von Neumann Institute for Computing, Jülich (2003), pp. 217–268.