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A POD-DEIM REDUCED ORDER MODEL WITH DEFORMING MESH FOR AEROELASTIC APPLICATIONS

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Abstract. Reduced-order models (ROMs) of nonlinear dynamical systems are essential for broadening the scope of high-fidelity non linear CFD design optimizations or aeroelastic investigations. The wide variety of fluid dynamics ROMs reported in literature shares the aim of reducing the dimensionality of dynamical systems by performing a projection of the governing equations onto a basis that may be often constructed via the Proper Orthogonal Decomposition (POD). Unfortunately, the relevance of POD is still an open question for aeroelastic applications because it is impossible to define a spatial correlation for a domain that is moving and deforming during the simulation. Moreover, when addressing real-life applications, the non-linear compressible Navier-Stokes equations have to be considered, resulting in an additional difficulty.

In the present work, we propose a ROM based on the classical Galerkin projection onto a basis contructed via index-based POD. The discrete empirical interpolation method (DEIM) is adopted in order to efficiently deal with the compressible Navier-Stokes equations non-linearities. Numerical tests have been carried out to evaluate the performance of the present ROM. First, we validate the classical POD-DEIM technique for a time-dependent and slightly compressible flow around a NACA 0012 airfoil at high incidence. Then, the POD-DEIM ROM is used to reproduce the solution of a high-fidelity model based on the Arbitrary Lagrangian-Eulerian (ALE) formulation combined with a deforming grid for a flow around an oscillating supercritical airfoil. On the basis of the achieved results, we highlight the limits and the strengths of the proposed technique.

1 INTRODUCTION

Dynamical systems are the basic framework for modeling and control of a large variety of complex systems of scientific interest or industrial value. The aeronautic and aerospace fields are important examples in which some complex physical phenomena, such as fluid-structure interaction, shock interaction, flow separation, limit-cycle oscillations and so on, can be investigated with the aforementioned dynamical systems. Interactions between fluid and moving (deforming) structures are among the most important issues of aircraft design and numerical simulation has been one of the few available means for studying this kind of problems. However, the growing need for an improved accuracy attainable with a high fidelity (or full order) model leads inevitably to an unaffordable computational cost whose reduction represents one of the main motivation of this work. The projection-based reduced order models (ROMs) are physics based methods that can potentially yield very high speedups. These approaches involve the resolution of the dynamic equations of the system by projecting them on a suitably chosen low-dimensional sub-space. The effective dimension reducibility for these methods is usually limited to problems with linear or multi-linear terms. In fact, when a general non-linearity is present, the cost to evaluate the projected non-linear function still depends on the dimension of the original system, resulting in ROM simulation time of the same order as the original system time computation. Therefore, several techniques have been introduced to address this issue by approximating the non-linear term of the system equations. The Discrete Empirical Interpolation Method (DEIM), introduced by Chaturantabut et al. [1] provides an interpolation of the non-linear quantities in a reduced domain including only a small number of elements. An efficient implementation of the DEIM for non-linear compressible flows ROMs is reported in [2].

Considering aeroelastic dynamical systems solved on a moving/deforming mesh, Anttonen et al. [3] outlined that an additional level of difficulty arises since the POD involves spatial correlations that require special care in a continuous deformable framework. The loss of the spatial correlation and the increasingly important problems of stability and accuracy make the aeroelastic ROMs study widely challenging. In a discrete framework, Freno et al. [4] provide an accurate ROM requiring that the projection basis is defined taking into account the mesh deformation. They use dynamic basis functions to take into account the domain deformation dynamics by defining a dynamics for the projection basis related to the instantaneous deformed configuration. In the particular case of a rigid body motion, an interesting manner to avoid this issue is proposed by Lewin and Haj-Hariri [5] and Placzek et al. [6]. They perform the projection of the governing equations in a non-inertial reference frame in order to preserve the consistency of the POD formulation. Anyway, stability problems appear when considering non-linear flows. Bourguet et al. [7] propose a Hadamard formulation to take into account small wall deformations. Then, an *a posteriori* calibration is proposed to improve the stability of the model. Liberge and Hamdouni [8] implement a multiphase formulation that allows to define the POD on a deforming domain using characteristic functions to follow the fluid-structure interface. Stankiewicz et al. [9, 10] deepen the study of Anttonen et al. [3] with test cases of increasing complexity also considering parametric changes. Although the aforementioned aeroelastic studies do not consider systems with a general non-linearity, Freno et al. [4, 11] tackle this problem, but the non-linear term is evaluated at each time step by the full order model (FOM) on the whole domain. This limitation may be prevented by the use of the DEIM to improve the ROM efficiency.

In the present work, we propose a ROM based on the classical Galerkin projection onto a basis constructed via the index-based POD formulation. The discrete empirical interpolation method (DEIM) is adopted in order to efficiently deal with the compressible Navier-Stokes non-linearities.

2 THEORETICAL FRAMEWORK

The FOM considered in the present paper for an aeroelastic CFD problem is governed by the Arbitrary Lagrangian-Eulerian (ALE) formulation of the non-linear Navier-Stokes equations. In the semi-discrete form the ALE formulation for the basic cell of the computational domain reads:

$$\frac{d}{dt}\int_{\Omega(t)} \boldsymbol{W} d\Omega = -\sum_{i=1}^{6}\int_{\sum_i(t)} \boldsymbol{F_c}(\boldsymbol{W},\boldsymbol{s})\cdot\boldsymbol{n}d\Sigma - \sum_{i=1}^{6}\int_{\sum_i(t)} \boldsymbol{F_d}(\boldsymbol{W})\cdot\boldsymbol{n}d\Sigma = -\boldsymbol{R}_\Omega(\boldsymbol{W},t) \quad (1)$$

where $\boldsymbol{W}$ is the vector of the conservatives variables ρ, $\rho\boldsymbol{U}$ and ρE. $\boldsymbol{F_c}$ and $\boldsymbol{F_d}$ are, respectively, the convective and diffusive fluxes and Σ_i represents the i face of the hexahedral cell considered in the structured mesh. The arbitrary motion of the computational mesh is taken into account by the definition of $\boldsymbol{c} = \boldsymbol{U} - \boldsymbol{s}$ as the convective velocity, where $\boldsymbol{s}$ is the mesh velocity with respect the spatial domain. The flow equations are solved by a finite-volume method with ONERA's in-house software *elsA* [12]. Once spatially discretized, we can write:

$$\frac{d\boldsymbol{W}}{dt} = -\frac{\boldsymbol{R}_\Omega(\boldsymbol{W},t)}{\mathcal{V}(\Omega(t))} = \widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W},t) \quad (2)$$

where $\mathcal{V}(\Omega(t))$ is the volume of the related cell and $\widetilde{\boldsymbol{R}}_\Omega$ is a non-linear residual operator that, on the basis of the concerning instantaneous aerodynamic field, computes the flux balance for each cell and divides it for the related cell volume.

2.1 Proper Orthogonal Decomposition

Empirical modes resulting from Proper Orthogonal Decomposition [13, 14] are the most widely used for fluid dynamics reduced-order modeling. This method consists in looking for the deterministic function that is most similar in an average sense to an ensemble of representative systems solutions (snapshots). Sirovich [13] introduced the so-called "method of snapshots" for computing a POD basis. Assuming that the snapshots are collected in a matrix $[\mathbf{U}] \in \mathbb{R}^{N_x \times N_t}$ with N_x the number of degrees of freedom and N_t the number of collected snapshots, the POD is equivalent to a singular value decomposition (SVD) of the matrix $[\mathbf{U}]$, so that:

$$[\mathbf{U}] = [\boldsymbol{\Phi}][\boldsymbol{\Sigma}][\mathbf{V}]^T \quad (3)$$

where the matrix $[\mathbf{\Phi}] \in \mathbb{R}^{N_x \times N_r}$ is an orthonormal matrix that contains the POD mode vectors, with $N_r = \text{rank}(\mathbf{U}) \leq \min(N_x, N_t)$. The diagonal matrix $[\mathbf{\Sigma}] \in \mathbb{R}^{N_r \times N_r}$ contains the singular values of $[\mathbf{U}]$ listed in order of decreasing magnitude. If we define the diagonal element of $[\mathbf{\Sigma}]$ as σ_i the rate of "energy" captured by the first N_m modes is given by $E_{N_m} = \sum_{i=1}^{N_m} \sigma_i^2 / \sum_{i=1}^{N_r} \sigma_i^2$. This quantity is an indicator of the energy neglected by retaining only the first N_m POD modes.

The POD is based on the assumption that there is a correlation between successive snapshots of the flow. As a consequence, if a non-deforming mesh is considered, the snapshots collected in $[\mathbf{U}]$ are easily spatially correlated as each finite volume preserves the spatial position during the simulation. On the contrary, the spatial domain correlation is lost when a deforming domain is considered as each finite volume is moving and deforming during the numerical simulation. In this case, Antonnen et al. [3] propose an index-based POD for which the snapshots in $[\mathbf{U}]$ are collected retaining the index numbering order so that a fixed computational index-based domain is taken into account to preserve the consistency of the POD process. This approach allows to directly deal with discrete projection and discrete inner product for the construction of the ROM, in contrast with the continuous formulation that requires the use of characteristic functions to follow the moving fluid-structure interface [8].

2.2 Reduced Order Model

The basis of the ROM consists in approximating the conservative field by a base solution $\boldsymbol{W_0}$ (steady or time-average flow) and a linear combination of the POD modes $\mathbf{\Phi}_i$:

$$\boldsymbol{W}(t) \approx \boldsymbol{W_0} + \sum_{i=1}^{N_m} a_i(t) \mathbf{\Phi}_i \tag{4}$$

In this work, the Galerkin projection [9, 15] is used to develop the ROM, which means that the system is projected orthogonally onto the space spanned by the modes used to approximate the solution. Substituting eq.(4) in eq.(2) and projecting (taking into account the orthogonality of the POD basis) the following ROM is obtained:

$$\frac{d\boldsymbol{a}}{dt} = \mathbf{\Phi}^T \widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W_0} + \sum_{i=1}^{N_m} a_i(t) \mathbf{\Phi}_i, t) \tag{5}$$

This is a system of N_m ODEs, with $N_m \ll N_x$, where the unknowns are the time amplitudes associated to the POD modes of the ROM.

It should be noted that, for the compressible Navier-Stokes equations, the operator $\widetilde{\boldsymbol{R}}_\Omega$ is non-linear and cannot be expressed as the product of constant terms and amplitudes unless specific approximations or formulations are introduced to obtain a multilinear form [16, 17]. To tackle this issue a *hyper reduction technique* introduced by Chaturantabut et al. [1], called Discrete Empirical Interpolation Method (DEIM), is used. The DEIM involves the approximation of the non-linearity as:

$$\widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W}, t) \approx \sum_{j=1}^{N_l} c_j(t)\boldsymbol{\Psi}_j \tag{6}$$

where, analogously to eq.(3), a POD of the non-linear term snapshots is adopted to obtain the basis $\boldsymbol{\Psi}$ of size $N_x \times N_l$, where N_l is the number of retained modes after cutting the POD basis of the non-linear term. The coefficients $c_j(t)$ represent the DEIM amplitudes. The system of eq.(6) is still overdetermined because it involves N_x equations in N_l unknowns. The DEIM provides a greedy algorithm to select the N_l equations of eq.(6) so to obtain a determined system. More specifically, if $e_{\wp_i}$ is the $\wp_i$-th column of the identity operator of size $N_x \times N_x$, the DEIM algorithm computes the operator $\mathbf{P} = [e_{\wp_1}, \dots, e_{\wp_l}] \in \mathbb{R}^{N_x \times N_l}$ so that the subsystem

$$\mathbf{P}^T \widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W}, t) \approx \sum_{j=1}^{N_m} c_j(t)(\boldsymbol{P}^T\boldsymbol{\Psi})_j \tag{7}$$

of size $N_l \times N_l$ can be inverted. So the DEIM approximation of the RHS of eq.(5) reads:

$$\widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W}, t) \approx \sum_{j=1}^{N_m} \underbrace{\left((\boldsymbol{P}^T\boldsymbol{\Psi})^{-1}\mathbf{P}^T \widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W}, t)\right)_j}_{c_j(t)} \boldsymbol{\Psi}_j \tag{8}$$

Projecting eq.(8) and substituting in eq.(5) yields to the POD-DEIM ROM:

$$\frac{d\boldsymbol{a}}{dt} = \boldsymbol{\Phi}^T \sum_{j=1}^{N_m} \left((\boldsymbol{P}^T\boldsymbol{\Psi})^{-1}\mathbf{P}^T \widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W}_\mathbf{0} + \sum_{i=1}^{N_m} a_i \boldsymbol{\Phi}_i, t)\right)_j \boldsymbol{\Psi}_j \tag{9}$$

The term $\mathbf{P}^T \widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W}_\mathbf{0} + \sum_{i=1}^{N_m} a_i \boldsymbol{\Phi}_i, t)$ changes during the time integration as it depends on t. Moreover, if the time integration corresponds to the sampled time period, the term $\widetilde{\boldsymbol{R}}_\Omega$ is *a priori* known for the sampled time instants and can be interpolated for the intermediates ones by using the collection of the non-linear term snapshots. In the present work, this particular case is investigated although it is quite restrictive. In fact, when we want to integrate the ROM for a longer time period or for a different flow parameter the operator $\widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W}, t)$ must be correctly evaluated. Since this operator evaluates componentwise the conservative variable vector $\boldsymbol{W}$, we can observe that:

$$\mathbf{P}^T \widetilde{\boldsymbol{R}}_\Omega(\boldsymbol{W}_\mathbf{0} + \sum_{i=1}^{N_m} a_i \boldsymbol{\Phi}_i, t) = \widetilde{\boldsymbol{R}}^*_\Omega \left(\mathbf{P}^T \boldsymbol{W}_\mathbf{0} + \sum_{i=1}^{N_m} a_i (\mathbf{P}^T\boldsymbol{\Phi})_i, t\right) \tag{10}$$

The new operator $\widetilde{\boldsymbol{R}}^*_\Omega$ evaluates exactly the RHS of eq.(2) only for the N_l indexes (cells) selected by the DEIM algorithm.

The results of the present paper are related to the formulation of eq.(9), but the DEIM amplitudes $\boldsymbol{c}(t)$ are computed directly by projecting the non-linear snapshots on the non-linear POD modes, thus they are *a priori* known at the time instants where the snapshots have been sampled. For this reason, the ROM system of ODE in eq.(9) is easily integrated. Furthermore, when the POD-DEIM ROM is used to reproduce a longer time period or the solution for a different flow parameter, there are problems concerning two aspects. The first one is related to the definition of suitable POD basis [18, 19]. The second one is related to the construction of the operator $\widetilde{\boldsymbol{R}}^*_\Omega$. In fact, it evaluates the instantaneous non-linearity of the FOM for the indexes selected by the DEIM algorithm. This operation is not straightforward when a deforming grid is considered since the position and the deformation of each cell selected by the DEIM (as well as for its related neighbors in order to complete the stencil) must be updated at each time step in order to take into account the metric changes in the computation.

3 Results

The ROM resulting from the aforementioned formulation is used to model the flow for two different configurations. First, the POD-DEIM technique is validated for a time dependent and slightly compressible flow around a NACA 0012 airfoil at high incidence. Then, the index-based POD-DEIM ROM is used to reproduce the flow induced by a forced oscillation prescribed to a NLR7301 supercritical airfoil.

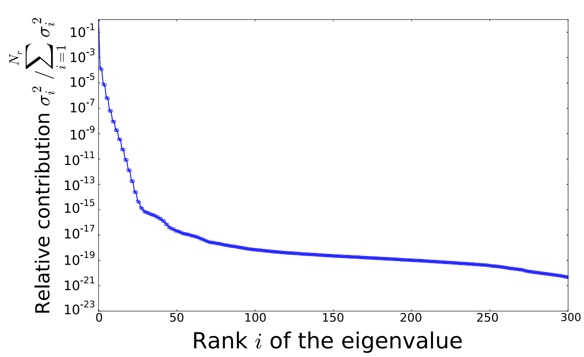

Figure 1: Proper orthogonal eigenvalue spectrum.

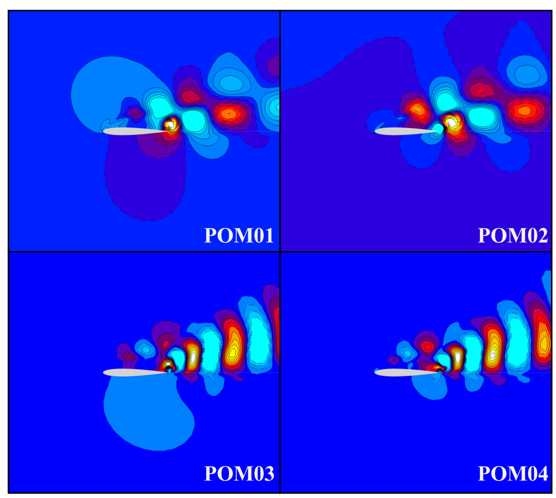

Figure 2: Density fields of the first four proper orthogonal modes (POMs).

3.1 High incidence flow around a NACA 0012 airfoil

For the first test-case, a fixed NACA 0012 airfoil is set at an incidence of $\alpha = 20°$ with the flow parameters $M = 0.2$ and $Re = 1000$. A structured C-grid is used with about 30×10^3 finite volumes. The farfield distance is set to 10 chord lengths. The flow can be considered as slightly compressible and laminar. With a preliminary steady

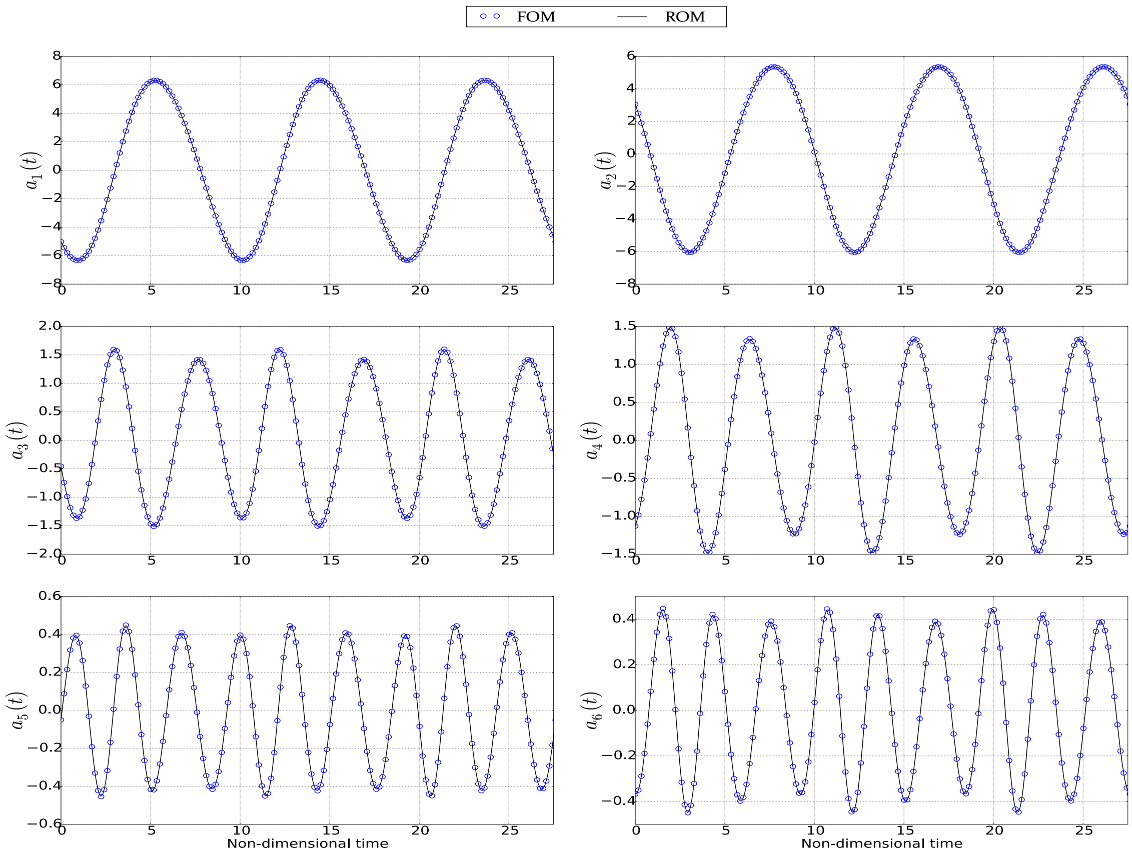

Figure 3: Comparison of the first six modal coordinates $a_i(t)$ computed as the solution of the ROM equations and the FOM coordinates defined by the projection of the snapshots on the POD modes.

configuration as initial condition, an unsteady simulation is performed with the Backward Euler scheme and the Dual Time Step acceleration technique [20]. The maximal number of sub-iterations is set to 50. After a transitory time period a periodic vortex shedding phenomenon is observed. We employ a sampling of 300 snapshots on 3 vortex-shedding cycles to construct the POD basis for the flow solution and for the non-linear residual term. The energy distribution of the solution POD modes is plotted in Fig.1. The value of $E_{N_m} = \sum_{i=1}^{N_m} \sigma_i^2 / \sum_{i=1}^{N_r} \sigma_i^2$ for $N_m = 10$ is already over 0.9999. The first four density POD modes are depicted in Fig.2.

The ROM is then constructed for a basis with $N_m = 10$ POD modes and $N_l = 40$ non-linear POD modes. The time integration of the ROM provides the time histories of the modal coordinates $a_i(t)$ on the sampled time interval. The comparison of the modal coordinates resulting from the time integration of the ROM and from the projection of the snapshots on the POD modes is shown on the plots in Fig.3 for the first six amplitudes. Good agreement can be found between the time-domain ROM and the full-order reference. The achieved result validates the formulation of the POD-DEIM ROM for the case of a non-deforming spatial domain.

3.2 Turbulent flow around a NLR-7301 supercritical airfoil in forced motion

The ROM aeroelastic capabilities are investigated on a NLR-7301 supercritical airfoil in forced motion. The free-stream conditions are $\rho_\infty = 0.383$ kg/m^3 and $T_\infty = 277$ K with a mean angle of attack $\alpha_0 = 0.5°$.

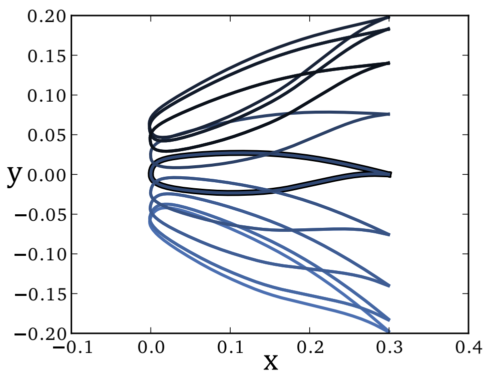

Figure 4: Displacement of the airfoil fluid-structure interface during one cycle of vibration. The initial position of the undeformed airfoil is the black thick line and the airfoil contours are the colored from the black to the blue as the time increases during the cycle of vibration.

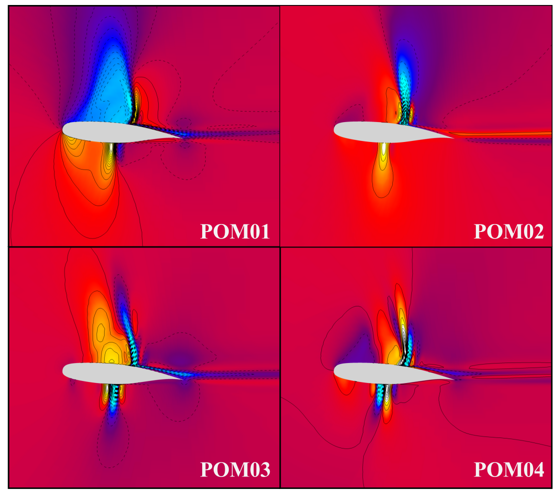

Figure 5: Density fields of the first four proper orthogonal modes (POMs) for the flow around a NLR-7301 supercritical airfoil in forced motion.

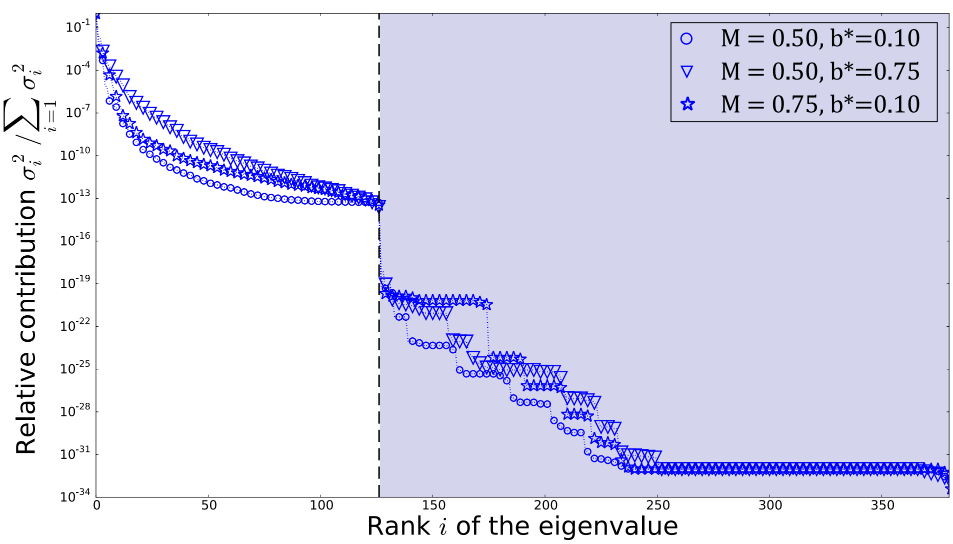

Figure 6: Proper eigenvalue spectrum of the three test cases for the NLR-7301 airfoil in forced motion.

Fig.4 illustrates the airfoil's fluid-structure interface motion prescribed using a single structural mode combining pitch and heave with a frequency $f = 10$ Hz, multiplied by a generalized coordinate q defined by the harmonic law $q(t) = b^* \cos(2\pi f t)$. Three different numerical simulations are performed for different pair of parameters $(M_\infty, b^*) \in [(0.50, 0.10); (0.50, 0.75); (0.75, 0.10)]$. The corresponding Reynolds numbers based on the chord length $c = 0.3$ fall in the interval $Re \in [1.1 \times 10^6; 1.7 \times 10^6]$. A structured C-grid is used with about 20×10^3 finite volumes. The Reynolds-averaged Navier-Stokes equations (RANS) are solved with the $k - \omega$ turbulence model of Kok [21]. The farfield distance is set to 10 chord lengths. The time integration for unsteady computations with a prescribed harmonic motion is performed with the backward Euler scheme and the Dual Time Step acceleration technique. The maximal number of sub-iterations is set to 50. The time step $\delta t = T/n_T$ is set such that the vibration period $T = 1/f$ is split into $n_T = 128$ samples. Unsteady simulations are performed to cover $n_{per} = 20$ periods of vibration to reach a stable periodic state and the resulting number of temporal iterations is $N_{it} = n_{per} n_T$. The snapshots for the computation of the POD modes are regularly sampled on the last three periods of vibration, resulting in 384 snapshots. The eigenvalues associated to the POD modes are plotted in the spectrum in Fig.6. In this case, a significant gap is observed in the spectrum for the eigenvalue $\lambda_i = 127$ since the flow field is periodic and the snapshots sampled in the last two periods are redundant with those computed on the first period of vibration. The number of independent snapshots is therefore 128 minus one since the centering process of the snapshots before the POD decreases by one the rank. The value of E_{10} is already over 0.9999. The first four density POD modes are depicted in Fig.5. In contrast with the previous example, in this case the local value of each POD mode has not a related spatial location as the mesh is deforming during the sampling process. For this reason Fig.5 is only illustrative because the index-based POD provides index (not spatial) POD modes. In this paper, ROM integration results for the case $(M_\infty, b^*) = (0.75, 0.10)$ are shown. Similar results have been obtained for the other two investigated cases, for which the smaller Mach number do not lead to any shock formation on the airfoil surface. The ROM is constructed for a basis with $N_m = 10$ POD modes and $N_l = 40$ non-linear POD modes. The time integration of the ROM provides the time histories of the modal coordinates $a_i(t)$ on the sampled time interval. The comparison of the modal coordinates resulting from the time integration of the ROM and from the projection of the snapshots on the POD modes is shown on the plots in Fig.7 for the first six amplitudes. Then, the instantaneous conservative fields are reconstructed so that the non-dimensional aerodynamic force coefficients are integrated. The comparison of the aerodynamic force coefficients of the FOM and the ROM is shown in Fig.8. Good agreement can be found between the time-domain ROM and the full-order model of reference.

4 Conclusion

This paper presented the implementation of a POD-DEIM ROM that uses an index-based computational domain in order to deal with deforming grid in aerodynamic systems. This method was motivated by the desire to build a reduced order model for non-linear flows with deforming meshes. The POD basis was used to approximate the solution and to project the FOM equations, while the non-linearities associated to the Navier-Stokes equations were efficiently dealt with by the DEIM. Firstly, the method was validated for a slightly compressible flow around a NACA 0012 airfoil at high incidence for a non-deforming mesh. Then, this method was applied and validated for the flow induced by

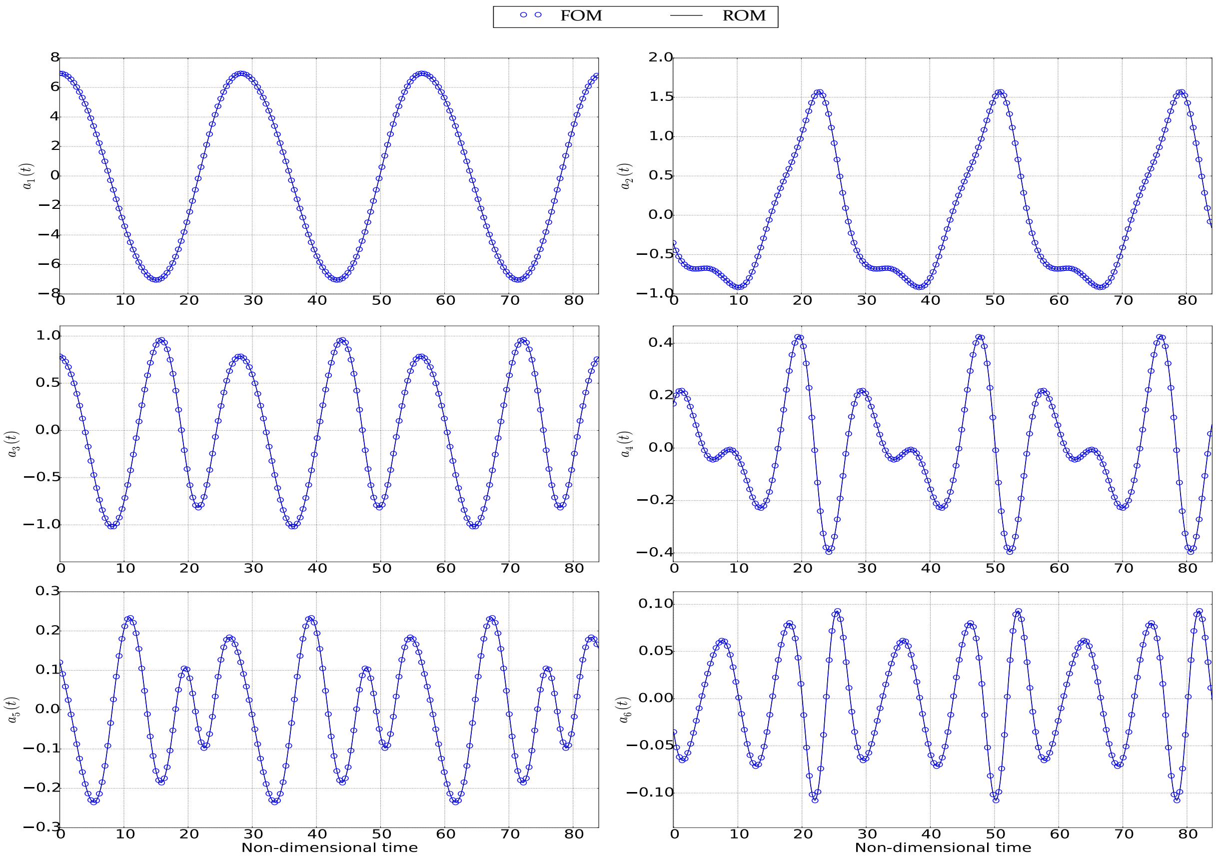

Figure 7: Comparison of the first six modal coordinates $a_i(t)$ computed as the solution of the ROM equations and the FOM coordinates defined by the projection of the snapshots on the POD modes.

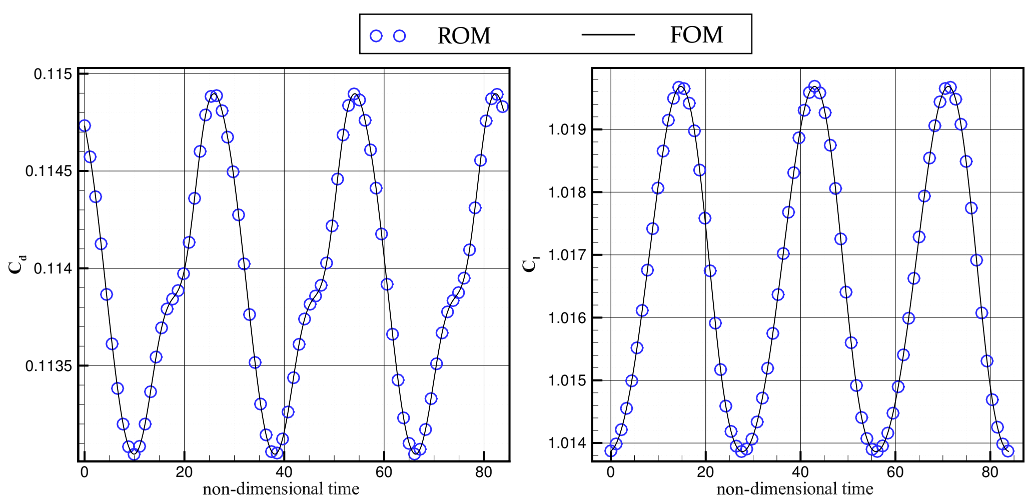

Figure 8: Comparison of the non-dimensional aerodynamic force coefficients related to the the ROM and the FOM conservative fields.

a forced oscillation of a NLR-7301 supercritical airfoil at high *Re* number. The index-based POD-DEIM ROM properly reproduces the aerodynamic fields in both cases. A clarification was provided about the preliminar implementation of the operator for the non-linear term since in the present paper this issue was avoided by integrating the ROM in the sampled time interval.

Future work will extend the presented method to a longer integration time interval or to a different flow parameter configuration that implies an appropriate definition of the POD basis and a proper implementation of the non-linear term DEIM operator. Finally, the perspective will be the integration of the presented technique in a coupled fluid-structure problem in which the structure motion depends on the aerodynamics loads.

References

[1] S. Chaturantabut and D. C. Sorensen, *Nonlinear model Reduction via Discrete Empirical Interpolation*, SIAM Journal on Scientific Computing, 32 (5), pp. 2737–64, 2010.

[2] M. F. de Pando, P. J. Schmid and D. Sipp, *Nonlinear Model-order Reduction for Oscillator Flows Using POD-DEIM*, Procedia IUTAM, 14, pp. 329 – 336, 2015. IUTAM ABCM Symposium on Laminar Turbulent Transition.

[3] J. S. R. Anttonen, P. I. King and P. S. Beran, *POD-Based Reduced-Order Models with Deforming Grids*, Mathematical and Computer Modelling, 38 (1-2), pp. 41–62, 2003.

[4] B. A. Freno, N. R. Matula, R. L. Fontenot and P. G. A. Cizmas, *The use of Dynamic basis functions in proper orthogonal decomposition*, Journal of Fluids and Structures, 54, pp. 332–60, 2015.

[5] G. C. Lewin and H. Haj-Hariri, *Reduced-Order Modeling of a Heaving Airfoil*, AIAA Journal, 43 (2), pp. 270–283, February 2005.

[6] A. Placzek, D.-M. Tran and R. Ohayon, *Nonlinear POD-Based Reduced-Order Models for Aeroelastic Compressible Flows*, *International Forum on Aeroelasticity and Structural Dynamics*, Seattle (WA), 21–25 juin 2009.

[7] R. Bourguet, M. Braza and A. Dervieux, *Reduced-order Modeling of Transonic Flows Around an Airfoil Submitted to Small Deformations*, J. Comput. Phys., 230 (1), pp. 159–184, January 2011.

[8] E. Liberge and A. Hamdouni, *Reduced order modelling method via proper orthogonal decomposition (POD) for flow around an oscillating cylinder*, Journal of Fluids and Structures, 26 (2), pp. 292–311, 2010.

[9] W. Stankiewicz, M. Morzyński, R. Roszak, B. R. Noack and G. Tadmor, *Reduced order modelling of a flow around an airfoil with a changing angle of attack*, Archives of Mechanics, 60 (6), pp. 509–526, 2008.

[10] W. Stankiewicz, R. Roszak and M. Morzyński, *Arbitrary Lagrangian-Eulerian approach in reduced order modeling of a flow with a moving boundary*, Progress in Propulsion Physics, 5, pp. 109–124, 2013.

[11] B. A. Freno and P. G. A. Cizmas, *A proper orthogonal decomposition method for nonlinear flows with deforming meshes*, International Journal of Heat and Fluid Flow, 50, pp. 145–59, 2014.

[12] L. Cambier, S. Heib and S. Plot, *The Onera elsA CFD software: input from research and feedback from industry*, Mechanics & Industry, 14 (3), pp. 159–74, 2013.

[13] L. Sirovich, *Turbulence and the Dynamics of Coherent Structures, Parts I-III*, Quarterly of Applied Mathematics, XLV (3), pp. 561–90, 1987.

[14] P. Holmes, J. L. Lumley and G. Berkooz, *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge University Press, 1996.

[15] B. R. Noack, K. Afanasiev, M. Morzynski, G. Tadmor and F. Thiele, *A hierarchy of low-dimensional models for the transient and post-transient cylinder wake*, Journal of Fluid Mechanics, 497, pp. 335–63, 2003.

[16] A. Iollo, S. Lanteri and J. A. Désidéri, *Stability Properties of POD-Galerkin approximations for the compressible Navier-Stokes equations*, Theoretical and Computational Fluid Dynamics, 13 (6), pp. 377–96, 2000.

[17] A. Placzek, J.-F. Sigrist and A. Hamdouni, *Numerical simulation of an oscillating cylinder in a cross-flow at low Reynolds number: Forced and free oscillations*, Computers & Fluids, 38 (1), pp. 80–100, 2009.

[18] D. Amsallem, J. Cortial, K. Carlberg and C. Farhat, *A method for interpolating on manifolds structural dynamics reduced-order models*, International Journal for Numerical Methods in Engineering, 80 (9), pp. 1241–58, 2009.

[19] K. Carlberg, C. Farhat, J. Cortial and D. Amsallem, *The GNAT method for nonlinear model reduction: Effective implementation and application to computational fluid dynamics and turbulent flows*, Journal of Computational Physics, 242 (0), pp. 623 – 647, 2013.

[20] A. Jameson, *Time Dependent Calculations using Multigrid, with Application to Unsteady Flows past Airfoils and Wings*, *10th Computational Fluid Dynamics Conference*, n^{o} 1596, Honolulu (HI), 24–26 Juin 1991.

[21] J. C. Kok, *Resolving the dependence on free-stream values for the k-ω turbulence model.*, Technical report NLR n^{o} 99295, 1999.