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NUMERICAL SIMULATION OF 3D FREE SURFACE FLOWS IN TIME DEPENDENT CURVILINEAR COORDINATES

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Abstract. *We propose a three dimensional non-hydrostatic shock-capturing numerical model for the simulation of wave propagation, transformation and breaking, which is based on an original integral formulation of the contravariant Navier-Stokes equations, devoid of Christoffel symbols, in general time-dependent curvilinear coordinates.*

1 INTRODUCTION

In recent years, the numerical simulation of the wave motion and the associated hydrodynamic phenomena in coastal regions has been the subject of several publications. One of the most used approaches is based on the depth averaged motion equations[1-5], which are obtained by assuming a simplified distribution of the hydrodynamic quantities along the vertical direction (depth averaged models). This approach proves to be valid only in the cases in which a fully three dimensional representation of the motion is not needed.

A different approach for the wave motion simulation is based on the numerical integration of the three-dimensional Navier-Stokes equations. Some of the most recent models based on this approach use a coordinate transformation in the vertical direction, named sigma coordinate transformation, by which the Cartesian vertical coordinate is expressed as a function of a moving vertical coordinate, σ, which adjusts to the free surface motions[6-8]. This coordinate transformation does not concern the horizontal coordinates x^1 and x^2, consequently does not allow to accurately represent the coastal regions complex geometries.

In order to overcome the limits that are imposed by Cartesian grids, the numerical simulation of the fluid motion in three-dimensional form on domains characterised by complex geometries can be carried out by using boundary conforming curvilinear coordinate systems and by expressing the governing equations in contravariant formulation.

A complete differential contravariant formulation of the Navier-Stokes equations in time varying, curvilinear coordinates was achieved by Ogawa and Ishiguro[9] and Luo and Bewley[10]. Such differential form includes the covariant derivatives of contravariant vectors which imply the presence of the Christoffel symbols, that prevents the convective terms of the motion equations from being expressed in conservative form[11]. It is known[12] that the numerical methods for the solution of conservation laws in which the convective terms are expressed in non-conservative form, do not guarantee the convergence to the weak solution, i.e. the solution that may contain discontinuities. In order to obtain a numerical model for the solution of conservation laws which is able to converge to the weak solution, it is necessary to express the convective terms of the differential motion equations in conservative form or express the motion equations directly in integral form[12].

In order to realise a three dimensional numerical model which is able to simulate the discontinuities in the solution related to the wave breaking on domains that reproduce the complex geometries of the coastal regions, we propose an integral contravariant form of the Navier-Stokes equations, devoid of the Christoffel symbols, in a time dependent curvilinear coordinate system. The resulting equations represents the general integral contravariant formulation of the momentum equation in a time dependent curvilinear coordinate system. Indeed, taking the limit as the volume approaches zero, with simple passages it is easy to obtain the complete differential formulation of the contravariant Navier-Stokes equations in a time dependent curvilinear coordinate system, which is the same as the one obtained by Luo and Bewley[10].

The motion equations are numerically solved in order to realize a three dimensional non-hydrostatic shock capturing numerical model which is able to simulate wave propagation and the nonlinear hydrodynamic wave phenomena related to it.

2 MODEL FORMULATION

Let $H(x^1, x^2, t) = h(x^1, x^2, t) + \eta(x^1, x^2, t)$ be the total water depth, where h is the undisturbed water depth and η is the free surface elevation with respect to the undisturbed level. We indicate the acceleration due to gravity by G and we split the pressure p into a hydrostatic part, $\rho G(\eta - x^3)$, and a dynamic one, q.

In order to accurately represent the complex geometry of a curved shaped coastal region and to follow the wave induced free surface evolution, we consider the following time dependent transformation from the Cartesian system of coordinates, (x^1, x^2, x^3), to the moving curvilinear system of coordinates, (ξ^1, ξ^2, ξ^3)

$$\xi^1 = \xi^1(x^1, x^2, x^3) \ ; \ \xi^2 = \xi^2(x^1, x^2, x^3) \ ; \ \xi^3 = \frac{x^3 + h(x^1, x^2)}{H(x^1, x^2, t)} \ ; \ \tau = t \qquad (1)$$

where the horizontal curvilinear coordinates ξ^1 and ξ^2 conform to the horizontal boundaries of the physical domain and the vertical coordinate ξ^3 varies over time to adapt to the free surface movements. This coordinate transformation basically maps the irregular, varying domain in the physical space to a regular, fixed domain in the transformed space, where ξ^3 spans from 0 to 1.

Let $\vec{g}_{(l)} = \partial\vec{x}/\partial\xi^l$ be the covariant base vectors and $\vec{g}^{(l)} = \partial\xi^l/\partial\vec{x}$ the contravariant base vectors $(l = 1,3)$. The covariant and contravariant metric coefficients are defined, respectively, by $g_{lm} = \vec{g}_{(l)} \cdot \vec{g}_{(m)}$ and $g^{lm} = \vec{g}^{(l)} \cdot \vec{g}^{(m)}$ $(l, m = 1,3)$. The Jacobian of the transformation is given by $\sqrt{g} = \sqrt{|g_{lm}|}$, where $|\ \ |$ denotes the determinant of the covariant metric coefficients g_{lm}. The transformation relationships between the components of the generic vector $\vec{b}$ in the Cartesian coordinate system and its contravariant and covariant components, b^l and b_l, in the curvilinear coordinate system are

$$b^l = \vec{g}^{(l)} \cdot \vec{b} \quad ; \quad \vec{b} = b^l \vec{g}_{(l)} \quad ; \quad b_l = \vec{g}_{(l)} \cdot \vec{b} \quad ; \quad \vec{b} = b_l \vec{g}^{(l)} \tag{2}$$

in which (and hereinafter) the summation convention, where repeated indices are automatically summed over, is employed.

In the model here presented, the physical domain occupied by the fluid is described by moving curvilinear coordinate lines and is represented by computational grid cells. The upper boundary of the computational grid moves rigidly with the free surface, while internal grid nodes move in order to preserve a given distribution along the water depth. Thus, the grid cell faces that lie on the free surface move with the same fluid velocity, while the other grid cell faces move with a different velocity. Since the computational grid cells are used as control volumes, the Navier-Stokes equations are need to be written in integral form on a control volume whose boundary surfaces move with a velocity different from the fluid velocity. The integral form of the continuity equation, in the time varying curvilinear coordinate system can be written as

$$\frac{d}{d\tau}\int_{\Delta V(\tau)} \rho dV + \int_{\Delta A(\tau)} \rho(u^m - v^m) n_m dA = 0 \tag{3}$$

where u^m $(m = 1,3)$ are the contravariant components of the fluid velocity vector, v^m are the contravariant components of the velocity vector with which the points belonging to the surface area $\Delta A(\tau)$ move and n_m are the covariant components of the outward unit vector normal to the surface of area $\Delta A(\tau)$. Since the surfaces $\Delta A(\tau)$ lie on curved surfaces (in the physical space) which are used as coordinate surfaces of the curvilinear coordinate system, v^m is equal to the contravariant components of the velocity vector of the moving coordinates, v^m_{MC}. It is easy to demonstrate that the m^{th} contravariant component of this velocity vector is $v^m_{MC} = -[\partial \xi^m(x^1, x^2, x^3, t)/\partial t]|_{\vec{x}=cost}$.

In the curvilinear coordinate system, in order to express the momentum conservation law in integral form, the rate of change of the momentum of the material volume and the net force acting on it must be projected in a physical direction. The direction in space of a given curvilinear coordinate line changes, in contrast with the Cartesian case. Thus, the volume integral of the projection of motion equations onto a curvilinear coordinate line has no physical meaning since it does not represent the volume integral of the projection of the aforementioned equations in a physical direction[11]. We identify a physical direction with the direction of a constant and parallel vector field $\vec{\lambda}$. This vector field is represented in the Cartesian coordinate system by constant and uniform components and in the curvilinear coordinate system by constant and space-varying (not uniform) components, λ_l. Indeed, since the base vectors of the curvilinear coordinate system vary from point to point, also the relative values of vector components λ_l must vary in order to represent the same physical direction at every point. As a constant and parallel vector field $\vec{\lambda}$, we choose the one which is normal to the coordinate line on which the ξ^l coordinate is constant at point $P_0(\xi_0^1, \xi_0^2, \xi_0^3) \in \Delta V$. The contravariant base vector at point P_0, indicated by $\vec{g}^{(l)}(\xi_0^1, \xi_0^2, \xi_0^3)$, is by definition normal to the coordinate line on which ξ^l is constant and is used in this work to identify the above vector field. Let $\lambda_k(\xi^1,\xi^2,\xi^3)$ be the covariant component of $\vec{g}^{(l)}(\xi_0^1,\xi_0^2,\xi_0^3)$, given by $\lambda_k(\xi^1,\xi^2,\xi^3) = \vec{g}^{(l)}(\xi_0^1,\xi_0^2,\xi_0^3)\cdot\vec{g}_{(k)}(\xi^1,\xi^2,\xi^3)$, and let indicate $\vec{g}^{(l)} = \vec{g}^{(l)}(\xi_0^1,\xi_0^2,\xi_0^3)$ and $\vec{g}_{(k)} = \vec{g}_{(k)}(\xi^1,\xi^2,\xi^3)$. The momentum conservation law reads

$$\frac{d}{d\tau}\int_{\Delta V(\tau)}\vec{g}^{(l)}\cdot\vec{g}_{(k)}\rho u^k dV + \int_{\Delta A(\tau)}\vec{g}^{(l)}\cdot\vec{g}_{(k)}\rho u^k(u^m - v^m)n_m dA = \int_{\Delta V(\tau)}\vec{g}^{(l)}\cdot\vec{g}_{(k)}\rho f^k dV + \int_{\Delta A(\tau)}\vec{g}^{(l)}\cdot\vec{g}_{(k)}T^{km}n_m dA \tag{4}$$

where f^l $(l = 1,3)$ are the contravariant components of the external body forces per unit mass vector and T^{lm} are the contravariant components of the stress tensor.

Let $\sqrt{g_0} = \vec{k}\cdot(\vec{g}_{(1)}\wedge\vec{g}_{(2)})$, where $\vec{k}$ indicates the vertical unit vector and $\wedge$ indicates the vector product. It is not difficult to verify that in the specific case of the above mentioned transformation, the Jacobian of the transformation can be written in the form $\sqrt{g} = H\sqrt{g_0}$. This makes it possible to write an original three-dimensional integral contravariant conservative form of the momentum, in which the conserved variables are given by the cell averaged product between the water depth H and the three contravariant components of the punctual velocity u^l with $l = 1,3$. To this end, we define the cell averaged values in the transformed space as $\overline{H} = \frac{1}{\Delta A_0^3\sqrt{g_0}}\int_{\Delta A_0^3} H\sqrt{g_0}d\xi^1 d\xi^2$ and $\overline{Hu^l} = \frac{1}{\Delta V_0\sqrt{g_0}}\int_{\Delta V_0}\vec{g}^{(l)}\cdot\vec{g}_{(k)}u^k H\sqrt{g_0}d\xi^1 d\xi^2 d\xi^3$, where a restrictive condition has been introduced on the control volume $\Delta V(\tau)$ that defines $\Delta V(\tau)$ as the volume of a physical space that is bounded by surfaces lying on the curvilinear coordinate surfaces. In the curvilinear coordinate system $\Delta V(\tau) = \int_{\Delta V_0}\sqrt{g}d\xi^1 d\xi^2 d\xi^3$, where $\Delta V_0 = \Delta\xi^1\Delta\xi^2\Delta\xi^3$ indicates the corresponding volume in the transformed space. Analogously, in the curvilinear coordinate system, the area of a surface of the physical space that lies on the coordinate surface in which ξ^α is constant is $\Delta A^\alpha(\tau) = \int_{\Delta A_0^\alpha}\left|\vec{g}_{(\beta)}\wedge\vec{g}_{(\gamma)}\right|d\xi^\beta d\xi^\gamma$, where $\Delta A_0^\alpha = \Delta\xi^\beta\Delta\xi^\gamma$ indicates the corresponding area in the transformed space. It must be noted that the volume $\Delta V(\tau)$ and the surfaces $\Delta A^\alpha(\tau)$ are functions of time, because they are expressed as functions of the base vectors, $\vec{g}_{(l)}$, and the Jacobian of the transformation, $\sqrt{g}$, whose values change over time as the curvilinear coordinates follow the displacements of the free surface. Conversely, the volume ΔV_0 and the areas ΔA_0^α are not time dependent.

By adopting the volume $\Delta V(\tau)$ defined above as control volume and by using the definition of the cell averaged values $\overline{H}$ and $\overline{Hu^l}$, in the transformed space, equations 3 and 4 can be rewritten in an integral contravariant expression of the three-dimensional motion equations, in which the Christoffel symbols are absent, in the time dependent coordinate system, (ξ^1,ξ^2,ξ^3,τ)

$$\frac{\partial\overline{H}}{\partial\tau} = \frac{1}{\Delta A_0^3\sqrt{g_0}}\sum_{\alpha=1}^{2}\left[\int_0^1\int_{\Delta\xi_0^{\alpha+}}u^\alpha H\sqrt{g_0}d\xi^\beta d\xi^3 - \int_0^1\int_{\Delta\xi_0^{\alpha-}}u^\alpha H\sqrt{g_0}d\xi^\beta d\xi^3\right] \tag{5}$$

$$
\begin{aligned}
\frac{\partial \overline{Hu^l}}{\partial \tau} = & -\frac{1}{\Delta V_0 \sqrt{g_0}} \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A_0^{\alpha+}} \left[\vec{g}^{(l)} \cdot \vec{g}_{(k)} Hu^k (u^\alpha - v^\alpha) + \vec{g}^{(l)} \cdot \vec{g}^{(\alpha)} GH^2 \right] \sqrt{g_0} d\xi^\beta d\xi^\gamma \right. \\
& \left. - \int_{\Delta A_0^{\alpha-}} \left[\vec{g}^{(l)} \cdot \vec{g}_{(k)} Hu^k (u^\alpha - v^\alpha) + \vec{g}^{(l)} \cdot \vec{g}^{(\alpha)} GH^2 \right] \sqrt{g_0} d\xi^\beta d\xi^\gamma \right\} \\
& + \frac{1}{\Delta V_0 \sqrt{g_0}} \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A_0^{\alpha+}} \vec{g}^{(l)} \cdot \vec{g}^{(\alpha)} GhH \sqrt{g_0} d\xi^\beta d\xi^\gamma - \int_{\Delta A_0^{\alpha-}} \vec{g}^{(l)} \cdot \vec{g}^{(\alpha)} GhH \sqrt{g_0} d\xi^\beta d\xi^\gamma \right\} \\
& + \frac{1}{\Delta V_0 \sqrt{g_0}} \sum_{\alpha=1}^{3} \left\{ \int_{\Delta A_0^{\alpha+}} \vec{g}^{(l)} \cdot \vec{g}_{(k)} \frac{T^{k\alpha}}{\rho} H \sqrt{g_0} d\xi^\beta d\xi^\gamma - \int_{\Delta A_0^{\alpha-}} \vec{g}^{(l)} \cdot \vec{g}_{(k)} \frac{T^{k\alpha}}{\rho} H \sqrt{g_0} d\xi^\beta d\xi^\gamma \right\} \\
& - \frac{1}{\Delta V_0 \sqrt{g_0}} \int_{\Delta V_0} \vec{g}^{(l)} \cdot \vec{g}^{(m)} \frac{\partial q}{\partial \xi^m} H \sqrt{g_0} d\xi^1 d\xi^2 d\xi^3
\end{aligned}
\tag{6}
$$

where the continuity equation has been integrated over a vertical water column which is bounded by coordinate surfaces, between the bottom (impenetrability condition $u^3 = 0$) and the free surface (kinematic condition $v^3 = u^3$). In equation 5 $\Delta\xi_0^{\alpha+}$and $\Delta\xi_0^{\alpha-}$ (with $\alpha = 1,2$) indicate the contour line of the surface element ΔA_0^3 on which ξ^α is constant and which are located at the larger and at the smaller value of ξ^α respectively, and the last two terms on the right hand side are calculated, respectively, at the free surface ($\xi^3 = 1$) and at the bottom ($\xi^3 = 0$). In equation 6 $T^{k\alpha}$ is now the stress tensor in which the pressure is omitted, the gradient of the hydrostatic pressure is split into two parts by using $\eta = H - h$ and the last integral on the right hand side of equation 6 is related to the gradient of the dynamic pressure.

The resulting motion equations are numerically solved, on a time dependent curvilinear coordinate system, by a finite volume shock capturing scheme, which uses an approximate HLL-type Riemann solver[13]. The advancing in time of the numerical solution is carried out by a second order accurate Strong Stability Preserving Runge-Kutta (SSPRK) fractional-step method in which, at every stage of the Runge-Kutta method, a predictor velocity field is obtained by the shock-capturing scheme and a corrector velocity field is added to the previous one, in order to produce a non-hydrostatic divergence-free velocity field and to update the water depth. The corrector velocity field is obtained by solving a Poisson equation, expressed in integral contravariant form. This equation is solved through a multigrid method which uses a four-colour Zebra Gauss-Seidel line-by-line method as smoother.

3 MODEL RESULTS

In this section, the results obtained by numerically reproducing the laboratory experiment carried out by Hamm[14] are shown and discussed. This test simulates breaking waves propagation on a plane sloping beach of 1:30 with a rip channel excavated along the centerline (Figure 1). Incident regular waves are considered with a period of $T = 1.25$s and wave height of $H = 0.07$m. The simulation is carried out on a curvilinear computational grid (Figure 1) and is run for 520s with a time step of 0.0025s. Figure 2 shows a three dimensional detail of an instantaneous wave field (contour) carried out with the curvilinear computational grid (lines), at the time when the breaking induced circulation is fully developed. It is possible to notice that, when approaching the beach, the wave fronts rotate and an increment of the wave height in correspondence with the channel location due to the wave-current interaction.

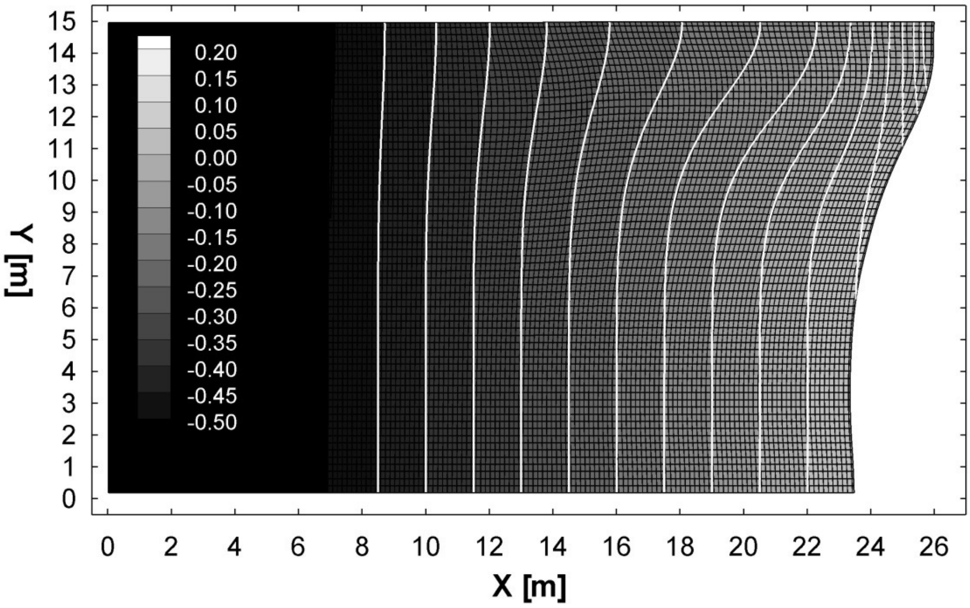

Figure 1: Bathymetry (contour) and curvilinear grid (lines) of the computational domain (only one out of every three coordinate lines is shown)

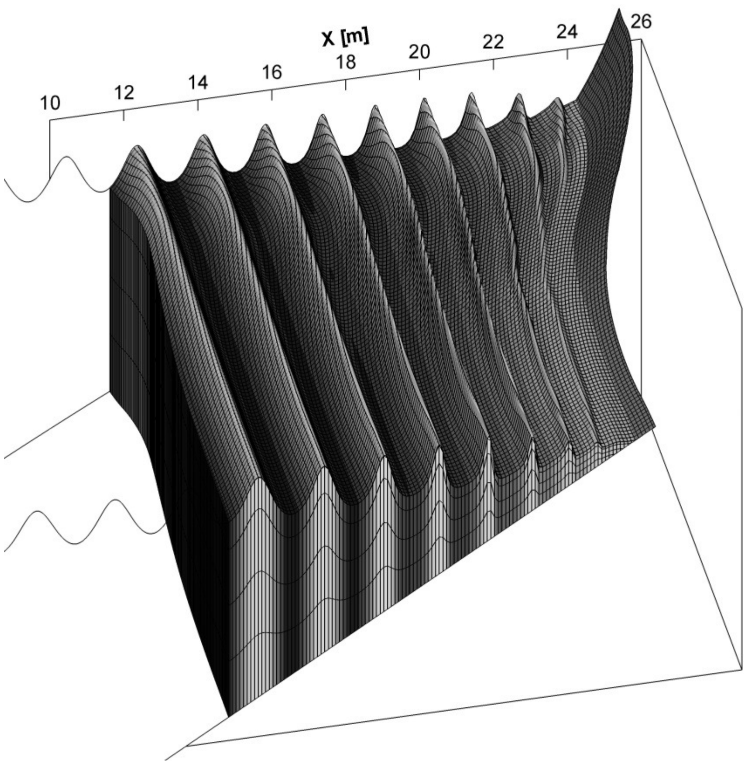

Figure 2: Three dimensional detail of an instantaneous wave field (contour) at the time when the breaking induced circulation is fully developed and of the curvilinear computational grid (lines)

4 CONCLUSIONS

A three dimensional non-hydrostatic shock-capturing numerical model for the simulation of wave propagation, transformation and breaking, has been presented. The proposed model is based on an original integral formulation of the contravariant Navier-Stokes equations, devoid of Christoffel symbols, in general time-dependent curvilinear coordinates. The contravariant Navier-Stokes equations are numerically solved, on a time dependent curvilinear coordinate system, by a finite volume shock capturing scheme, which uses an approximate HLL-type Riemann solver. A coordinate transformation maps the time-varying irregular physical domain to a fixed uniform transformed computational one.

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